Selected Topics In Decision Modeling
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Lecture – 04
An Investment Problem (Contd.)

So, in lecture 4 of our Selected Topics in Decision Modeling, we shall continue the investment problem which we have taken up in our previous lecture and after that we take another project allocation problem and see how this investment problem and the project allocation problem has got a similar structure.

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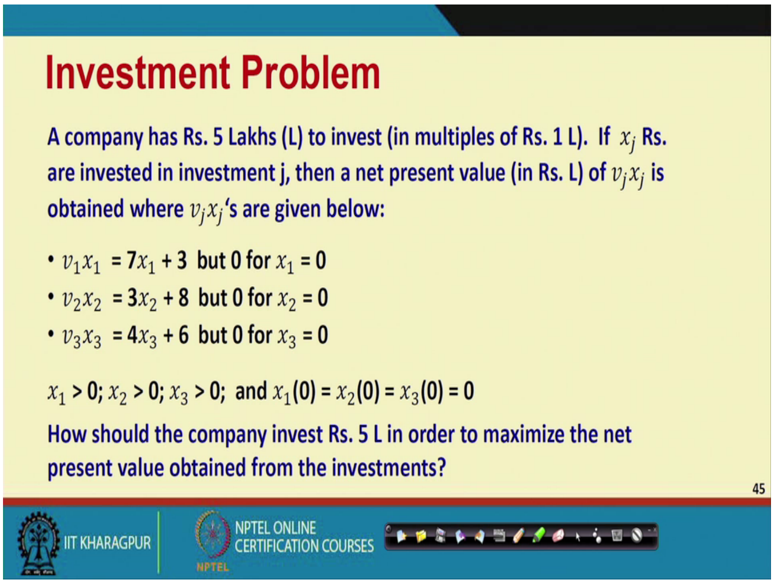

So, in our previous class we have taken up this investment problem very quickly, if we just reiterate a company has rupees 5 lakhs to invest in multiple sub rupees 1 lakhs, if x j rupees are invested in investment j and net present value of v j x j is obtained which are given by these three equations and to know that if we invest nothing at a given stage or given investment options.

Then there is no NPV, they time the formula should not be used. The question is therefore, how should the company invest that rupees 5 lakhs in multiples of rupees 1 lakhs, so as to maximize the net present value obtained from all the investments.

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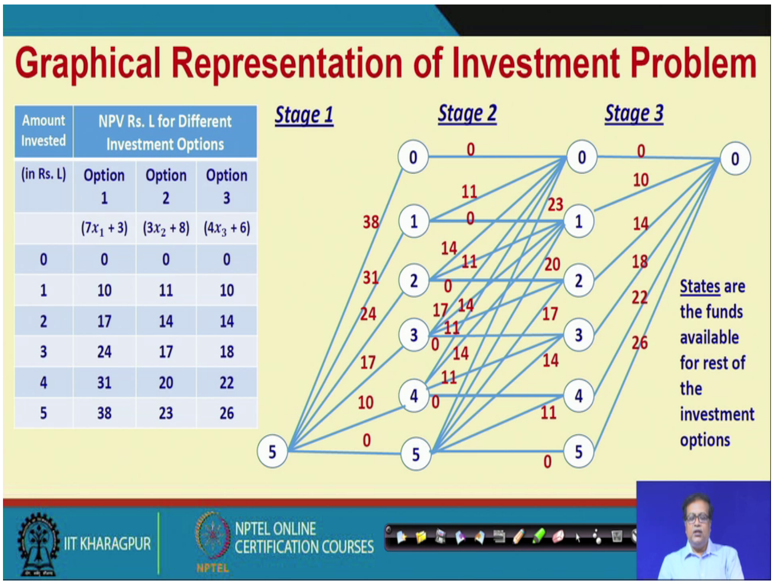

What you did if we recall we had actually constructed you know the following graphical representation and this graphical representations really combined the stages, there are 3 stages at every stage you know the state is determined by the fund that is available for rest of the investment options.

In the beginning the entire 5 lakhs is available, after stage 1 it could be either 0 1 2 3 4 or 5 depending on how much you have invested and at stage 3 also that is 0 1 2 3 4 5. But finally, obviously we have nothing left if we invested rest of the thing in the 3 investment options and the decision was that x 1, x 2, x 3 which are the amount invested in a particular investment option.

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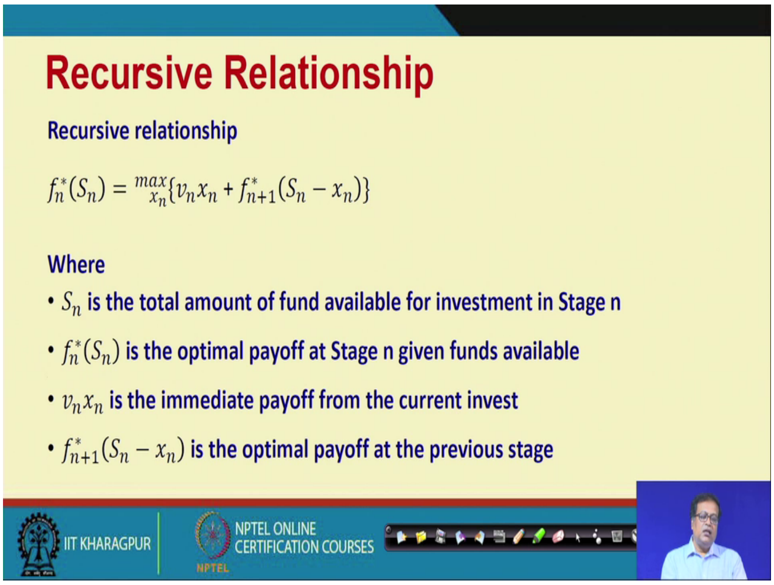

What we said further that there has to be a recursive relationship which we shall used which that relationship should help us in solving the problem, you know it should not be just equal to a brute search for an exhaustive search method which will really examine all possible paths of these plot.

See this is still a simple problem, so for a such a simple problem you can still do and exhaustive search and you can find the solution, but think of not 3 stages but 30 stages and think of not just see you know options. But 30 options and you can imagine how complex the graph could be.

So, in such a complex graph obviously we must need method which could be an exact method, but at the same time finds the solution quickly by less number of iterations that is the essential part and in order to do that we need to use a recursive relationship which form the from the Bellman's principle; which basically says f n star S n is the maximum value for a given x n value v n x n plus f star n plus 1 S n minus x n, where f n is the stage n f n plus 1 relates to the stage n plus 1 and these are the different values which I already explained in our previous lecture.

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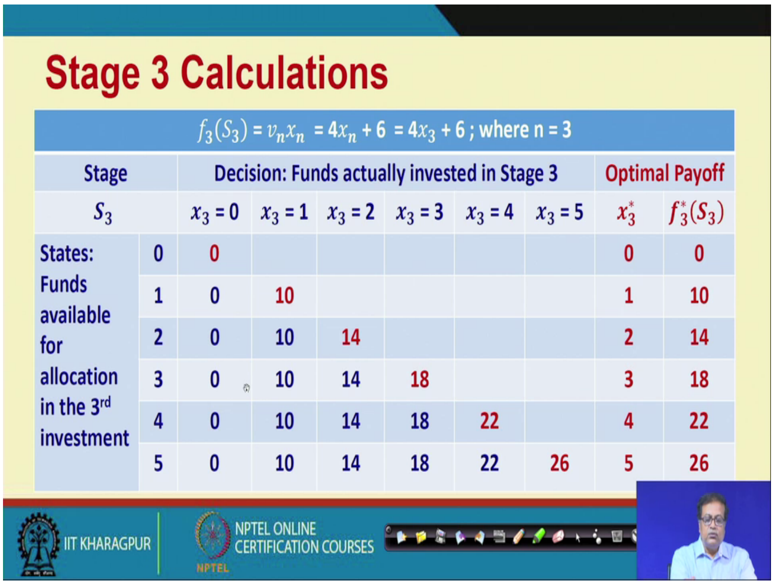

So, how to iterate what we do we start with the stage 3 calculations. So, at the stage 3 if you really look the stage 3, the f 3 S 3 that is the investment that you do will be v n x n equal to 4 x n plus 6 because, that is the guiding formula which is equal to n equal to 3 for n equal to 3 it is written twice.

Now, at state at stage S 3 at this stage you know you have 1 2 3 4 5 6 investment options, obviously at stage 3 1 may not like to make use of these investment options 1 may say that whatever remaining we might spend that is what has come out also. And as the optimal but let us say if we say for the sake of thing that if we really go for it. Now, the at this stage 1 can have 1 of the 6 situations, that is either we have the fund that is available it could be 0 it could be 1 it could be 2 3 4 or 5 1 of these.

But if you spend only 0 out of that and if 0 is available, then the return should be 0 look here these are the option 3 values 0 10 14 18 22 and 26. So, you know 0 10 14 18 22 and 26 similar values are there, now if we invest nothing then irrespective of whatever fund was available.

If you invest nothing you should get 0 that is the first column second column, if you invest only 1 based on the formula your return should be 10 irrespective of whatever was the amount that was available because, unless you invest you are not going to get return, so if you invest 1 these are going to be your returns.

If you invest 2 your returns will be 14 14 14 14 if you invest 3 your returns will be 18 18 18 if you invest 4 it will be 22 and if you invest 5 it will be 26. So, having these kind of calculations what we can get at the next stage therefore, that what is the optimal value out of this. So, although all values are same so obviously, the first value I have taken that is the 0 1 2 3 4 or 5 that is the optimal and the values are 0 10 14 18 22 and 26.

So, at stage 3 things are very easy to really compute we do not have to do too much thinking and let us go over to the second stage. See here things are not as simple as you can see, so we have to continuously refer to the optimal value of at the third stage. See the optimal values are 0 to 0 1 10 2 14 3 18 4 22 and 5 26. These are the x 3 star and the pay of at the third stage.

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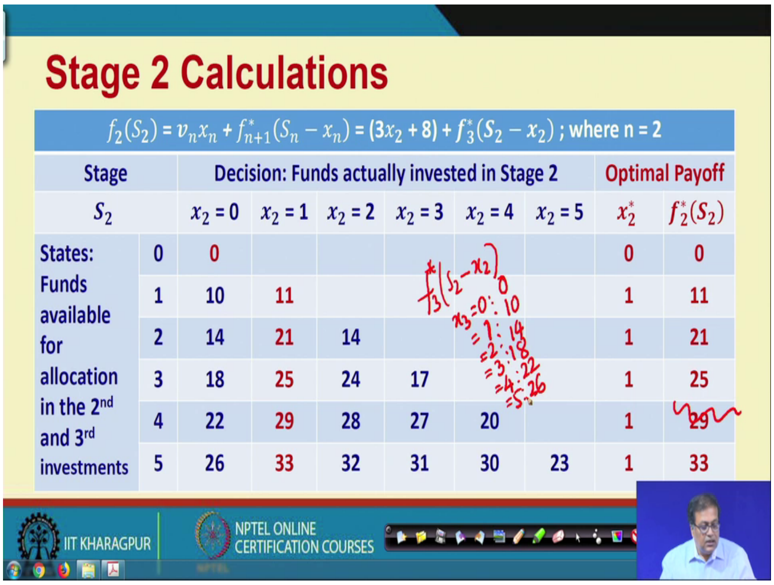

Fortunately for us these values are available here right these values are incidentally available here because, at the second stage if you invest nothing then whatever profits you should get those profits are actually guided by whatever profits you have got at the previous stage right. So, those values are only going to we coming up and not the contribution of the second stage.

So, in order to understand this let us look at the recursive relationship once again, the payoff at a given stage is equal to the payoff at the previous page plus the return which accumulates from these stage right. So, this is the optimal payoff from the previous stage this is the optimal payoff from these stage and this is the contribution of these stage. So, for even these other equation anyhow where we wrote this thing is quickly.

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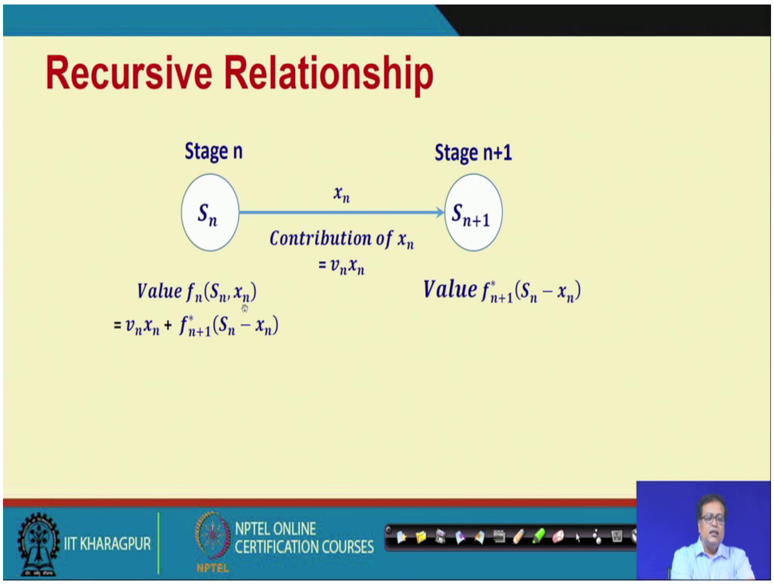

You know these value at a given stage is equal to the investment at this stage and the optimal from the previous stage. So, having known that now yeah let us lets come to the specific discussions that we are doing, that is we are at the stage 2 of the calculations.

So, at this stage you know what we really find this formula that v n x n is really 3 x 2 plus 8 plus f 3 star S 2 minus x 2, which is the optimal from the previous stage you are n equal to 2. So, these are the stage 2 calculations now you see if you invest nothing then whatever profits because, these portion will be 0 then the only the optimal portion from the previous stage will be remaining, what is a S 2 minus x 2 that is the remaining money.

So, remaining money if there is 0 the remaining money is 0 if x 2 equal to 0 if s 2 is 1; that means, remaining money was 1 that 1 must have been spent at the third stage. So, this is the backward different what you call dynamic programming, at stage 3 you have spent say certain amount the remaining amount you spend at the second stage.

So, at second stage if you spend nothing and the remaining money is 0 that means, in the third stage also you have spend nothing. But if the remaining money is 1 and the second stage you have spend 0, that means the remaining 1 you must have spent in the third stage going 1 step up if you spent 1 amount in the x 3 the optimal profit is 10. So, what will be your optimal profit at this stage if you have remaining money 1 and you spend nothing at this particular investment options it must be 10 because, this portion is 0 this portion is 10; so that gives us 10.

So, using that you can fill up the all the values for an investment of 2 should be 14 for an investment yeah for an investment of 2 it should be 14 for an investment of 3 should be 18 for an investment of 4 it should be 22 and for an investment of 5 it should be 26, so that is for the no investment.

If you invest 1 if you invest 1 what is the return that you are going to get the return for these investment is 3 x 2 plus 8 that should be 11 because x 2 equal to 1. So, 3 x 2 plus 8 should be 11 now these 11 is this 1 because if you spend 1 at the second stage; that means, that the third stage you have nothing left because total fund is 1 1 you have spend in this stage for x 3 therefore, it should be 0.

So, if it is 0 then that earning will be 0 so that means, 11 plus 0 that should be 11 only is it all right, what will be the earning if you had a spend up 2 you haves 2 you had only 1 you have spend here; that means, 1 more is remaining that is on x 3 go back 1 slide for 1 the optimal was 10; that means f 3 star s 2 minus x 2 is 10, so 11 for 1 and 10 for your next stage in should be 21. So, these in work calculation has to be made and the optimal values that has to be really brought in on these calculations.

So, if we if we write those optimal calculations, you can see that these you know at x 2 equal to 1 stage the next value should be 25; why it should be 25 because at x 2 equal to 1 it should be 11, but the remaining 2 will fetch profit of 14 because you know f 3 star if we write all these f 3 star S 2 minus x 2 values then x 3 equal to 0 it should be 0.

So, equal to 1 it is 10 equal to 2 it is 14 equal to 3 it is 18 into 4 is 22 equal to 5 it is 26 , so these are the values look here once again , so 0 11 21 25 so that is the stage 2 calculations yeah. So, these are the stage 2 calculations so the stage 2 calculations, you can therefore obtain all those values based on by making use of the x 2 equal to 0 optimality values and then you can get the 11 21 25 29 and 33 once again 5 33 because x 2 equal to 1 therefore it should be 4.

So, 11 plus 22 that is thirty 3 similarly at x 2 equal to 2 what will be the optimal values this 1 is 14 and 2 so it is 0, so it should be 14.

Next 1 14 plus 10 24 next one 14 plus 14 28 next one is 14 plus 18 32. So, like this if you complete this chat and then you see row wise what are some optimal values. So, clearly speaking you know if you see these row wise the optimal value is 0 , if you see this row wise the optimal value is 11, if you see these way optimal value is 21 , if you see 25 this row 29 this row 33.

Now, you may be m s 2 see you that why all the optimal values are at x 2 equal to 1 , there is no pattern in this incidentally; it has happened here does not mean it will happen in the next problem also. In fact, we have taken a second problem it does not happen there. So, there is nothing to think he just because, here certain column will become optimal nothing like this.

So, anyhow if you therefore, write the optimal payoffs at this level then for 0 it is 0 value for 1 it is 11 for 1 it is 21 I mean for next one the optimal value is 1 and it is 25 next one the optimal value is 1 and it is 29 and the last 1 the optimal value is 1 and it is 33. So, those are the values of x 2 starts right for what is the optimal payoff at this stage that has been given and what is the corresponding x 2 star which is also noted and that is highlighted in this chat.

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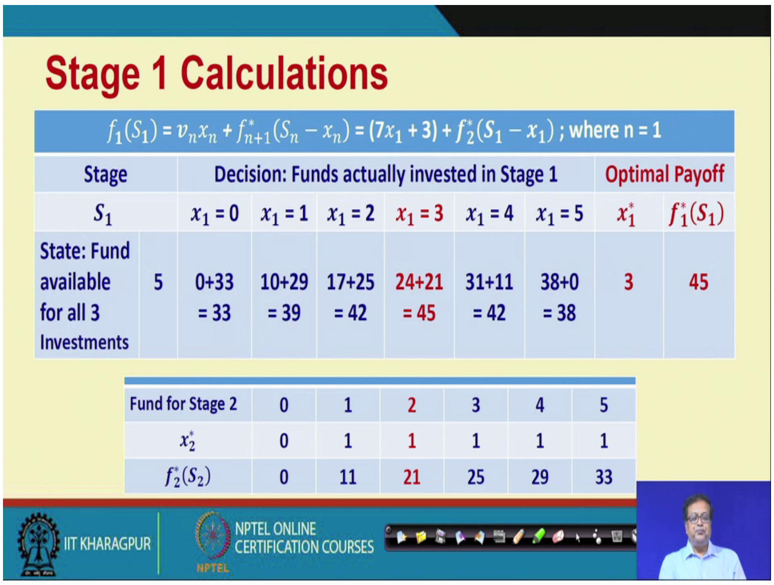

So, next go to the stage 1 if you have come to stage 1 at stage 1. So, you see now the fund available for all 3 investments has to be the entire amount that is 5, now out of these entire amount that is 5.

Now obviously you have 5 6 choices that is x 1 equal to 0 x 1 equal to 1 x 1 equal to 2 3 4 and 5 and at the stage 2 funds are all written here right; stage 2 just previous slide if you see these are your our optimal value 0 11 21 25 29 and 33 so same values are written here. So, these are the funds first stage 2 the optimal values of stage 2 they are all written here.

So, supposing you have total fund available at stage 1 5 and you spend nothing at this stage; that means, what that means that remaining; that means, the entire 5 you have spent in the remaining stages. If you spend the entire amounts 5 at the remaining stages, then you are going to get an optimal profit of 33 out of that you have spend 1 in x 2 and obviously remaining 4 in the x 3.

So, any how those are different but you have 33, so that means with x 1 0 you get a 0 profit because if you invest nothing you get nothing from that, but 33 from the remaining amount. So, total profit in this case would become 33 what would be happening if you are spending only 1 at first stage; that means, 1 at x 1 4 at the previous 4 optimal value is 29 so 29 and 10 39.

What will be at the next 17 because if x 1 equal to 2 these will become 17 and this 1 will then become your you know 17 plus 25 equal to42 y 25 because, 2 here remaining 3 at the stage 2. Then if you spent 3 then you know the remaining 2 you spend here that is the stage 2 where the optimal was 21 so net total is 45 and finally 4 and 5 the values at 42 and 38 respectively by similar method of calculation.

So, now if you look at what is which 1 is the optimal at the these level, then we see that x 1 is the optimal value is it all right. So, look your how Bellman’s principle is working that irrespective of how you have got optimal value at this stage the remaining distances much constitute and optimal path. So, remaining distances are at this stage that is constituting an optimal path. So, the total optimal therefore should be 45; that means, that is the maximum that we can achieve is it ok. So, we achieve a maximum of 45 and this has been illustrated further.

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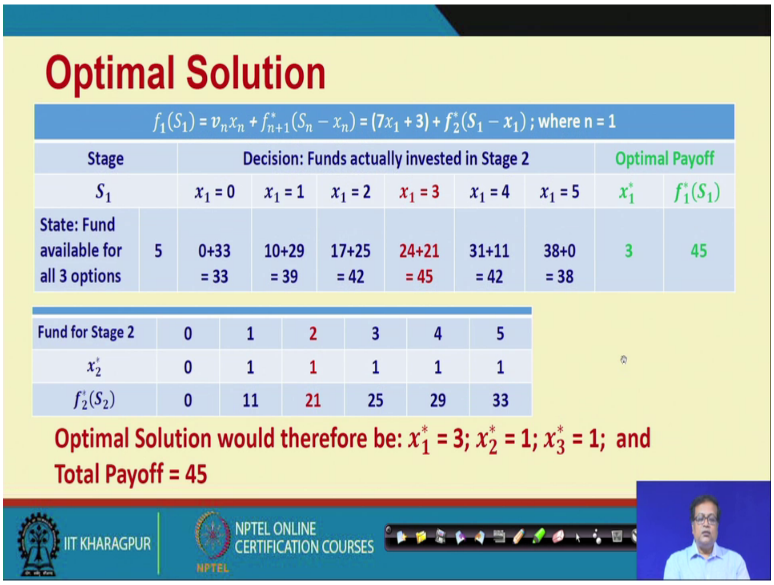

So, this is the chart once again that whatever calculations we have done. So, what becomes our optimal solution the optimal solutions will be x 1 should be equal to 3 x 2 should be equal to 1 and the remaining 1 should go to x 3.

So that means you should spend because look further into these 2 let us go back and see how these 2 has been obtained see this 2 that is this 21 if you see further, then x 2 equal to 1 and you know the total money was 2. That means, 1 was from the previous stage, that means that stage 3 that 1 has get give us 10 at stage 2 another 1 would have given 11 plus 10 21 and in the final stage 3 the remaining 3 would have given as 24 and the whole total we got is 45 right.

So, that is our optimal solution for this problem x 1 star equal to 3 x 2 star equal to 1 and x 3 star equal to 1 and therefore the total payoff will be 45 and this 45 pay off his shown look here in this diagram how this has happened.

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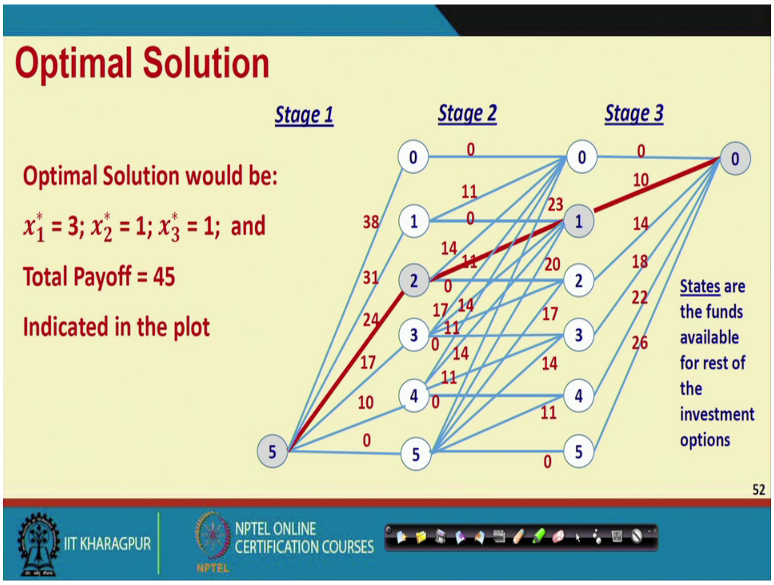

So that means, from stage see if we come from backward from stage 1 we haves , you know invested 1 here we got a return of 10 we invested another 1 here we got a return of 11 and then we invested how much 24 here I mean invested 3 here and got a return of 24.

So, 24 11 10 45 that is the our total payoff and we found this path by really evaluating some of these paths and not all the paths that is where the advantage of this particular dynamic programming method. So, having understood that how this problem is sometimes what happens, if you do only one problem the understanding may become incomplete.

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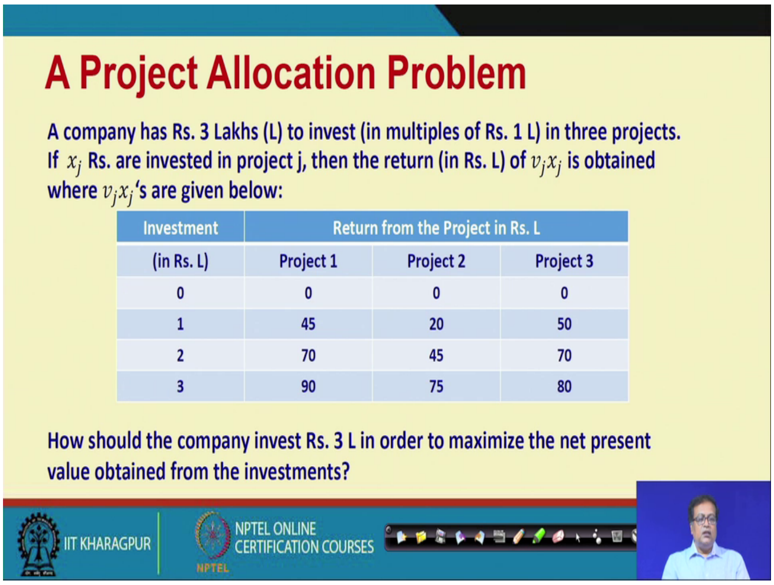

So, let us take one more problem of similar type let us call it project allocation problem. So, let us say company has 3 lakhs to invest in multiples of rupees 1 lakhs in 3 projects, now if x j rupees are invested in project j and the return therefore, is v g x j which are given in this manner.

So, if we invest 0 1 2 or 3 for project 1 your profits are 0 45 70 and 90 project 20, 20, 45 and 75 and project 30 50 70 and 80 is it all right. So, how should the company invest rupees 3 lakhs in order to maximize the net present value obtained from the investments, so let us look at how do we go ahead with the problem.

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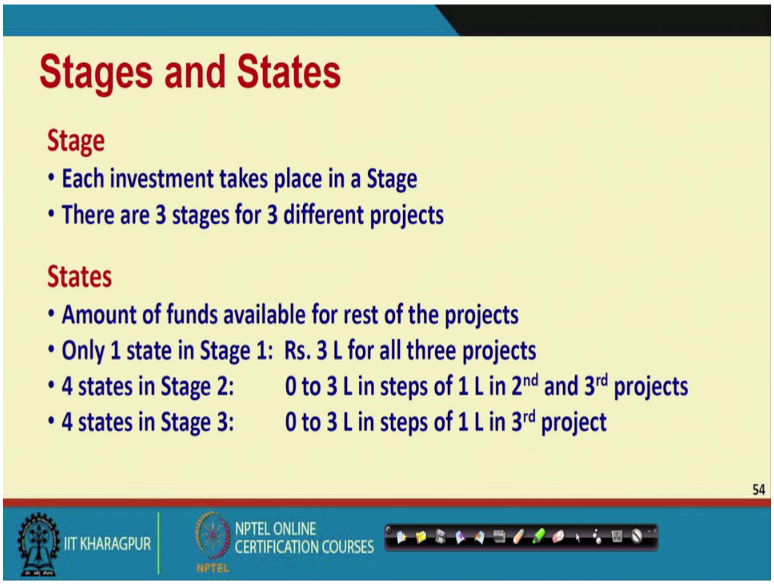

So, at the very first again we can take there are 3 stages there are 3 different projects. So, each investment takes place in a stage right and there are 3 stages, what should be the states the amount of fund available for rest of the projects. So, for stage 1 entire 3 lakhs for stage 2 and stage 3 there are 4 states anything between 0 to 3 lakhs in steps of 1 lakhs in the second and third project.

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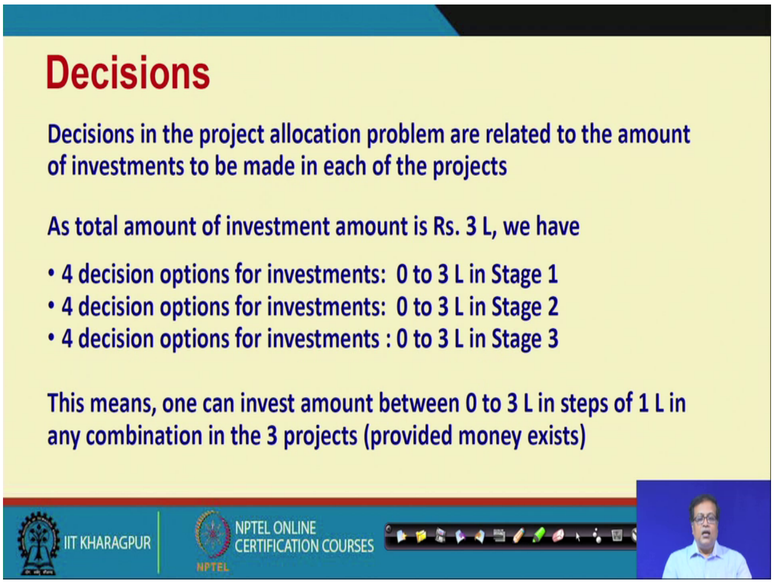

So, what are some decisions should be the amount that you invest this 3 lakhs how you invest, so obviously at stage 1 stage 2 and stage 3 at each stage you have up to 4 options

0 to 3 l in steps of 1 lakhs these are the decision options at a given this thing. So, this means one can invest between 0 to 3 lakhs in steps of 1 lakhs in any combination in the 3 projects provided money exists; Now, this is our graphical representation.

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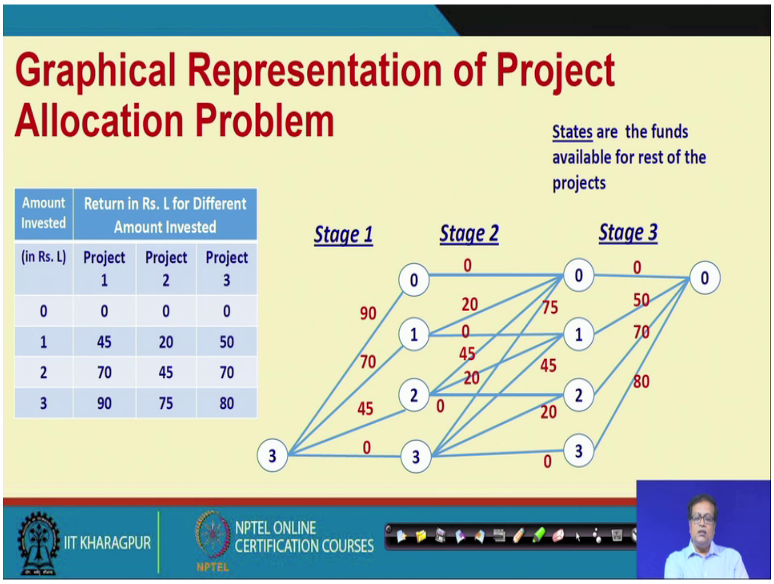

So, once again if you look that from stage 1 if you just you can see those returns and you can combine them here, if you invest nothing at the first stage your return is 0 if you invest 1 your return is 45 invest 2 your return is 70 invest 3 return is 90. Now, depending on what you invest you can reach one of those 4 stages again, if you invest nothing profit is 0 the return is 0 20 45 or 75 depending on these values and at the project third level, if you invest nothing obviously only the remaining amount can be invested.

So, it could be 0 50 70 or 80 because here 3 to 0, that means has the state changes from 3 to 0 3 is the remaining amount. So, you must have invested 3 amount and for 3 amount the return is 80.

So, once again you know the problem really is find the path here which maximizes our return. So, what is that path which maximizes the return; obviously, you know if you this problem is easier. So, if you look a little carefully can you find what is that path which gives us the maximum possible profit.

So, can you see here that if you take here this is 45 this is 20 and this is 50. So, it is 115 this could be a candidate or there are different other paths this one is 80 this one is 90, so this one is again 90 this one is again 90 this is again 90. So, all these different paths are available this 1 is 45 plus 20 plus 50 so 115.

So, all these different parts are available which are candidates, but once again the exhaustive search method requires much more computation for a simple problem, it does not look but for more complicated problem obviously it will take much more iterations and time.

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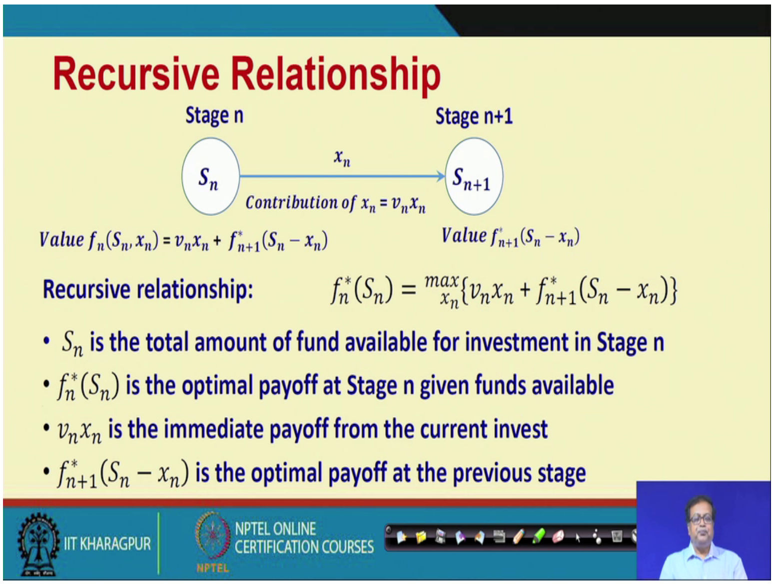

So, once again what is our recursive relationship the recursive relationship would be depending on the at stage n, what happens and at stage n plus 1 what happens and what is the contribution of x n that is v n x n and the value of f n x n S n x n will be v n x n plus f n star n plus 1 S n minus x n and the optimal value of that should be the maximum value of these expression given x n; where S n is the total amount of fund for investment f n star x n is the optimal payoff at stage n v n x n is the immediate payoff from the current investment and f n star n plus 1 S n minus x n is the optimal payoff at the previous stage.

So, I hope by this time you are able to understand the recursive relationship idea and but please remember this recursive relationships are problem dependent for another problem the recursive relationship will be different. So, how your stages and the states are combined, how you have you know really computing for a given problem based on which the recursive relationship will depend. For investment problems of this type this structure is there right, but this structure may not be valid for another problem, so we must know that how much of it is generalized and how much of it is not.

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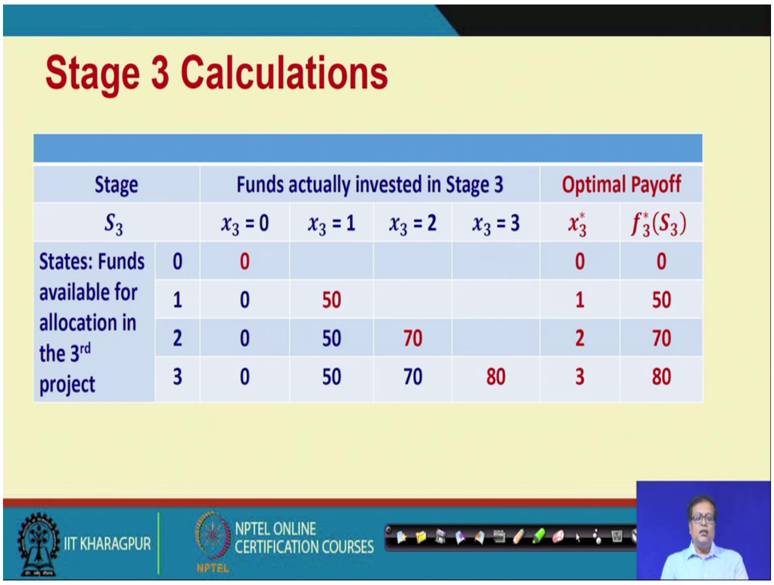

So, having known that let us see what we do at the stage 3 calculations, see at stage 3 values once again let us see for project 3 you know we have the 0 1 2 3 and the returns as 0 50 70 and 80. So, look here same 0 50 70 and 80 returns are both here at stage 3 if you have 0 amount available; obviously, you can invest 0 and you know a return will be 0.

If you have 1 available you can you know either invest 0 or 1 depending on what you invest you get either 0 or 50 return, at 2 you can invest 0 1 or 2 and get 0 50 or 70 return and if you have 3 then 0 50 70 or 80. Now, what are some optimal values 0 1 2 3 and 0 50 70 and 80 those are our optimal values.

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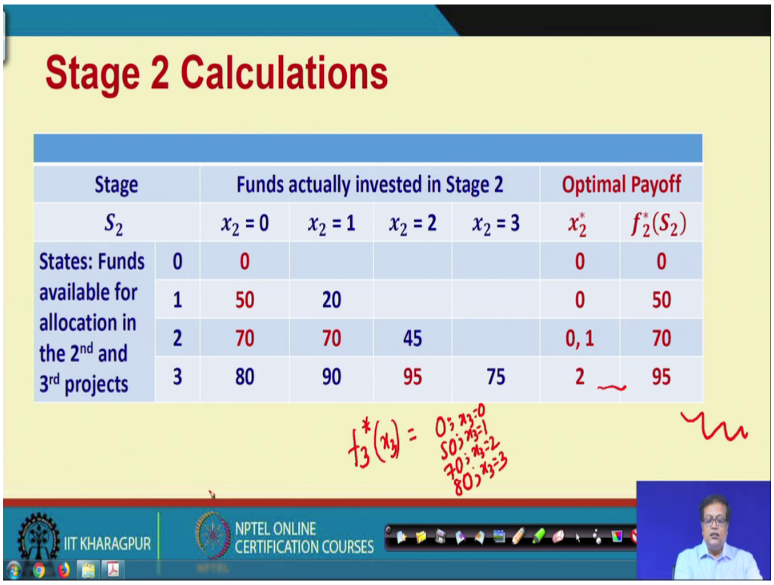

So, having this optimal values now the next one is can be therefore computed. So, what is what is there now all already we know that the sum of the optimal values are here that is 0 1 2 3 and 0 50 70 and 80. Fortunately for us you know this x 3 star f 3 star these values are 0 50 70 and 80 for x 3 equal to 0 , x 3 equal to 1 x 3 equal to 2 and x 3 equal to 3 for different values these are the optimum values.

So, what we do if it is 0; obviously, for the second fund the second fund calculation you know second fund calculation; obviously, the formula let us see the formula the these returns us 0 20 45 and 75; so, 0 2045 and 75. So, if these are the values then you see if you have invested nothing then the same 0 50 70 80 which are optimal for the third stage there will be put here, but if you put 1 then you get 20 and 0 from the third stage.

If you give you know 1 and the total fund available is 2 then when you invest in the third stage and you get 50 to 70. So, like this if you 1 and 2 then you get 90 what is that 90 20 plus 70 that is 90, why these we make the calculations and this time you see the optimal values are different. Optimal values if x 2 equal to 0 optimal value is 0 if x 2 is you know sorry the optimal values are the if you have 1 available 0 available the optimal is x 2 star 0 return is 0.

If you have 1 available then optimal is do not invest anything put that 1 for the third stage and get a profit of 50, if you get 2 then either invest 0 or 1 you get a return of 70, if you put 3 I mean total is 3 then you invest 2 and 1 you invest at the third stage and get it total optimal payoff a 95.

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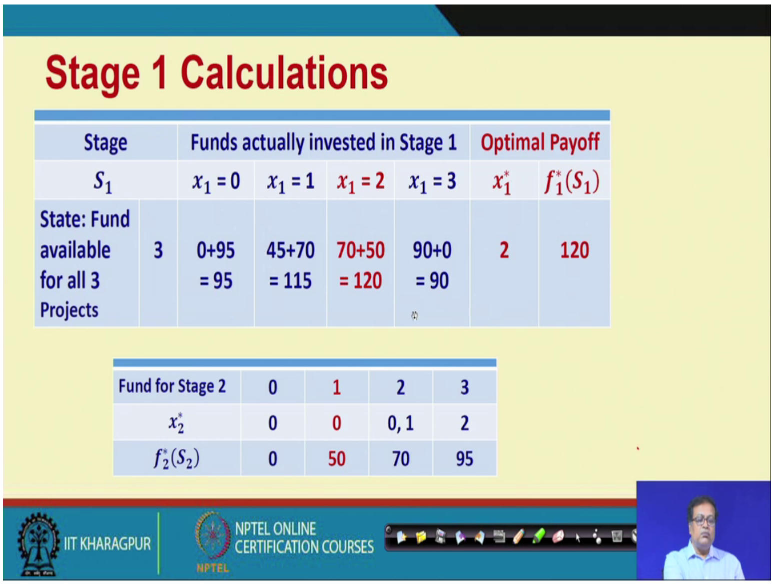

At the stage 2 calculations and that stage 2 calculations when you combine to the stage 1 calculations based on the appropriate values which are again the calculations those are they are available for stage 1, then you get 95 115 1 to 0 and 90 right. So, that 1 to 0 all though we saw 115, but there really there is an 120 also. So, that 120 is the optimal profit right.

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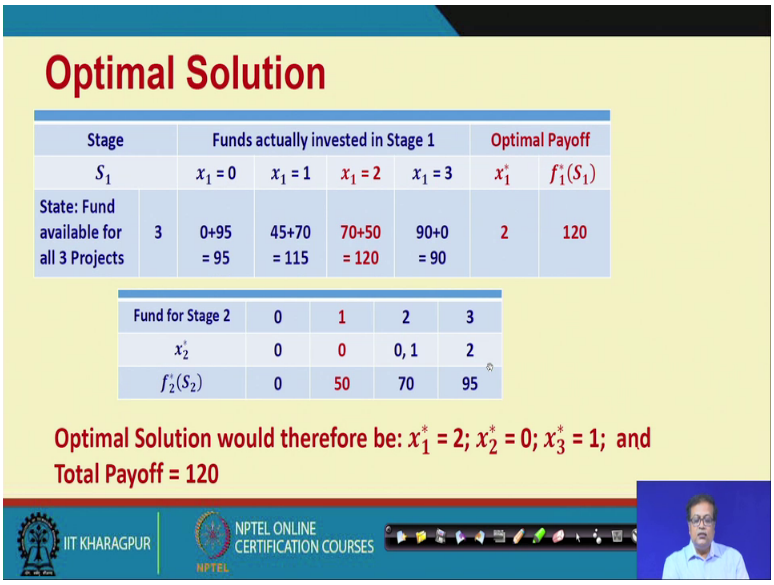

And this 120 will be invested in this way x 1 star as equal to 2 x 2 star equal to 0 and x 3 star should be equal to 1 and look how this is shown here.

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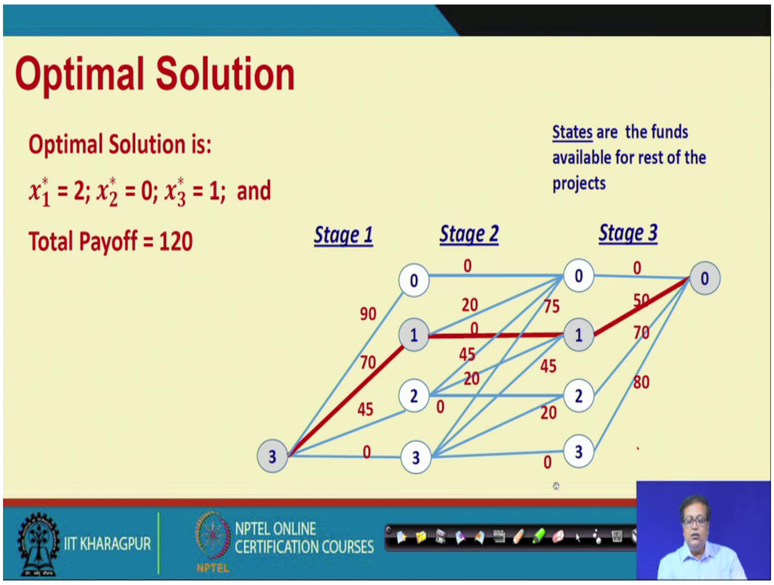

This is shown in this one that is from 3 to 2 you invest at this stage 1 invest nothing at the stage 2 invest the remaining 2 at the stage 3, so here you get a profit of 70 here you get nothing and here you get 50. So, you get a total profit of 120 which is our optimal solution right.

So, we have done what is known as the investment problems and a kind of problem that is project allocation problem, for dynamic programming and we have seen how through a recursive relationship we can apply dynamic programming and solve such problems right.

So, in our next lecture we shall see more such problems.