Selected Topics in Decision Modeling
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Lecture - 09
Probabilistic Dynamic Programming (Contd.)

So, let us begin lecture 9 our course Selected Topics in Decision Modeling. If you recall in our previous lecture, we have started probabilistic dynamic programming. So, the same probabilistic dynamic programming we shall continue in this lecture also right. So, in this particular lecture as you can see our topic is probabilistic dynamic programming.

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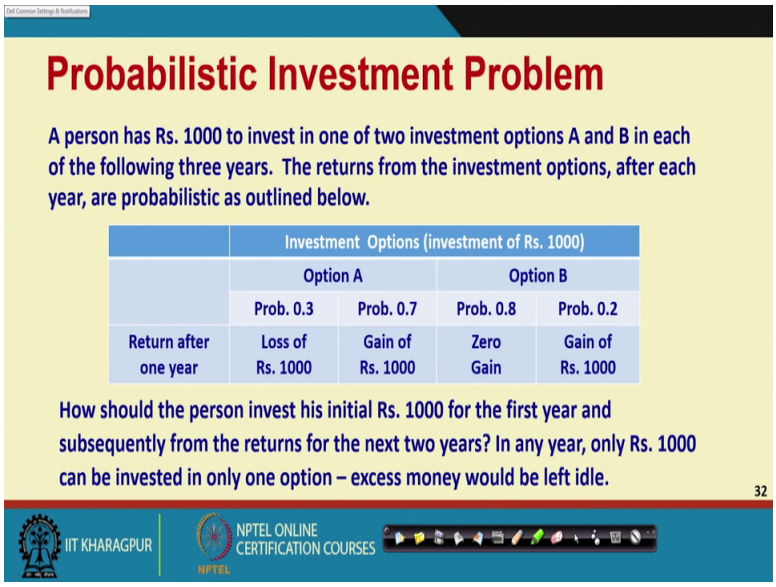

So, in our previous class we have considered this particular problem. So, I just read it one more time; A person has rupees 1000 to invest in one of two investment options A and B in each of the following three years.

The returns from the investment options after each year at probabilistic as outlined below. So, what are those 2 options if you recall that, if you put rupees 1000 in option A after 1 year you will lose all your money that is the entire 1000 is gone or you gain 1000 rupees; that means, you get 2000 rupees at the end of 1 year, but the probabilities are 30 percent and 70 percent for these 2 options. Whereas, for the option B either you will have 0 gain or you have a gain of rupees 1000 you know with probabilities 0.8 and 0.2 respectively.

So, if you follow this for the next 3 years I mean in total 3 years. So, may be first year and then 2 more years, where will you invest your rupees 1000 you know so, that you get the maximum return. So, this is really the probabilistic investment problem you know to make the problem simple, we have made some assumption one assumption is that in any year, you have only you know 3 options either you invest nothing or you invest rupees 1000 in option A or you invest rupees 1000 in option B. So, these are the our 3 options.

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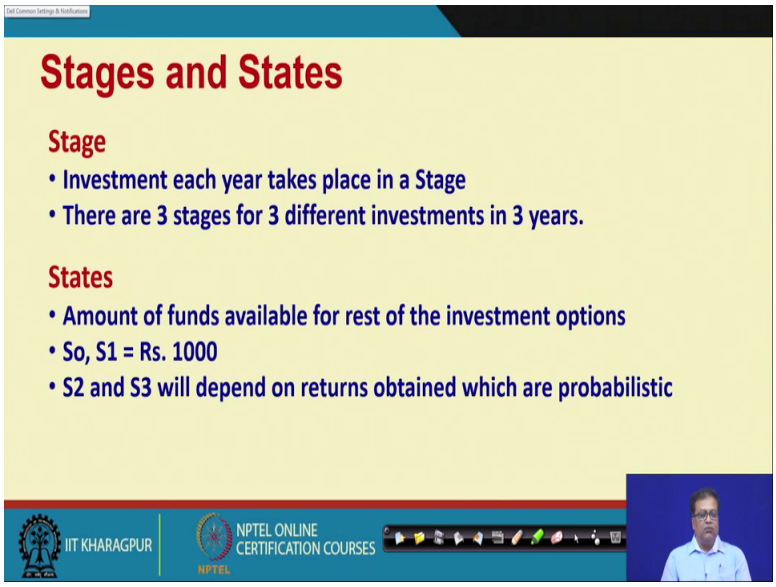

So, if we have to solve the problem by dynamic programming, we divide the problem in 2 stages. Since there are 3 years so, we make 3 stages in the first stage you know stage 1 is the year 1, stage 2 is year 2, and stage 3 is the year 3 of investment. So, these are stages. What is the state the states are amount of fund available for rest of the investment options; that means, you see because at the stage 1, you have all the 3 stages.

So, you have the entire 1000 rupees available for investment in stage 2 you have 2 stages that is stage 2 and stage 3, but you do not know exactly how much rupee is with you. So, let us assume you have S 2 available, which you will invest in second year and the return you will be investing in the third year. So, that is stage 2.

In stage 3 you have only the last stage that is stage 3 you know at the last year where you invest S 3 amount of money.

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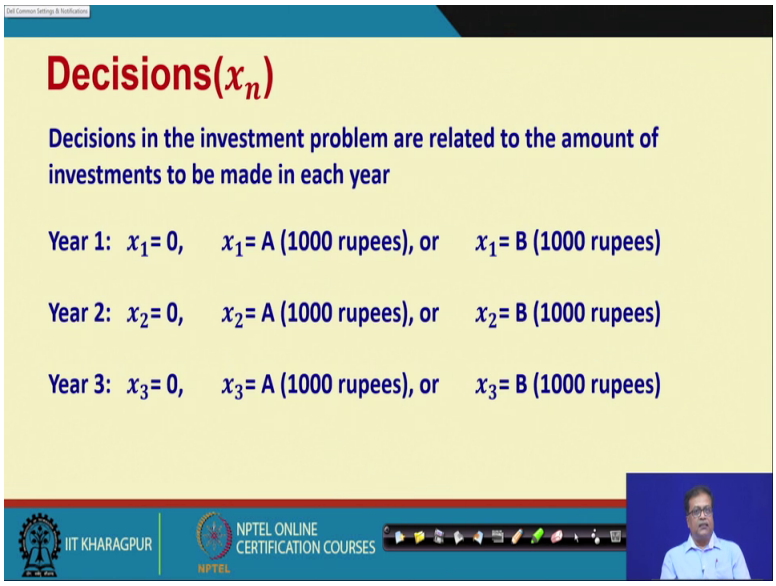

So, these are our states. Then, the what are the decisions as I already told that in every year you have basically 3 options either you do not invest at all. So, x 1 is equal to 0 or you invest in A 1000 rupees or you invest in B 1000 rupees. So, these are our decision.

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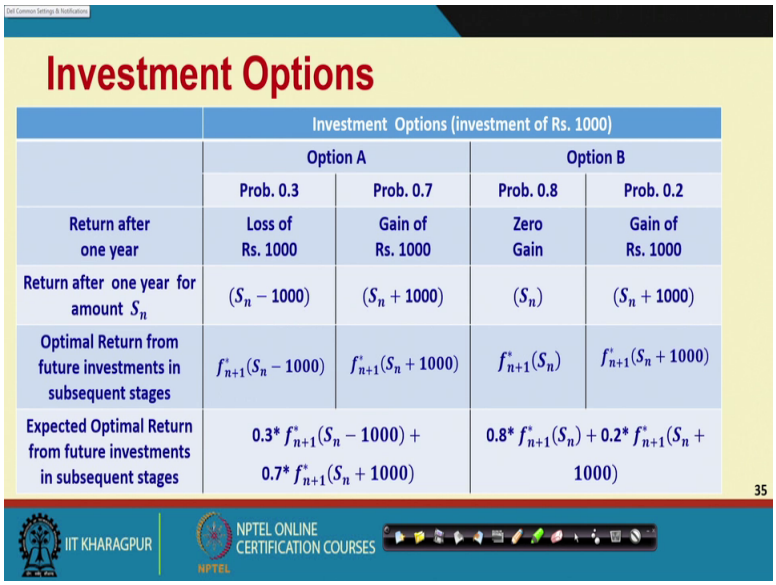

So, with that let us see how it works out? So, here is the worksheet. So, if you if you really look at the worksheet you will find that you know the return after 1 year is loss of rupees 1000 or gain of rupees 1000 this is the 2 options. For option B 0 gain or gain of rupees 1000 these are the 2 options.

So, if you really think in terms of S n, n is the stage then you have S n minus 1000 and S n plus 1000 and in option B S n and S n plus 1000. Now, the expected return will not be based on these two figures only, the these two figures you will then you know invest in further maybe one of the options and then you get money, say it does not matter you get a S n minus 1000, you may actually invest in option B there is it all right or you do not invest.

So, you know all those probabilities are also there in subsequent years. So, you do that you get some return again that return you put in subsequent stages. So, all those subsequent stages whatever you get return, after all those subsequent stages that we call as a f n star n you know n plus 1 S n minus 1000 in this case.

So, whatever amount you get after this one year, that you put in the subsequent stages and the final return from all those stages is f n plus 1 star S n minus 1000 is it all right. So, then the expected optimal return from future investments will be then 0.3 times the first term and 0.7 times the second term for option A, and 0.8 times the first term and 0.2 times the second term for the subsequent years. So, that is how you really calculate what is known as the expected optimal return.

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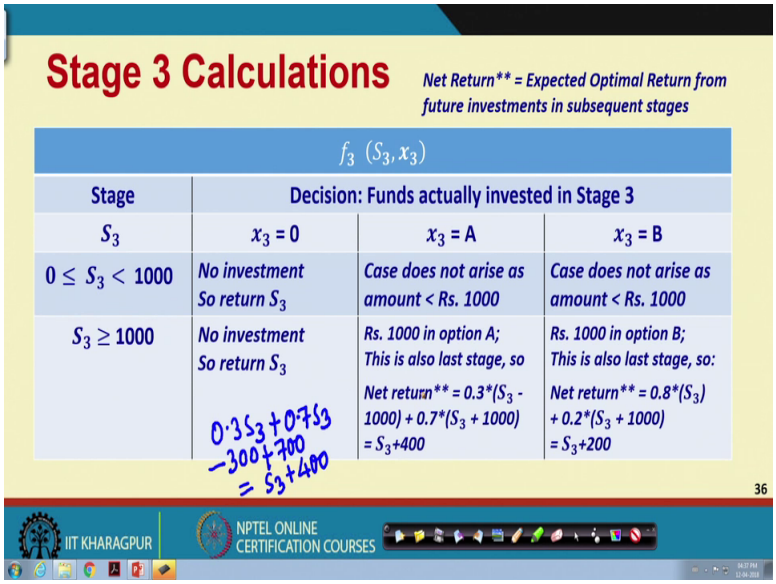

So, if this is what really are my investment options? So, I showed you in the previous class the stage 3 calculations. So, you see the stage 3 2 situations can actually happen is it not or either you have S 3 between 0 and 1000 or you had S 3 more than 1000.

So, only these 2 situations can result. So, really speaking even if the amount is more than 2000 it hardly matters, because even suppose you have really 3000 rupees at the beginning of stage 3, but you cannot invest 3000 rupees we can invest only 1000 rupees, that is the restriction we had already given. So, with that restriction what will do you will invest 1000, whatever return you will I get the remaining 2000 rupees will be with you and that plus that return that will be the final return.

So, really speaking if you have S 3 only two situations can occur you know if you invest in A either you have S 3 minus 1000 or S 3 plus 1000 in case of x is equal to A or if it is for x 3 equal to B, then your options could be either you know S 3 because there is no gain that is A first option or S 3 plus 1000.

So, those are the 2 options that you get for the option B. So, what will really happen you know you will have S 3.3 times, S 3 minus 1000, and 0.7 times S 3 plus 1000. So, if you compute these let us compute very simple computation really, it should be 0.3 S 3 plus you know 300 sorry 0.7 S 3 these are the S 3 part and here 0.3 into 1000 so, minus 300 and 0.7 into 1000 so, plus 700.

So, this is the total therefore, becomes S 3 plus 400. So, you understand this calculation. So, 0.3 times S 3 minus 0.3 times 1000, that is the minus 300 0.7 times S 3 plus 0.7 time 1000 that is 700 and the total becomes S 3 plus 400 so, that becomes the net return. But, please also understand this calculation was possible, because stage 3 is the last stage. Since this is the last stage no further returns will be obtained from the calculations from the investment amounts, which is available with us at that point of time. So, that will be our net return, S 3 plus 400.

What, will be the similar calculation for x 3 equal to B, it will be 0.8 S 3 because that is a no loss situation and 1000 rupees profit situation. So, that is a second part. So, 0.2 times S 3 plus 1000. So, total again comes to 0.2 into 1000 200 0.8 S 3 and 0.2 S 3. So, S 3 plus 200, that should be our return x t equal to B.

And, if you do not invest whatever be the case you know you will return will be S 3 itself whatever amount you had. So, those are the different values, which we are going to have you know in this situation, but if x 3 equal to A or x 3 equal to B will not arise when S 3 is between 0 and 1000. So, this is our stage 3.

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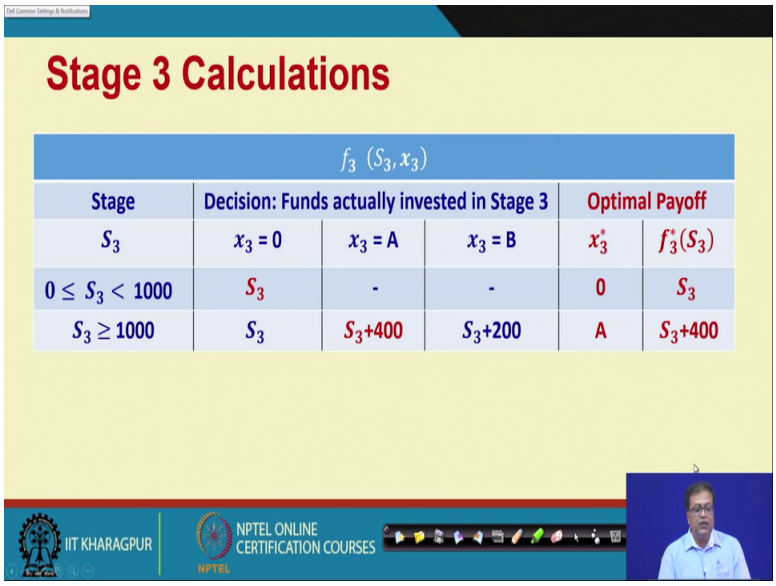

Now, let us you know move over to now you know stage 2 and this stage 3 things you know can be summarized in this manner. So, all those values only written here, is it alright and from here we can find out for the two cases that, if S 3 is less than 1000, but above 0 then optimal decisions is S 3 and investment none.

If, S 3 is greater than 1000 then you have 2 options 3 options S 3 S 3 plus 400 and S 3 plus 200. Out of that the S 3 plus 400, that is our S 3 plus 400 you know that is our maximum value. Since that is our maximum value so, therefore, optimal payoff will be that these A and that S 3 plus 400 is it ok.

So, I hope it is understood that up to stage 3 we have got the different options that we have. Now, these we shall carry forward this optimal payoff of portion into our stage 2 calculations. So, let us see how we carry on our stage 2 calculation.

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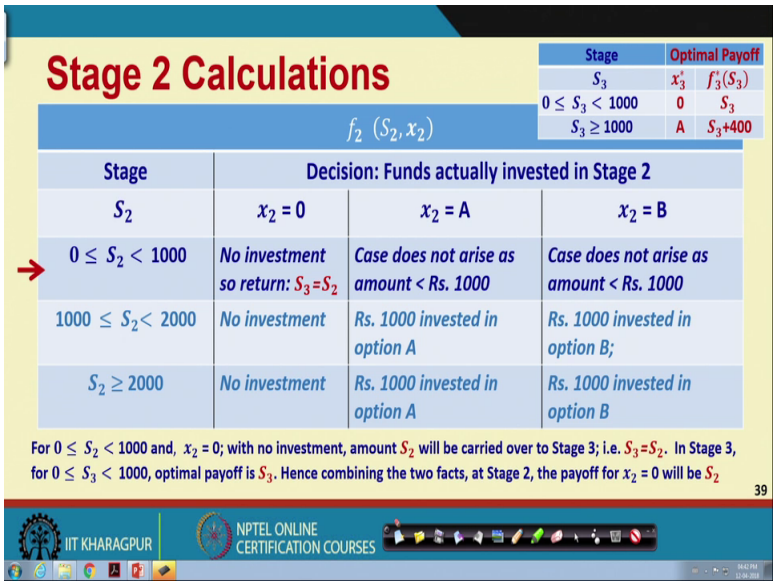

So, that is our stage 2 calculation. So, in our the stage 2 calculation, if you if you really see in our stage 2 calculation we have again 3 possible options. So, what are those 3 possible options in stage 2? See in stage 2 there is an additional option that we are having that is one is that S 2 is below 1000, but between 0 and 1000 another is 1000 plus that is 1000 to 2000, and the third option is S 2 above 2000 is it all right. So, these are the different options that we have.

So, now under that situation again we have 3 options x 2 equal to 0 x 2 equal to A and x 2 equal to B is it right. So, you can see that if it is below 1000 then all the 3 cases there are no investment. In fact, case does not arise x 2 equal to A or x 2 equal to B, but if S 2 is less than 2000, but more than 1000 that is 1000 plus or 1000 or more than 1000, then we can have no investment in the x 2 equal to 0 case x 2 equal to A invest in A x 2 equal to B invest in B and here you know x 2 less than equal to 2000 again only one investment of 1000 in A or B right. So, those are the different options that you have in our stage 2 calculation.

So, now if I go further into our stage 3 calculation if I really go into our further stage 2 calculations, then let us see the first row. If I see the first row then what I find is that in the first row these when S 2 is between 0 and 1000, when x 2 equal to 0 then no investment. So, return will be S 3 itself you see what is that S 3 equal to S 2 let us understand see at this stage we have no investment what was the money that was

invested S 2? Is it not or after you know this stage 2 is over in the beginning of stage 3, what will be the money that will be available; that means, what is the S 3?

The money that we have for investment at the beginning of stage 3, that will be S 2 itself, but then we also have to look at these chart what is that optimal decision if we have the case and now what will be your comment on S 2 S 2 is less than 1000.

We make low investment. So, you see S 2 is less than 1000, we make no investment and these particular S 2 which is you know becomes S 3 and both are equal if we invest in the stars stage 3, what will be the optimal return. You know the optimal return in that case would be nothing, but S 3 it itself, because you know that is the optimal payoff we have found from stage 3. So, S 2 goes to third stage, that is S 3 S 3 equal to S 2 and then that S 3 is giving return S 3 it itself.

That means the final return that is optimal return in this particular case where S 2 is between 0 and 1000, but less than 1000 is nothing, but S 2 it itself is not that is what is explained here for 0 S 2 between 0 and 1000 and S 2 equal to 0 with me no investment amount S 2 will be carried over to stage 3, that is S 3 equal to S 2 and in stage 3 for S 3 between 0 to 1000 optimal payoff is S 3, that you can see from that table and hence combining the 2 facts at stage 2 the payoff for next will be nothing, but S 2 itself.

Is all right; So, this is how you understand the first row. Now, once you understood the first row let us move over to the next one that is the next stage, that is stage 2 calculations and the second row.

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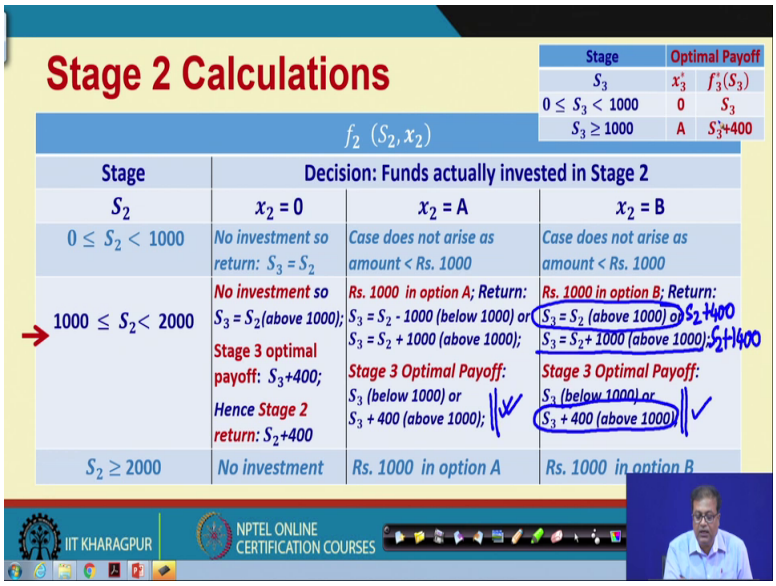

So, if you see the second row now, where S 2 is between 1000 to 2000 at that stage again you know when if you make no investment, then you see earlier we got S 3 equal to S 2. This time also the amount is more than 1000. So, we get again the same amount carried over to the third stage that is S 3, but this one is above 1000, because you know that is the case. So, if that is the case now look at the optimal table for you know the next stage at this stage, if you if you see the table then you find that if S 3 equal to 1000, then the optimal payoff is x 3 equal to A and that optimal value is S 3 plus 400.

So, this is something new from our previous slide. See, what has happened in your previous slide S 3 equal to S 2 and the case was below 1000. So, we use the first line and then I found whatever is the amount the same thing is written, but here now the optimal is A at stage 3 and the amount that you get back is S 3 plus 400, that is the expected return, but then S 3 equal to S 2 because that is the amount that is invested.

So, if stage 3 optimal payoff is S 3 plus 400 what will be the stage 2 return? What will be the stage 2 optimal? Because, the we see there is no investment same S 2 goes to S 3. So, therefore, the stage 2 optimal return in this particular situation will be nothing, but S 2 plus 400 right.

It is cannot be called optimal in that sense, because that is the return for x 2 equal to 0 the optimal will be the best result coming out of 0 A B all 3 consideration. So, this is going to be the return that is S 2 plus 400 and these value is obtained by considering the

amount of investment, that goes to the stage 3 and the optimal payoff that is coming at the stage 3 by both this consideration.

So, when an S 2 amount goes as S 3 in the stage 3, which is invested and you get and expected return of S 3 plus 400 and therefore, since S 3 equal to S 2 the stage 2 return would be S 2 plus 400 is it all right. So, I hope you understand. So, let us move to the next case x 2 equal to A.

So, can you really come out with specific discussion about it. See what happens your amount is less than 2000, but more than or equal to 1000. So, the 2 possible returns are there what are they one is that S 2 minus 1000 and the other is S 2 plus 1000. So, though the 2 different returns.

Now, you see your original value that is S 2 was less than 2000, if you invest 1000 then the remaining money will be below 1000 definitely. So, and in the other case where you get 1000 rupees more and you already had 1000 plus 1000 or more; obviously, the resulting amount is more than 1000 so; that means, the stage 3 optimal payoff will be two different case, in the first case because that was below 1000; obviously, or optimal thing will S 3.

So, look here this S 3, now this S 3 is put here how that we have an amount invested, that is S 3 which is below 1000 and we have made 1000 rupees loss our original money was less than 2000. So, now, our money is below 1000. So, that corresponding return is S 3. What will be my return if I make profit, suppose I put and amount of S 2 and you know out of S 2 1000 rupees in option A and I get A return of you know plus 1000.

So, my money now becomes S 2 plus 1000, which I invest further in stage 3 in the most optimal way. This case is this highlighted case and there my return is S 3 plus 400. So, therefore, there now this return will be S 3 plus 400, because it is above 1000 right. So, this is how we calculate stage by stage. So, you will now tell what will be the situation option B in option B please here the money is 1000 or more, you know we have 2 situations may come in stage 2.

When you put an amount of S 2 either it makes no gain no loss situation probability 80 percent or I make 1000 rupees profit, which is the profit situation. So, when I make no gain no loss situation then whatever amount S 2 I put that becomes S 3 or otherwise the

other option is I had S 2 and it becomes S 2 plus 1000 that is the optimistic situation. So, S 3 will have 2 cases both are S 2 and S 2 plus 1000 respectively and both are above 1000.

Since, both are above 1000; obviously, you know when we get to the stage 3 I will be then be guided by the optimal payoff here; that means, I put in A and I get an expected payoff of S 3 plus 400 is it all right. So, the what should be the stage 3 optimal payoff under those situations. So, under those situations then you know it will be not this it will be then S 3 plus 400 sorry these are the optimal payoff. So, this is these are the optimal payoff.

So, if these are the optimal payoff, what will be my I know the optimal payoff from these stage. So, can you calculate for both option A and option B. Let us calculate for the option B first just now we found that stage 3 optimal payoffs are S 3 and S 3 plus 400. Now, in both cases we have got above 1000. So, since in both cases we have above 1000; that means, in both cases this will be our return.

So, what will be the return here in this case S 3 equal to S 2, then the return will be S 2 plus 400. And, what will be my return here in this case our written will be then S 2 plus 1400, I hope it is clear. And in this case S 3 is below 1000 this is S 2 minus 1000 so, in this case since the optimal is S 3 itself.

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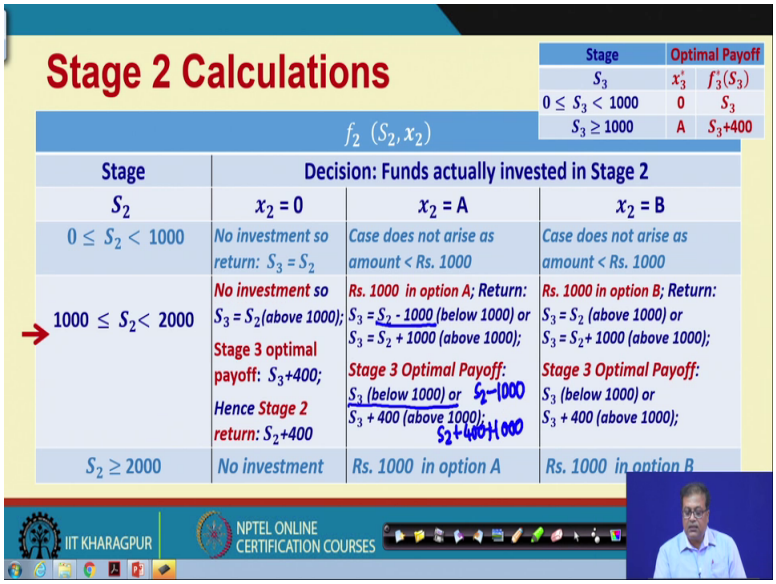

So, in this case the return will be in this case the return will be, because we had S 2 minus 1000 this return will be S 2 minus 1000, and in this case the return will be S 2 plus 400 plus 1000.

Right. So, those will be the returns. So, all of these are put in our next screen, you know that is our next screen.

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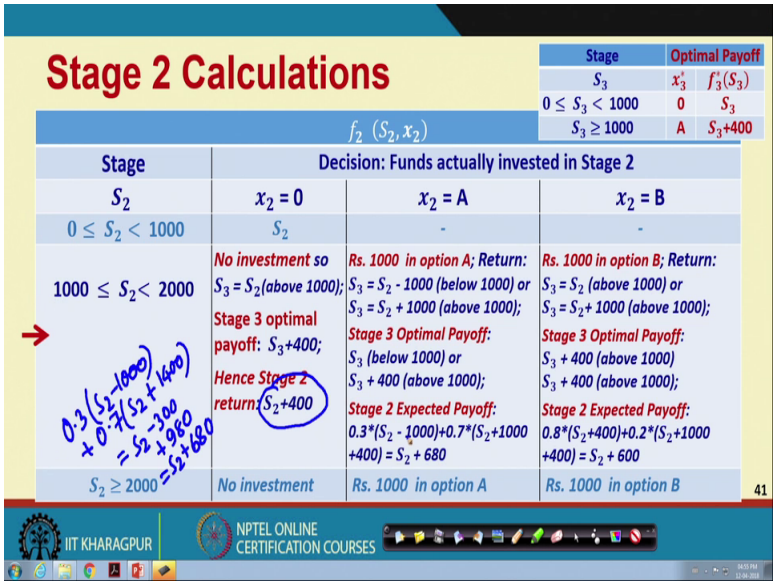

So, please look what really it has come out that is in the in the option A case option B we have already seen, that when it is between 1000 and 2000 that either it is S 3 plus 400 or S 2 plus 400, you know because S 3 plus 400 is the stage 3 S 2 return will then become S 2 plus 400. And, in this case we find that if it is below 1000 S 3 in that case it becomes S 2 minus 1000 itself or S 2 plus 1000 plus 400.

Because, that is the corresponding payoff it is above 1000. So, it becomes then 0.3 S 2 minus 1000; So, 0.3 S 2 minus 1000 plus 0.7 S 2 plus 1400. So, what is the resulting value it should be S 2 minus 3 00 plus 980. So, the resulting value should be S 2 plus 680. So, once again what is happen in x 2 equal to A our return is either S 2 minus 1000 or S 2 plus 1000, that is invested in stage 3 the stage 3 optimal returns are S 3 if it is below 1000 and S 3 plus 400 if it is above 1000.

Since, the first 1 is below 1000 it is remains as it is S 2 minus 1000 second one is the above 1000. So, it should be S 2 plus 1000 plus 400. So, those will be then the expected

payoff can be obtained by this calculation, which is S 2 plus 680; So, all right. So, this is how the calculation is done and we see for option B all the figures are you know above 1000, because there is no loss and the original amount was at 1000 or more. So, both the return figures after stage 3, you know they are above 1000 the optimal return from stage 3 or S 3 plus 400.

So, in both case 400 will be added. So, 1 is S 2 plus 400, another is S 2 plus 1000 plus 400 and the expected payoff because the probability is are 0.8 and 0.2 will then become S 2 plus 600.

So, that is how these calculations are done. So, once these calculations are done.

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Then we can you know put the try to do the last part of it that S 2 greater than 2000.

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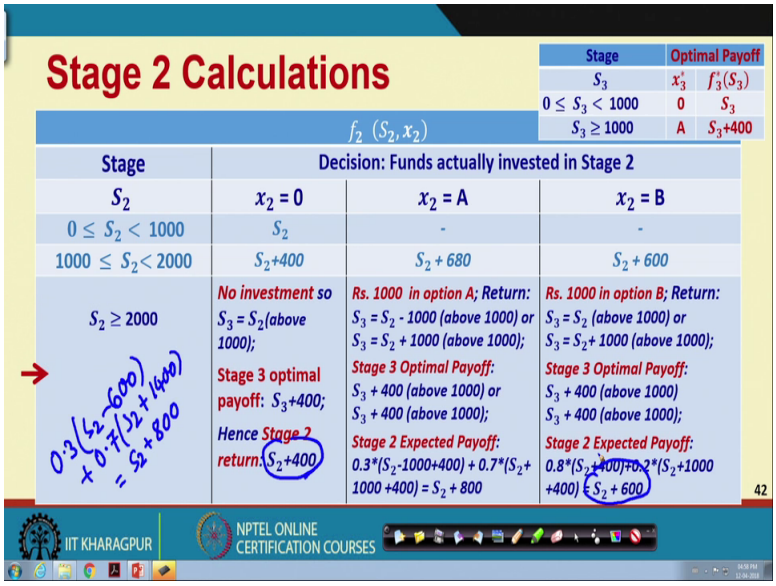

So, if S 2 greater than 2000 and we make no investment then we see that since S 2 is greater than 2000 no investment. So, S 3 will be S 2 itself and stage 3 optimal payoff for more than 1000 is S 3 plus 400. So, stage 2 return will be S 2 plus 400.

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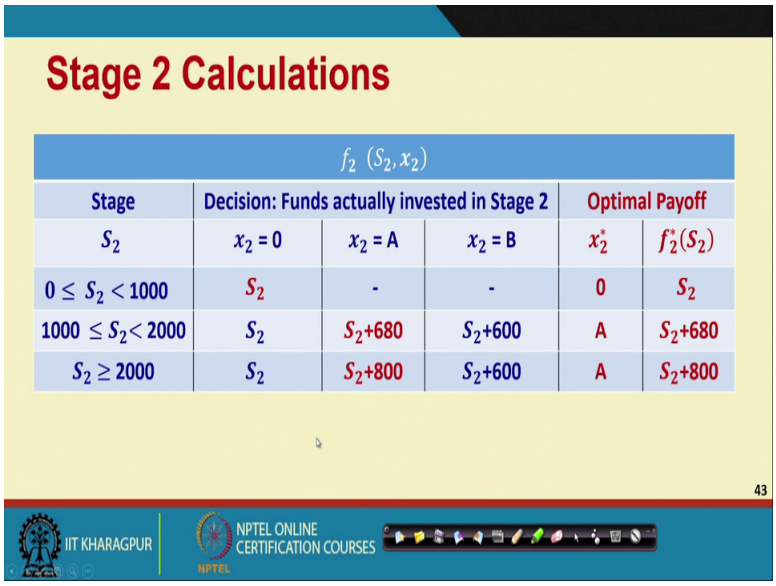

So, you see this is very straight forward. So, there will be no problem as such right. So, there will be no problem as such in these calculation this will be S 2 plus 400, but then what will happen here you know you had either S 2 minus 1000 or S 2 plus 000 the returns, because that we see in the option A case in both the cases the result is above 1000; So, if above 1000 the optimal payoff at stage 3, is S 3 plus 400 and therefore, both the cases will be guided by this.

So, both the cases will have another 400 rupees addition at the stage 3 level. So, when you add them it then become S 2 minus 1000 plus 400 and S 2 plus 1000 plus 400. So, it will be then 0.3 S 2 minus 600 plus 0.7 S 2 plus 14 00. So, when you calculate this it becomes S 2 plus 800 is it all right. So, this is 980 minus 180; So, S 2 plus 800.

And, in case of option B the similar returns will be S 2 plus 400 and S 2 plus 14 00 and 0.8 of these and 0.2 of this you get S 2 plus 600. So, that is how our stage 2 calculations will go and when you put them all together, then this is how it summarizes.

That is for the 3 cases you get the different way of figures and again you find out the optimal payoffs for all this different cases. So, these are my optimal payoff. So, if it is 1000 less than 1000 optimal is S 2, if it is between 1000 and 2000, but less than 2000 it is S 2 plus 680 and if it is above 2000 it is S 2 plus 800.

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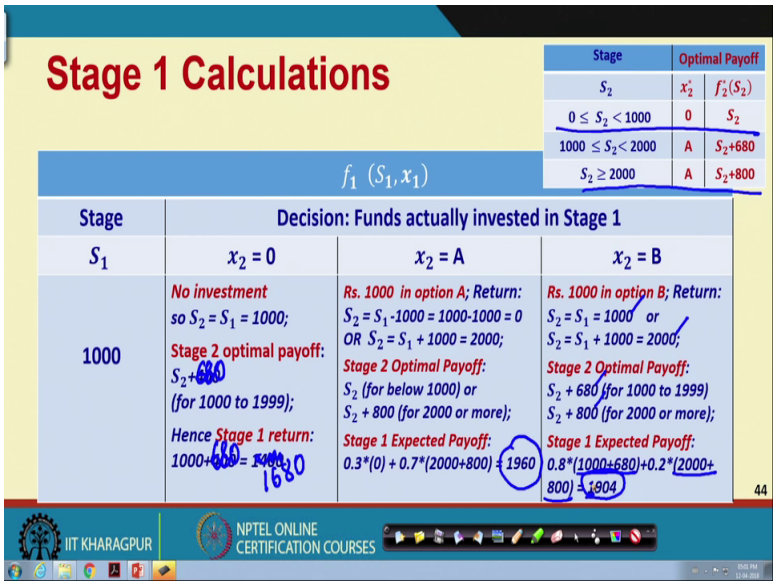

So, now if you add them all and then we move over to our stage 1 then you know you can come to our stage 1 calculation. So, now, you see in stage 1 you had just 1000 rupees. And, these 1000 rupees again you have those 3 options. Now in 1000 rupees if you do not invest at all then you know you have 1000 or more. So, the return for you in the subsequent stages will be for you know 1000 to 1000 to 2000 yeah that will be actually not 14 00 it will be 680 is it ok.

Because it was 680 so, this is not 400 you know just a minute yeah. So, it will be your 680 that is the return. So, it is 680. So, 1000 plus 680 and this will be then 1680. So, return will be 1680, that is the return. And, in this case the return will be 1000 you put so, the you know it if it is below 1000 then it is S 2 and if it is above 2000 or more because you see you have 1000 rupees at the beginning of S 1.

You put your entire 1000 rupees and if you make a loss then your return will be 0 and if you make A profit then your return will be 2000. So, the one case is here the other case is here right these are the two cases. So, return will be either S 2 or S 2 plus 800 so; that means, you know the two returns one is the return is no addition another is 800 addition. So, it is an expected payoff will be between that 0 and 2000 plus 800. So, it becomes 19 60. So, 0.3 into 0 nothing 0.7 into 20 800 comes to 1960 is it all right.

And then finally, at x 2 equal to B what you really find that your return will be either 1000 or 2000. So, in one case you add 680 and other case you add 800. So, when you add that the 1 will be 1680 the other will be 2080.

So, by making 80 percent of these and 20 percent of these you return will be 1904. So, those are my all the returns at the stage 1, then combining them all you can then have you know this 1 sorry.

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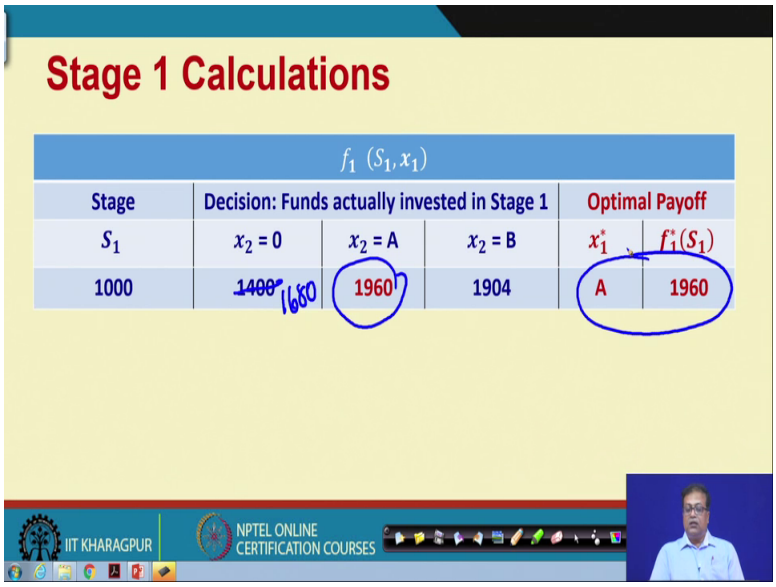

So, that one was 1680. So, there is A small change this was 1680 right. So, this is our final figure 1680 1960 and 1904 and the 5 the optimal value comes out to be 1960.

And therefore, you put an investment of 1960 in A, but then these optimal thinks has to be combined further.

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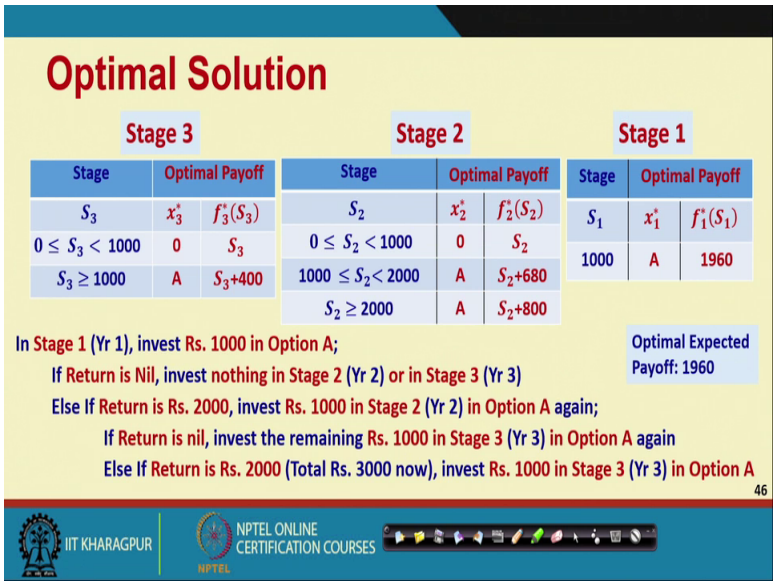

So, we will let us see that in our next slide, that is you know in stage 3 these are our optimal, in stage 2 these are our optimal, and in stage 1 these are our optimal. So, in stage 1 optimal decision is A. So, it says in stage 1 year 1 invest rupees 1000 in option A because that is optimal.

If return is nil invest nothing in stage 2, because if you put 1000 return could be 0 or 2000. If the return is nil invest nothing in stage 2 or stage 3, because you have no money you cannot invest anywhere, but if the return is rupees 2000, then invest rupees 1000 in stage 2 in option A again. Because, your return is 2000 which is this case; So, in this case optimum is A. So, invest rupees 1000 in stage 2 in option A.

Again your return could be nil, but if you have return is nil you had 2000. So, you have 1000 rupees available, which you again invest in A again because that is the case is here is it all right. And if the second stage your return is again 2000; that means, total you have now rupees 3000 is or not, because you had 2000 you invest in 1000 and gain 1000 more. So, again invest 3 1000 in stage 3 in option A right.

So, this is how you can solve the probabilistic dynamic programming problem is it ok. So, this was A little difficult exercise, but I am sure if you go through it carefully you will understand right.

Thank you very much.