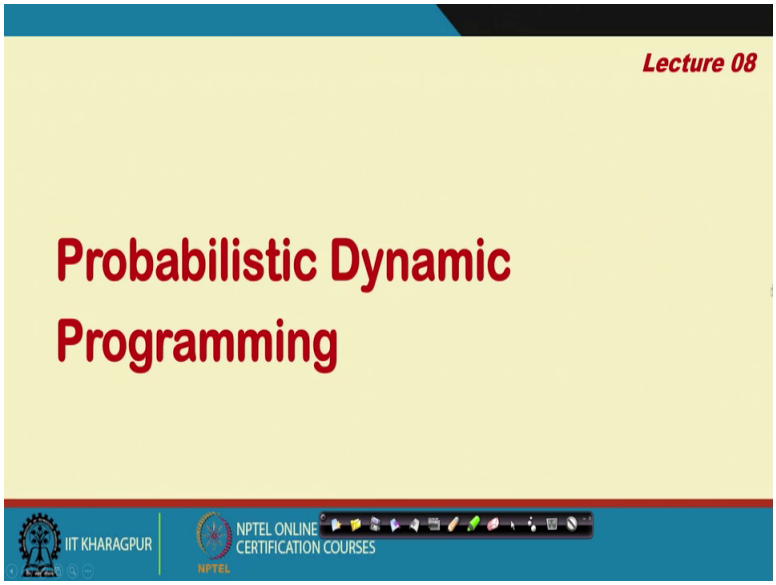
Selected Topics in Decision Modeling
Prof. Biswajit Mahanty
Department of Industrial and Systems Engineering
Indian Institute of Technology, Kharagpur

Lecture - 08
Probabilistic Dynamic Programming

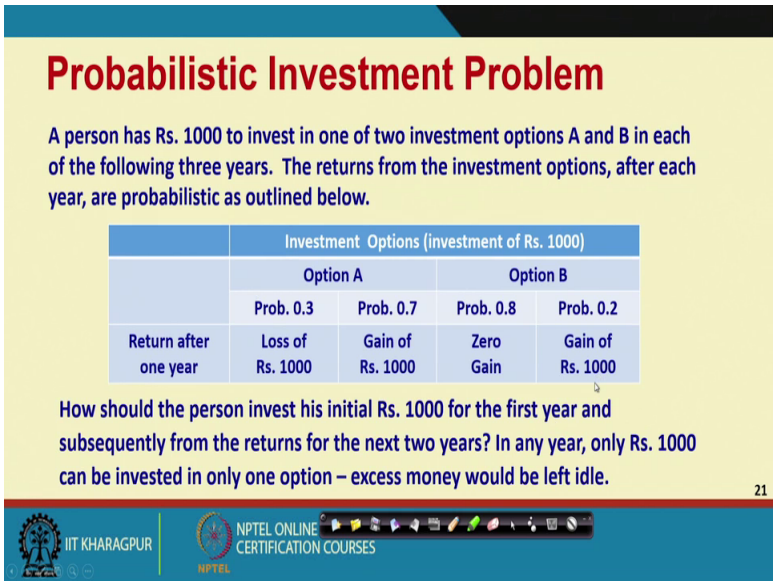
Good morning, we are in our course Selected Topics in the Decision Modeling. And the current lecture will be lecture 8, which we shall consider the Probabilistic Dynamic Programming right.

(Refer Slide Time: 00:30)



So, probabilistic dynamic programming.

(Refer Slide Time: 00:39)



Now, a probabilistic dynamic programming as you can see is you know is a departure from what we are discussing so far. If you really recall till such time the examples that we have taken, they were mostly from the deterministic dynamic programming problem. And in the beginning, we said that there are two types of dynamic programming problem; one is the deterministic and the other is the probabilistic.

So, what really varies in case of probabilistic dynamic programming? See, in the deterministic dynamic programming the decision that you have taken at any point of time the outcome of those decisions where clearly indicative the you know the resulting states will be finite or known deterministic.

But in the probabilistic decision situations, what really happens we do not know what is the state, that we are going to have based on the decisions that we have taken at a given point of time. So, essentially from a given state, we can reach you know not a single stage. So, it is not a stage state transition for a given state 1 to another state 2. From a given state 1, you can go 2, you can go to 3, you can go 4 also with some probability.

So, please understand also another thing what makes it a little more complicated that from state 1, if you go to state 2, from state 2, you can go to 5, 6, 7. From 3, you may actually go to may not be 5, 6, 8, but some other. So, when that happens you know you are really not very clear for a given state, what is the state that we are going to reach at evaluating the values at it any given point has to consider all those future states also. It is

not simply that you know I just considered immediate state because which optimal will you take there right, so that makes it somewhat complicated, but then dynamic programming methods are available for even this you know complications can be very easily handled.

So, instead of really going into the theory part of it, let us try to understand this with the help of a problem. So, the let us see a problem the person has rupees 1000 to invest in one of two investment options A and B in each of the following three years. The returns from the investment options, after each year, are probabilistic as outline below. What are they? Option A, if you invest rupees 1000, after 1 year either your entire 1000 rupees will be lost, is it all right; otherwise, you may gain another 1000 rupees.

In other words, you have invested 1000 rupees after one year either you may get 0 or you may get 2000 rupees, is it all right, so that has been written as loss of 1000 rupees or gain of 1000 rupees. The two you know eventualities are connected by probability values of 0.3 and 0.7 is it all right.

There is a second option also. The second option that is option B what really happens you put 1000 rupees, you get back 1000 rupees. So, there is 0 gain right. Now, the first one is you are you probably as you can understand that risk is involved you know either you lose your money or you gain 1000 rupees. But in the second, you do not lose anything. Either, you do not gain anything, you get back your 1000 rupees or you gain.

So, in a some sense option B looks better because you are not going to lose anything the chance is not there right, whereas option A is risky. But then option A, the probability of gain is higher there is 70 percent chance that you may gain and 30 percent that you lose all your money. Whereas option B only 20 percent chance of gain, 80 percent you do not get any gain right, but you get back your money.

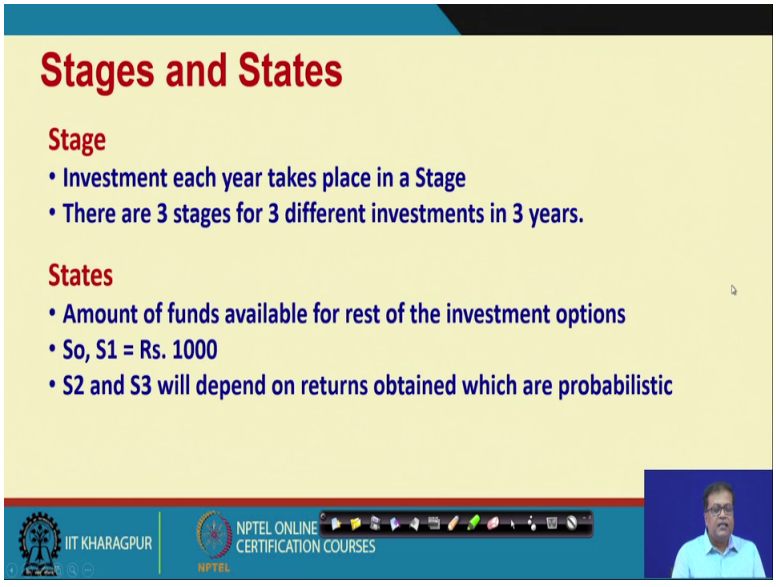
Now, you will be investing its thrice not once had it been only once you know you could simply calculate the expected value of your returns. We will come to expected value little later. If there was only one year as I said you could evaluate them by calculating one year returns and coming up with a decision. But what happens you do the same thing three consecutive years.

So, as you do the same thing three consecutive years, please understand one thing you have only rupees 1000 to start with right. And at a given point of time, one restriction has been given to make the problem slightly easier that you know you can make only one investment at any stage.

Even if you have let us say 2000 rupees, suppose you put 1000 rupees in the first year you gain another 1000 rupees and you have 2000 rupees, it does not mean next year you can invest the entire 2000 rupees, no. At any given year, you can invest only one investment or you may not invest also, is it all right.

So, either you have only three options at any given year, either you invest 1000 in A, or you invest 1000 in B, or you do not invest is all right, these are the only three options at any given year right. So, how will you invest your initial rupees 1000 for the first year and subsequently from the returns for the next 2 years, is it all right for the entire 3 years so that at the end of 3 years you have the maximum gain. So, this is the problem.

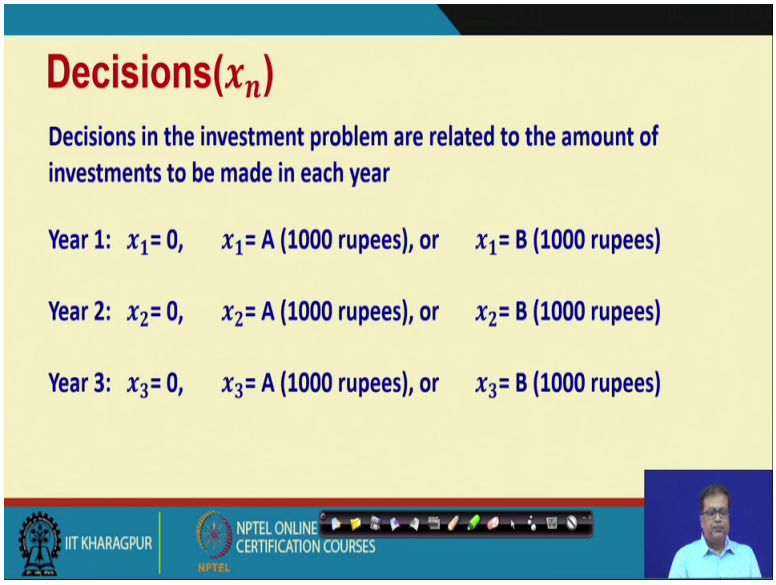
(Refer Slide Time: 07:50)



Now, what are the stages as you can clearly see that there are three stages first year investments, second year investments and the third year investments. What are the states the amount of funds available for the rest of the investment options. So, at the very beginning in the S 1, you have 1000 rupees. The S 2 and S 3 will depend on returns obtained which are probabilistic. You may not have 1000 rupees right, you have 1000 you may gain or you may not gain, is it all right. So, depending on whatever it may be

you know S 2 and S 3 values will be different than 1000 right. Or it could be 1000 also if you have a no gain situation, but then it could be different, is it all right. So, these are the states.

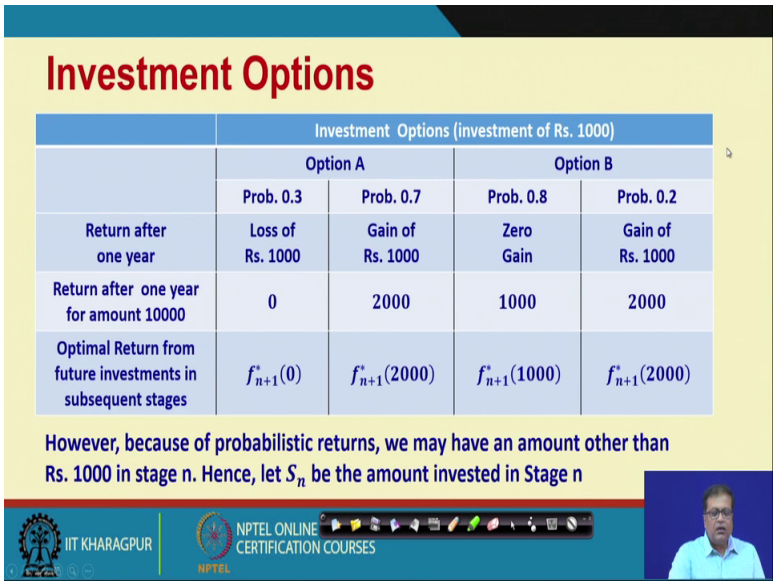
(Refer Slide Time: 08:45)



And what are your decisions? Right, the decisions again at any year either you do not invest. So, x 1 s it could be 0 or x 1 could be A that is putting 1000 rupees in A, because as I said you can put only 1000 rupees at a given point of time or it could be B let us say another not another or 1000 rupees.

And the same decisions you can take for all the 3 years provided you have money right. If you have money you can have the other two options; if you do not have money obviously you have only one choice that do not invest.

(Refer Slide Time: 09:30)



Having said that now let us look at you know what are the investment options, and what could be our optimal returns. So, assuming that you know just simple calculation if we put for only 1 year to make it understand it very simply, this kind of chart will return. The option A loss of rupees 1000, gain of rupees 1000 probability is 0.3 and 0.7. Option B, it could be 0 gain or gain of rupees 1000 with probabilities 0.8 and 0.2. So, in value terms, return after 1 year it could be for amount sorry not 1000, 1000, it should be 0 or 2000 and 1000 or 2000.

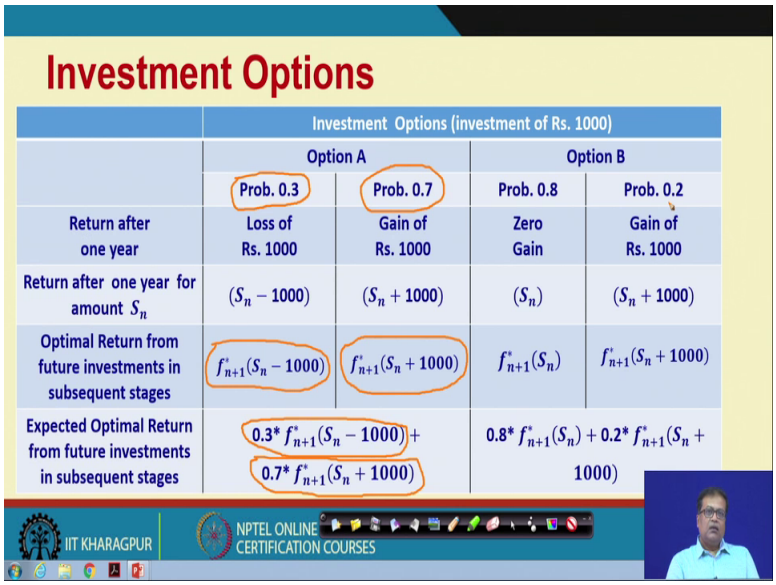
So, optimal return from future investments in subsequent stages, so look here these 0 or these 2000 or this 1000, you will be putting in the subsequent stages because we are considering only first year. So, remaining two years, let us say the optimal value that will get from those subsequent stages is f star n plus 1 with the 0 that you have right. So, here it should be then f star n plus 1 2000, f star n plus 1 1000, f star n plus 1 2000, because please recall what I said just few minutes ago that if you invest for only 1 year, then you can simply take these 0 and 2000, and find an expected value, and come out with your final return.

But exactly that is what is not happening, what is happening this 0, you will be putting in the final or these 2000 you will be putting in the subsequent stages. So, the return that will be coming will be not really 2000, the final return could be f star n plus 1 2000 is it

not, that is the optimal return from future investments in subsequent stages that is what we have to calculate.

So, that is how this table looks at this point. But there is one more thing to really think that is see look here we have 1000 rupees, but you really do not have these 1000 rupees in the subsequent stages. So, it is better not to consider 1000, but really have a value S n which is the amount invested in stage n right. So, because of probabilistic return you may not have really rupees 1000 rupees. So, let us have S n be the amount invested in a given stage. So, if that happens that is we have S n amount of invested let us then redraw this table.

(Refer Slide Time: 12:38)



So, when you redraw this table then you know you have this investment options in these manner that is you know here we have the return after one year will be a S n minus 1000 because you know it is a loss of 1000 rupees and S n was the amount. Suppose S n is say 1200 what does it mean S n minus 1000. So, you have put 1000 in the investment, it has loss, so entire 1000 rupees has gone. But you have 1200, so the remaining 200, so the remaining 200 will be with you is it all right that is why it is S n minus 1000, I hope you understand.

And here it will be then S n plus 1000. So, in the option A, if S n amount is invested after one year, the return will be S n minus 1000; after one year here the return will be S n

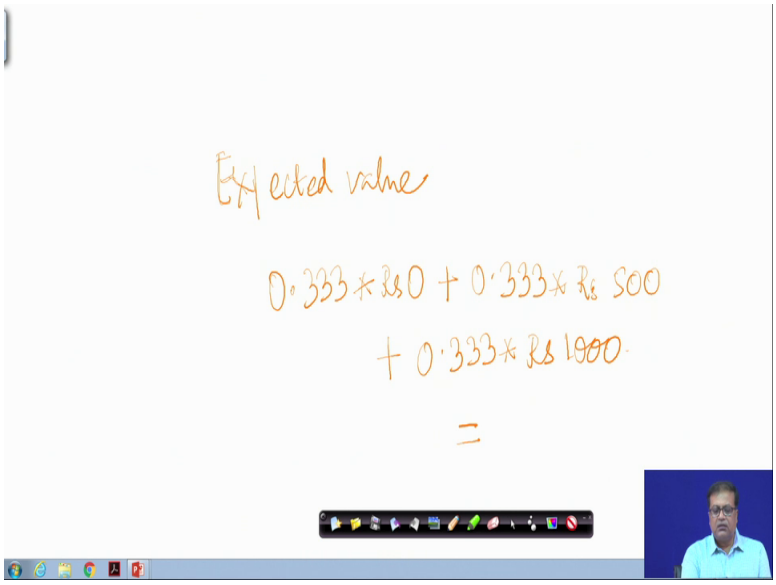
plus 1000. In option B case, it will be a S n and S n plus 1000 right. So, these are the different returns after one year.

Now, after one year; that means, you are from nth stage you know you have now come to n plus 1 you know n plus 1 stage. So, you will be again making further investment in subsequent stages what is the optimal return, let us call it f star n plus 1 S n minus 1000 right. So, let us call that you know so what these item this is the return coming from all the optimal returns from the all the future investments. And the other one is also possible. The first one is possible with probability 0.3 and the second one is possible with probability 0.7 right.

Now, the expected optimal return from all the future investments in subsequent stages we know that you know how we really calculate what is known as the optimal returns, the expected values. The expected value therefore, will be the 0.3 into you know this one S n minus 1000 plus 0.7 into S n plus 1000. So, what is this value this value is the expected optimal return is it all right. The expected optimal return in case of such situations can be calculated in this manner.

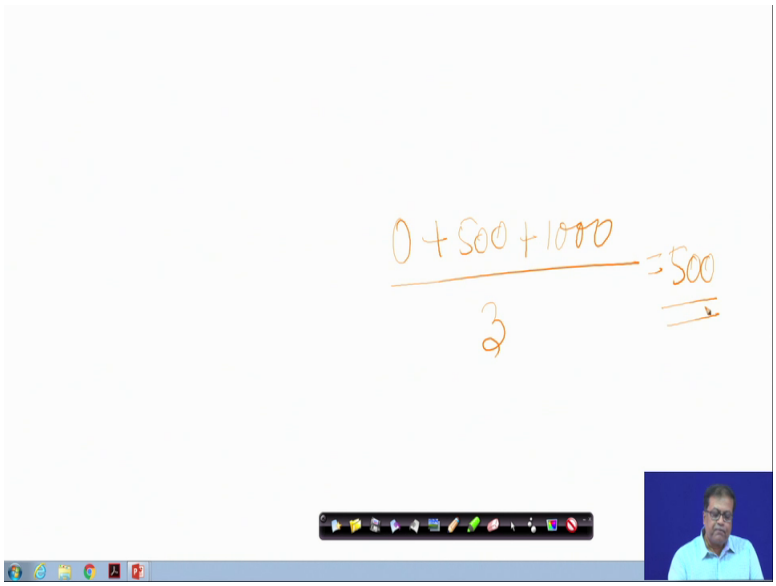
So, what is really expected value, if you really want to understand, let us put it this way that supposing I have a probabilistic situation and the probability really says that you know there are let us say three options that you have put 1000 rupees, the return could be nothing, return could be 500 rupees or return could be 1000 rupees right. So, there are three possible returns and the there are each are having a probability values. So, let us say the probability values are equal right, so around 0.333.

(Refer Slide Time: 16:42)



So, what happens what will be the expected value then the expected value, therefore could be found out as you know 0.333 expected value could be 0.333 into rupees 0 plus 0.333 into rupees 500 plus 0.333 into rupees 1000. Why it is so because you know we have equal probabilities, and therefore the expected value will be basically 0 plus 500 plus 1000 into you know it should be the 20.

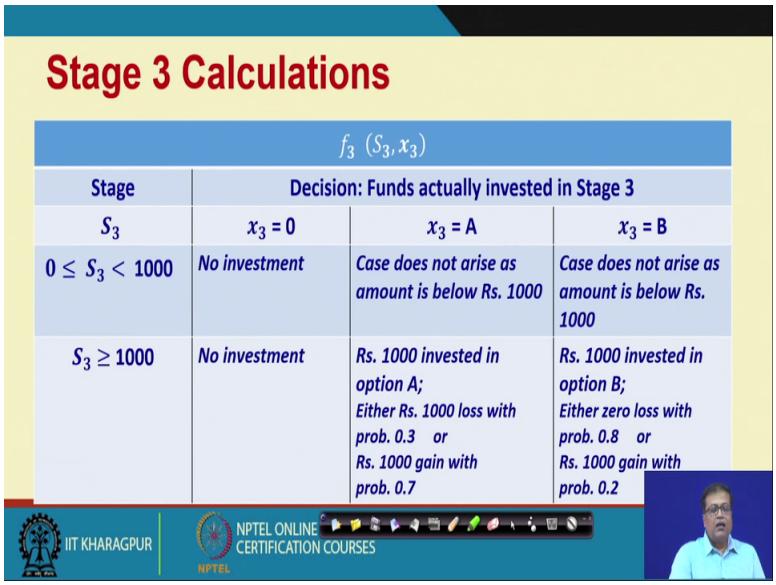
(Refer Slide Time: 17:39)



So, 0 plus 500 plus 1000 divided by 3 right; So, it should be then 500 rupees, why because they are all equally likely. So, we can calculate like this. So, if I do like this that

is our expected values, so you know we can therefore obtain such kind of calculations and find out the expected values out of the thing. So, similar thing has been applied here that is the expected optimal return from all the future investments can be obtained. Now, let us see how do I go ahead with such kind of calculation in further.

(Refer Slide Time: 18:30)



So, let us come out with the stage 3. So, what really happens in stage 3. At the final stage because we know that we usually do what is known as the backward calculations. So, at the stage 3, we have three possible options. What are the three possible options either we do not invest or we invest in A or we invest in B.

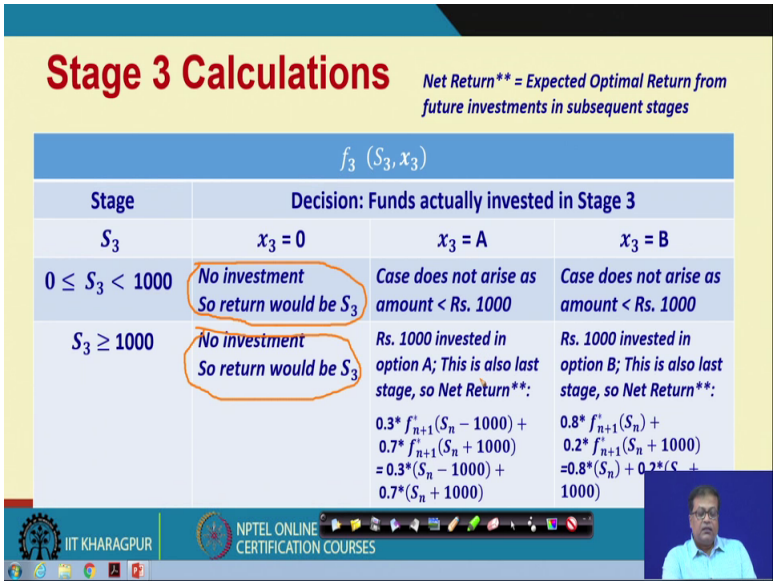
Now, what are those three possible situations that the I mean two situations that can happen either we have less than 1000 rupees or we may have more than 1000 rupees because you see stage three happens after you know we have invested certain money in the first stage and also invested certain other amount in the second stage.

So, when we do that essentially what we have in the first situation where we have you know S 3 is less than 1000 obviously, we cannot make any investment is it ok, so that means, there cannot be any investment because the amount of less than 1000.

So, it does not really matter whether you know you cannot choose any of them, there will be no investment in this if you have x 3 equal to 0. And x 3 equal to A or x 3 equal to B case does not arise thus the amount is below rupees 1000. But if S 3 is greater than equal

to 1000, then here again no investment because you have not invested anything, you have chosen that particular choice. But if x 3 is equal to either A or either B you know you can either get rupees 1000 loss with 30 percent probability or 1000 gain with 70 probability is it all right. And here either you get 0 loss with probability 0.8 or 1000 gain with probability 0.2. So, these are the different scenarios that can happen at the stage 3.

(Refer Slide Time: 20:49)



Further, if I you know calculate then we see that the second case, the S 3 was you know what we had. So, what will be the return, the return obviously, will be S 3 because we have the S 3 amount that has been put and we have made no investment. So, in this case, you know our S 3 will be our S 3 will be 0, I mean sorry not 0, it will be S 3 whatever amount S 3 we have had a same amount will be returned. Whereas, if S 3 is greater than 1000 and we have not invested anything again the return should be S 3 only because you have not invested anything. In these two, the case does not arise, so we do not take anything.

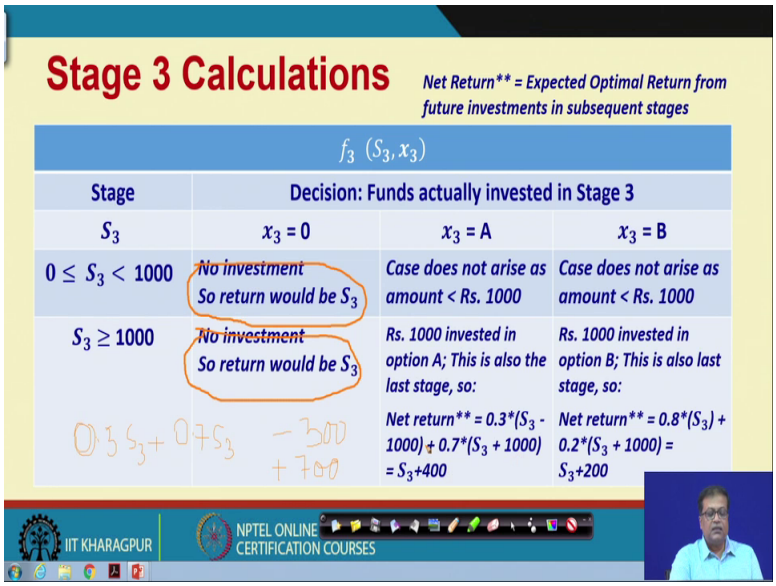
So, what will happen in this case, where we have you know invested rupees 1000 in this case, however this is the last stage right. So, what will be the net return, the net return will be 0.3 into f star n plus 1 S n minus 1000 and 0.7 f star n plus 1 S n plus 1000 plus 1000. And this net return is basically expected optimal return from the future investments in all subsequent stages. But what are these subsequent stage is here because please remember this is the last stage. So, in the last stage, there is no subsequent stages right.

So, the written that we are going to get will be this only; that means, this is the final one, we are not going to have any more subsequent stages.

So, if do not have any more subsequent stages, then this f star n plus 1 S n minus 1000 will be nothing but S n minus 1000. See, this point is very vital to understand. This point is this is happening only in stage 3, it is not going to happen in subsequent you know stage 2 or stage 1, stage 3 is the last stage.

We do not have any subsequent stages. So, any amount that we have you know got return is the full and final return right. We are not going to invest it in any further, is it all right. So, we can find out the expected value at this stage. And what is that expected value ok. If you if you calculate can you calculate what will be that expected value that is 0.3 let us do it here, it should be 0.3 f. So, it will be the 0.3 let us say.

(Refer Slide Time: 24:00)

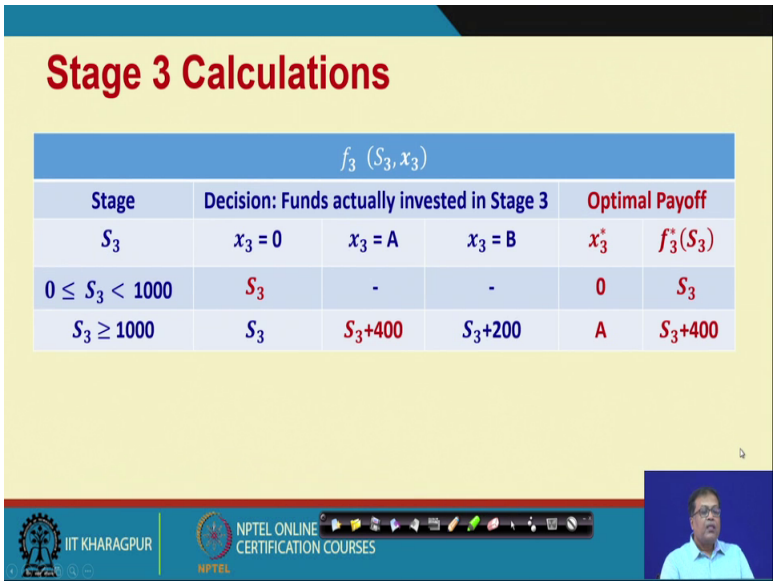


Let us take the pen. So, it will be 0.3 into S 3 plus 0.7 into S 3. So, what is 0.3 into S 3 plus 0.7 into S 3 yeah. So, it will be how much please tell me 0.3 S 3 plus 0.7 S 3, how much is that that is S 3 itself right. So, that one part of this is that 0.3 into S 3 plus 0.7 into S 3 that is nothing but S 3 that is the first part, is all right. So, however the other part that is the 0.3 into 1000 is 300 and 0.7 into 1000 that will be your 700 so that means, it will be minus 300 minus 300 plus 400 plus 700 right.

So, there are two parts. One part is your you know the 0.3 into S 3 plus 0.7 into S 3 which is nothing but S 3 itself. The other part is minus 0.3 into 1000 which is 300 and minus plus 0.7 into 1000 that is 700 and that value is 400 is it all right. So, you can see that we can actually calculate the net return in this case that will be S 3 plus 400.

Now, can you can you calculate the similar one for x 3 equal to B? Please understand this will be 0.8 times S 3 because there is no return. And in the second case, it will be 0.2 times S 3 plus 1000 is it ok. So, when that happens this term will be 0.8 S 3 plus 0.2 S 3 that means, we have S 3 and 0.8 here there is nothing but here 0.2 into 1000, so it will be 200, is it ok.

(Refer Slide Time: 26:43)



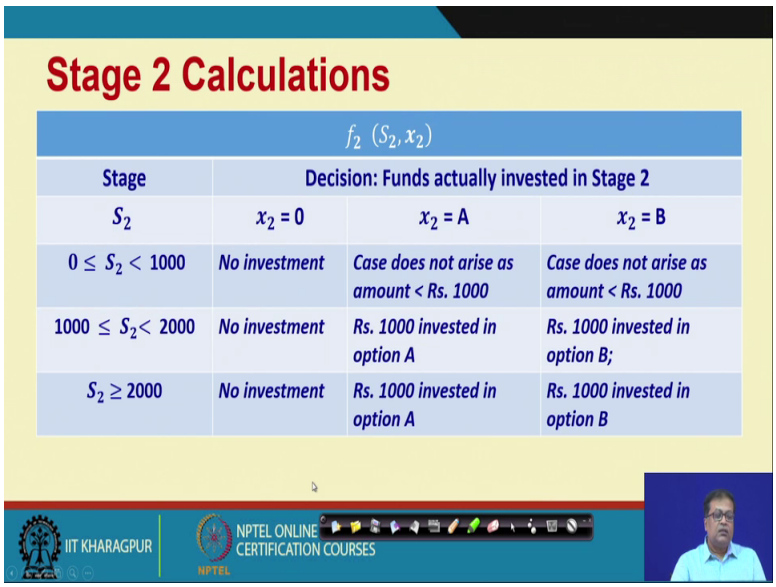
So, that means, if I combine them all what I get what I get is this particular chart that is at the stage 3 calculation level, we have those you know two options for the stage, one is that S 3 is between 0 and 1000. In that case if x 3 is equal to 0, because you are making no investment the S 3 will be returned right. If this happens irrespective of whether S 3 is between 0 to 1000 or above 1000 right.

So, irrespective of whether S 3 is between less than 1000 or more than 1000, it will be S 3 only. But if x 3 equal to A or x 3 equal to B, you know S 3 less than 1000 we cannot make any investment. So, those two cases does not arise, I mean they do not exist. But if S 3 was more than 1000 and we could make an investment, then the return would be S 3 plus 400 or S 3 plus 200 as the case may be, is it all right.

So, the important point that comes what is the optimal pay off then under those two situations right. So, out of these S 3 plus 400 and S 3 plus 200 we can clearly see that S 3 plus 400 is the higher value, so obviously that is our optimal pay off. And x 3 star will be A that means, in the third stage it is optimal to go for an investment into A and if we have more than 1000 rupees right. So, we have more than 1000 rupees, what we should do we should invest in S 3 all right. We should invest in option A and we get a return of S 3 plus 400 and our option will be A.

But if we had less than 1000 rupees, then we cannot invest any amount, and you know it will be 0 then. And in that case the our optimal payoff will be S 3 only because at this stage this one is nothing but S 3 this is so that is how the stage 3 calculations will go. These values could be really obtained in the earlier stages, I mean subsequent stages that is stage 2 and stage 1. ah

(Refer Slide Time: 29:38)



You see however one very interesting thing will be important that in stage 2. See in the stage 2, you know there will be a really three options once again that S 2 could be less than 1000, S 2 could be less than 2000 but more than 1000 I mean equal to or more, or S 2 could be more than 2000 right more than equal to 2000. And there will be different scenarios.

But what is very important to know I mean in S 2 you know in S 2 we cannot really like in S 3 what we did we had calculated in you know the net return that is 0.3 S 3 minus

1000 S 3 plus 1000 and we obtain that final value these simplification will not be possible because in that is not the last stage, there will be subsequent stages is it all right. So, calculations cannot be completed. So, what should be the calculations, and how do we proceed from there right that we shall see in our next lecture. And we continue this problem in our next lecture. So, stop here right.

So, thank you very much.