Selected Topics in Decision Modeling
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Lecture - 30
Example Problems on Constrained NLP

So, in our course Selected Topics in Decision Modeling we are in our 30th lecture, and in this lecture we are going to give some examples on Constrained Non-Linear Programming problems, right some examples of NLP.

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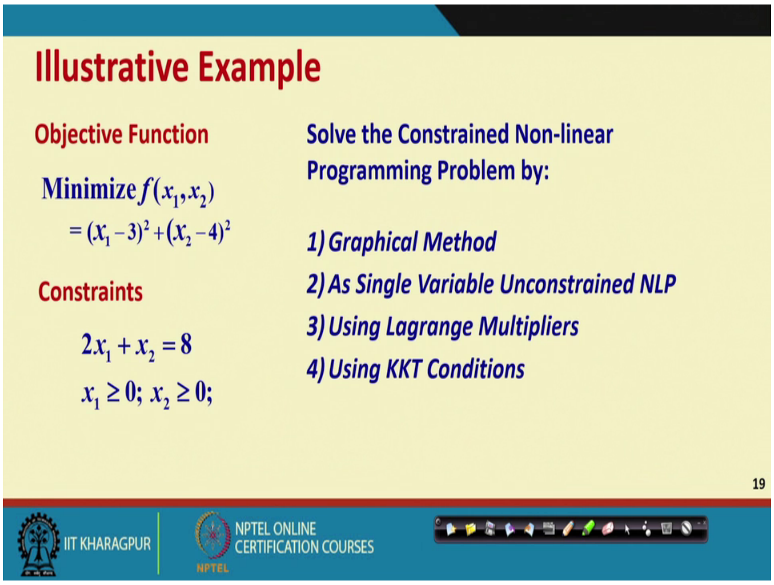

So, let us take very simple problem first. And to really understand the different methods that we have solved at different point of time can we solve this particular problem that is minimize a function of x 1 and x 2 x 1 minus 3 whole square plus x 2 minus 4 whole square and the constraints are 2 x 1 plus 2 x square is equal to 8 and x 1 and x 2 are greater than equal to 0.

So, can you solve this problem by graphical method as single variable unconstrained non-linear programming problem and using Lagrange multipliers and also using KKT conditions right? So, you see this is simple constraint that is equality constraints. So, if you solve the same problem by different methods then you can really understand the relative merits and demerits of these.

So, really speaking the KKT conditions are general conditions, they are utilized for inequality problems. But equality is a special case of inequality is it not? That when inequality constrained; usually say in this case would have been 2 x 1 plus x 2 less than equal to 8, but equal to is included in that. So, that the special case is it alright; so, all problem well Lagrangian multipliers used also be solved by KKT conditions; is alright.

So they are the general conditions can be applied for all problems with equality as well as inequality conditions. Now since an equality conditions is used the constraint is really inequality; sorry equality, you can put that into the objective function and transform it into a single variable problem right.

So, you can right in this case you can find out 2 x 1 plus x 2 equal to 8. So, you can find out x 2 equal to say 8 minus 2 x 1, put it in the objective function and you can also solve it as a single variable unconstrained NLP. This particular problem is simple; so it is possible, but it may not be possible for all problems suppose we have a 3 variable problem and we have only one constraint. So, one constraint can only reduce one variable, it will still remain a multi variable problem and cannot be solved really by single variable problem is alright.

So, that is not possible, but you can actually make it an unconstrained NLP by putting those constraints into the objective function is, alright. So, this is also possible when you have an equality constraint. So, let us proceed and see how we can solve these by all these 4 different methods.

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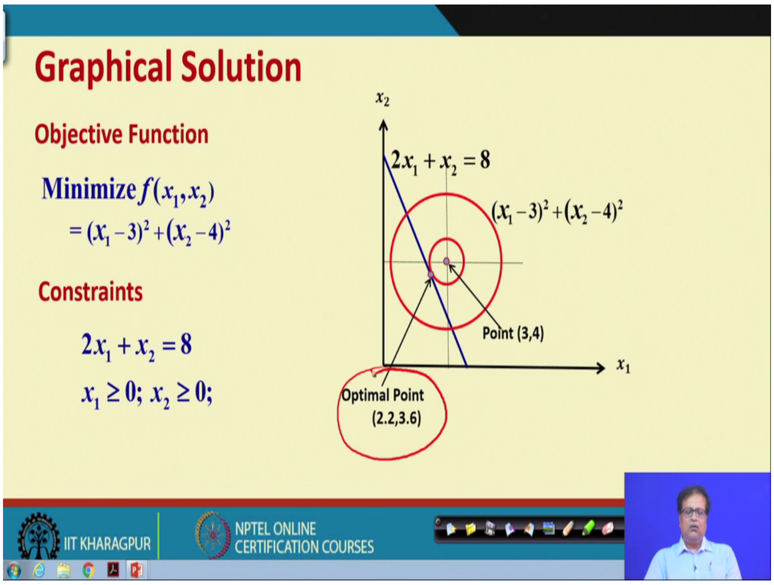

So, first let us look at the graphical method; so, you see first you draw the function these objective function the red one is really the objective function see I had only drawn 2 such circles, there could be more such circles is it not more circles, but then this is our constraint what is the constraint? 2 x 1 plus 2 x equal to 8.

So, this is the constraint line you know how to draw it? If you put x 2 equal to 0, then x 1 will be equal to 4. So, this point is 4 and if x 2 equal to 0 then sorry x 1 equal to 0 then x 2 should be equal to 8. So, we can draw line from this point to this point that will be our constraint line is it alright. Now you see if I draw different circles which are all and this is the point 3, 4 which is our center.

So, making these as a center if I draw different circles that circle which just touches this constraint line; just touches this constraint line that will be our the optimal point is it alright. And incidentally that optimal point is 2 by 2 and 3 by 6 is it alright; that is our optimal point. So, this is how you can solve this problem graphically; this is simple problem, so it is easy to solve, but it may not be. So, easy fine all such problems is it alright right. So, that is how you can do what is known as graphical solution. Let us see how we can solve this by an unconstrained optimal problem.

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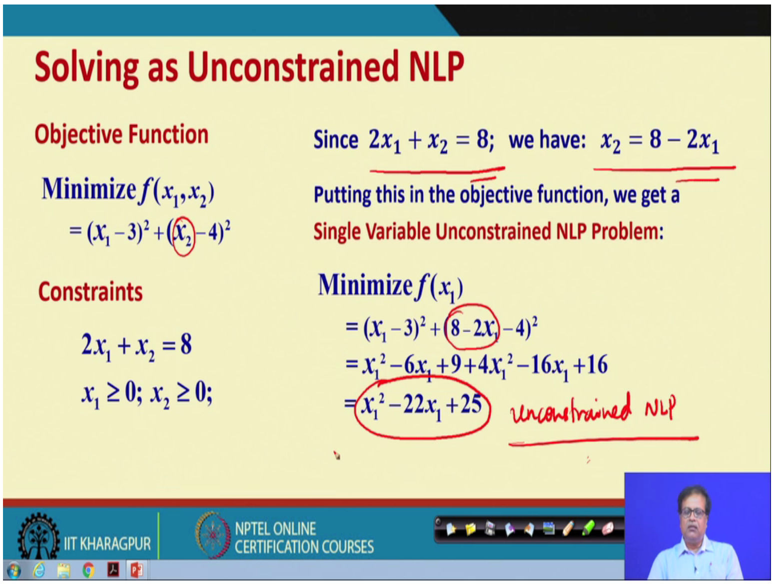

So, as I as we discussed earlier that the constraint line is 2 x 1 plus x 2 equal to 8; so, we have x 2 equal to 8 minus 2 x 1. So, if I put 8 minus 2 x 1 in this case instead of x 2, then I have a x 1 minus 3 whole square plus 8 minus 2 x 1 minus 4 whole square is it alright.

So, this becomes actually x 1 minus 3 whole square plus you know 4 minus 2 x 1 whole square. So, that is x 1 square minus 6 x 1 plus 9 and 4 x 1 whole square minus 16 x 1 plus 16, it comes to x 1 square minus 22 x 1 plus 25; so, that is our objective function. And this then become an unconstrained problem, unconstrained NLP because a constraint is taken inside.

So, what should be done in this case? Recall we have to take first differential for necessary condition and second differential for the sufficient condition. Now the second differential should be positive or negative? It is a minimization problem just recall, it should be negative right.

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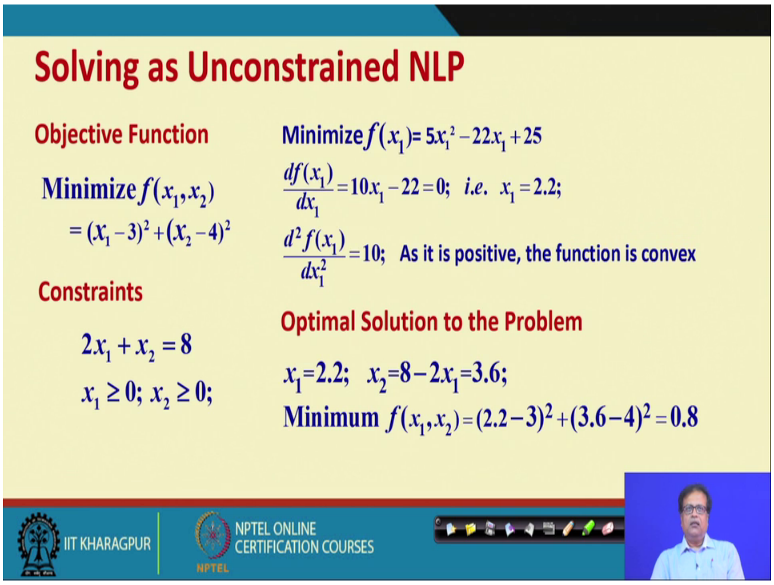

So, we remember that; so, once you remember then we can do that; that we take that it is a minimization problem. So, we take the first differential the fist differential is what we get is you know we take the first differential 10 x 1 minus 22 equal to 0. So, x 1 is 2.2. And second differential for minimization problem, it should be positive because we know if it is negative then it will be for maximization; if it is positive that will be for minimization.

So, minimization problem we must have second differential should be positive. So, second differential is coming to 10 which is positive. So, the functions is convex and minimization is possible so that means, x 1 is equal to 2.2 and x 2 equal to 8 minus 2 x 1 equal to 3.6.

So, the minimization function value is 0.8 and the optimal solution x 1 equal to 2.2, x 2 equal to 3.6. So, we have now seen that how to solve this problem by graphical method and also by unconstrained NLP method. So, how do you solve this same problem by the Lagrangian multipliers?

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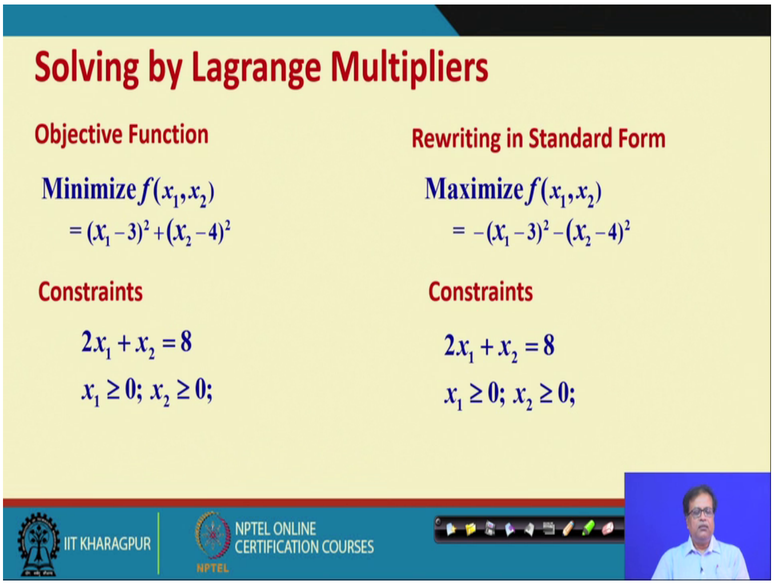

So, this is minimization problem it is better to put it in the standard form; so, if you put it into the standard form then it will be maximize f x 1 x 2 minus of x 1 minus 3 whole square minus of x 2 minus 4 whole square is it alright. So, that is how we can do and the constraint is x 2 x 1 plus x 2 equal to 8, x 1 and x 2 greater than equal to 0.

Actually for Lagrangian multiplier method these standard form is really not required, but we have done this once an once for all because we require this standard form for KKT conditions is it alright.

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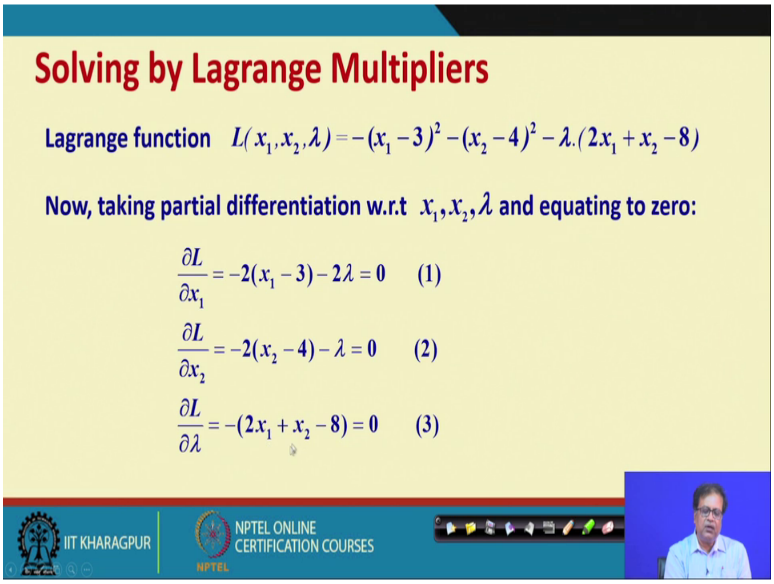

So, we wrote it in the standard form has maximization problem then we formulate what is known as the Lagrangian function. So, what would be the Lagrangian function? It should be minus of x 1 minus 3 whole square minus of x 2 minus 4 whole square that is of first part and minus lambda times 2 x 1 plus x 2 minus 8 that is our Lagrange's function. So, when we differentiate it with respective x 1 x 2 and lambda we get 3 equations what are they plus do it with respective x 1.

So, it should be minus 2 x 1 minus 3 and x 1 will be 0; so, 1; so multi 1. So, it is itself and here it should be 2 lambda is, alright. So, that is how it should be and second term again minus 2 of x 2 minus 4 and differential of these is 1; so that 1 is not required; so minus 2 x 2 minus 4 minus of lambda equal to 0 and the third term minus of 2 x 1 plus x 2 minus 8 equal to 0.

So, these are our 3 differentiation partial differentiation and then putting equal to 0 is alright. So, that would give us Lagrangian function differentiation.

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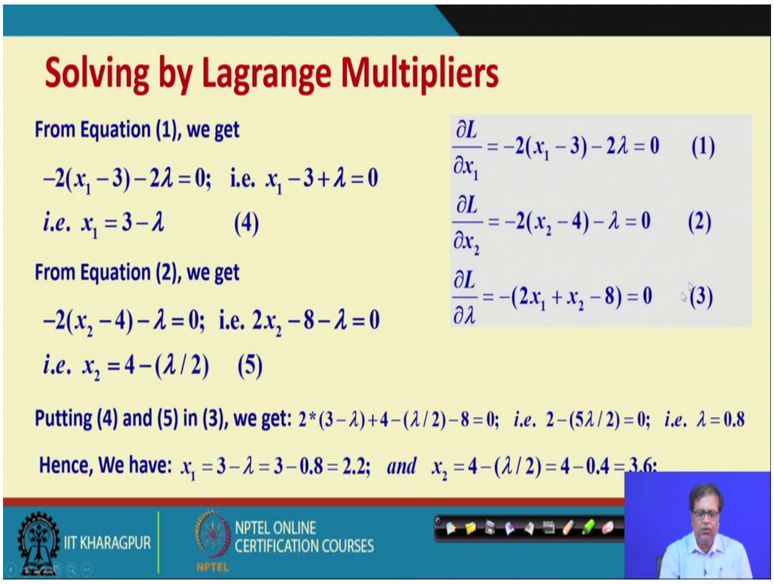

So, this is exactly what we have got now the first one will give minus 2 into x 1 minus 3 minus 2 lamda equal to 0 or x 1 equal to 3 minus lambda. Second one 2 x 2 minus 8 minus lambda equal to 0 or x 2 equal to 4 minus lambda by 2.

So, if put them into the third equation. So, 2 into 3 minus lambda plus 4 into min into x 2 that is 4 minus lambda by 2 minus 8 equal to 0. So, that gives 2 minus 5 lambda by 2 equal to 0 or lambda is equal to 0.8. So, that 0.8; if I put when x on is 2.2 and x 2 is 3.6. So, if you recall we have got the same value that is 2.2 and 3.6 by Lagrangian multipliers also.

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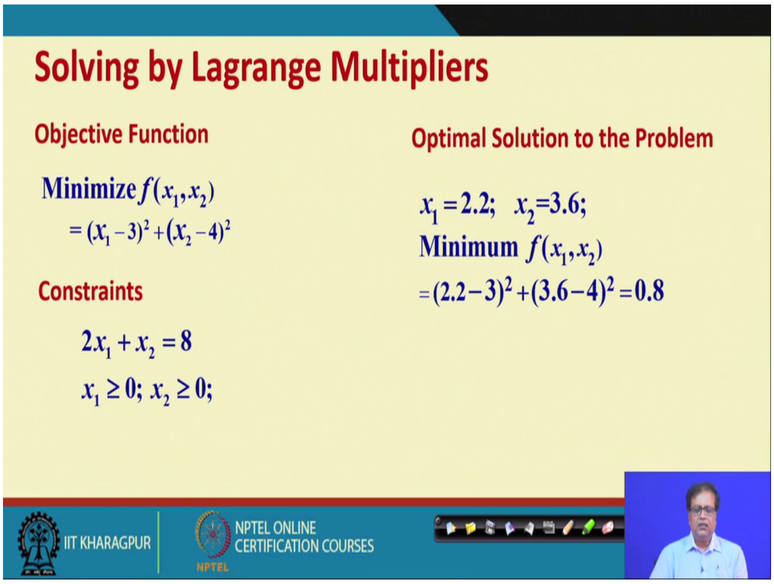

So, this is the solution then by Lagrangian multipliers that x 1 equal to 2.2 and x 2 equal to 3.6 is alright. So, how then this problem will be if you solved be by KKT conditions right?

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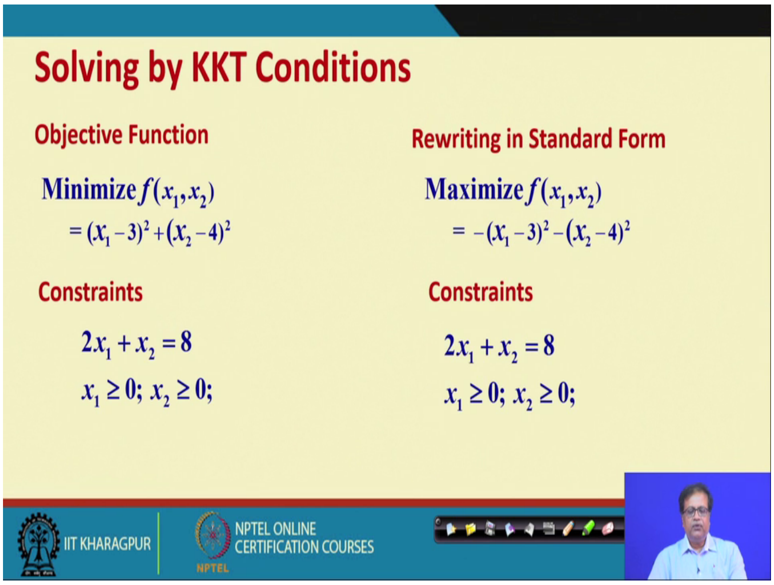

The same problem once again if we solve by KKT conditions; how it should look like? Look here the original problem is function of x 1 and x 2 equal to x 1 minus 3 whole square plus x 2 minus 4 whole square constraints 2 x 1 plus x 2 equal to 8.

So, putting in standard form minus x 1 minus 3 whole square minus x 2 minus 4 whole square 2 x 1 plus x 2 equal to 8, x 1 less than equal to 0 x 2 greater than equal to 0 x 1 and x 2 both greater than equal to 0.

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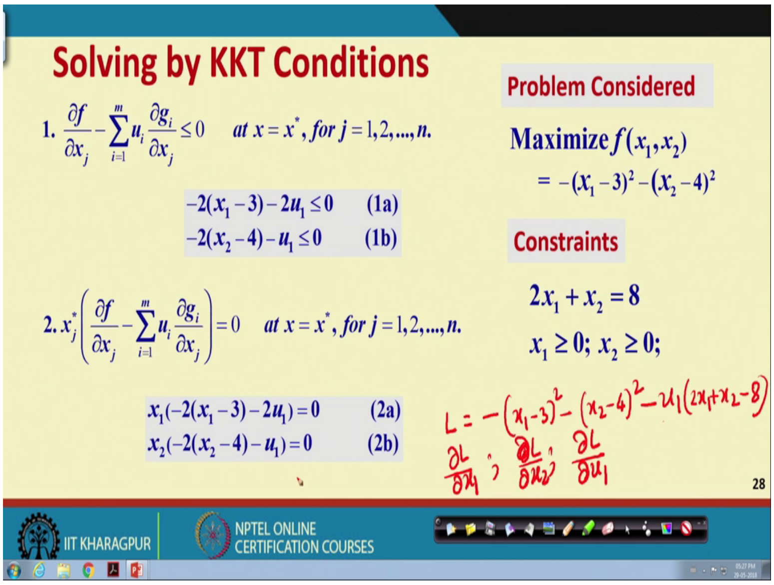

So, you know if I if I write; this is our original problem then this are these are the constraints that we should use; see either you can use then in these manner or as I just discuss in the previous classes what we can do? We can actually right down the Lagrangian function. So, the Lagrangian function could be equal to minus of x 1 minus 3 whole square minus of x 2 minus 4 whole square then minus u 1 2 x 1 plus x 2 minus 8.

So, we can right the Lagrangian function and then we can obtain del l, del x 1, del l del x 2 and del l del u 1. So, what will be del l del x 1? Minus 2 x 1 minus 3 minus 2 u 1; that is the first one. Second one minus 2 x 2 minus 4 and minus u 1 that is second term. So, that should be greater than equal to 0 and another term will be x 1 multiplied by del l del x 1 equal to 0; x 2 multiplied by del l del x 2 equal to 0 right.

So, this is how these 2 sets of conditions can be written. So, once again we can formulate the Lagrangian the del l del x 1 less than equal to 0 del l del x 2 less than equal to 0 x 1 times del l del x 1 equal to 0 x 2 time del l del x 2 equal to 0 right. So, this is how we can right all those 4 constraint.

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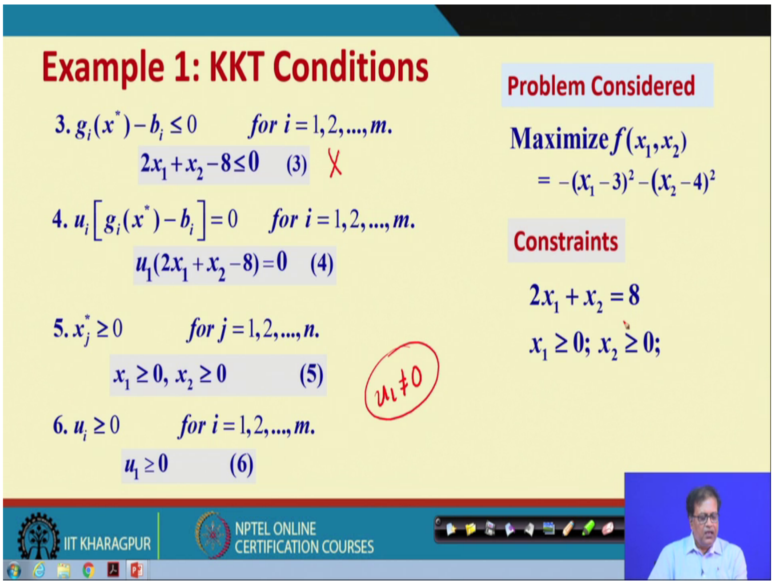

So, moving ahead we can right the constraint line itself 2 x 1 plus x 2 minus 8 less than equal to 0 and u 1 into 2 x 1 plus x 2 minus 8 equal to 0. So, this is another set of constraint and the last ones are x 1 greater than equal to 0; x 2 greater than equal to 0 and u 1 greater than equal to 0. Now once we have written all these KKT conditions, now we see that in this particular case we have x 1 equal to x 2; 2 x 1 plus x 2 equal to 8.

So; that means, these time equal to 0. So, that gives us that u 1 equal to not 0 right; so, we fine that u 1 is not equal to 0. So, that is one thing that we find out directly. And there is no need to right really condition 3, it is condition 3 is not really required because directly we have 2 x 1 plus x 2 equal to 8.

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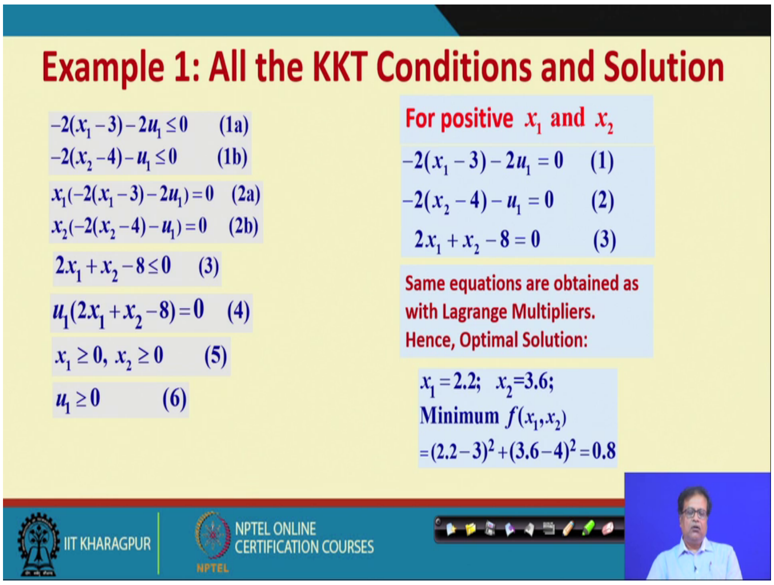

Now apart from this we can also observe that by writing all these constraints that the x 1 and x 2 are going to be positive because you know really speaking we are have a non zero solution.

So, since that is so automatically from here since x 1 and x 2 are positive x 1 and positive x 2. So, we can find out that these terms are really equal to 0; so, right. So, really have 3 equation that these equal to 0 these equal to 0 and these equals to 0. So, if we solved them these become exactly same as those equations obtained with Lagrangian multipliers.

So obviously, the solutions also would be same that is x 1 equal to 2.2 and x 2 will be equal to 3.6 and the minimum function would be 0.8 alright. So, see we have really seen that the KKT conditions can be utilized for almost all NLP problems and particularly for problems also where equality constraints is also used is alright.

So, these particular example is I had taken really to explain the equivalence of the KKT conditions and the Lagrangian multipliers. And as a special case for equality constraint we have also seen how a similar problems can be solved by equality constraint for 2 variable problem as an unconstrained problem also; additionally the graphical explanation is also seen right

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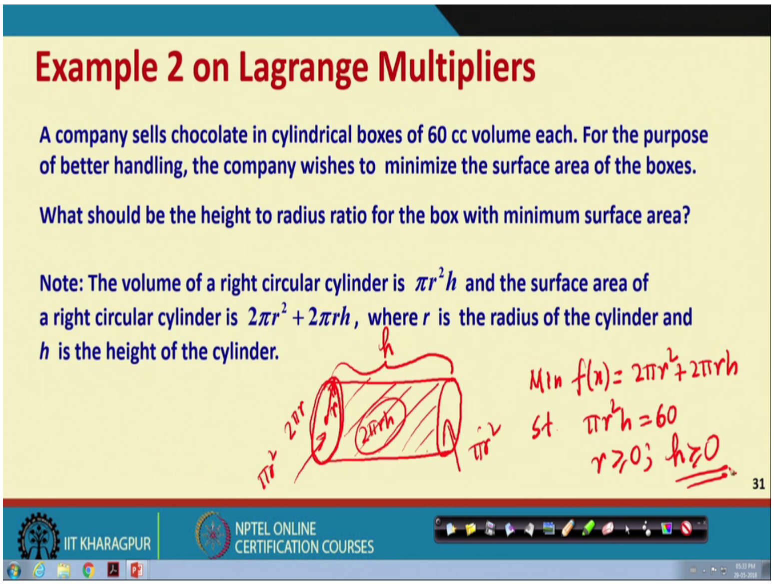

So, as more exercise let us let us see some problems on Lagrangian multipliers again. Let us see a problem on this type; say a company sells chocolate in cylindrical boxes of 60 cc volume each. For the purpose of better handling the company wishes to minimize the surface area of the boxes right. So, what should be the height to radius ratios for the box with minimum surface area is alright.

So, you know what the company wants? The cylindrical boxes the volume is 60 cc right that is given, but then how do you make a height to radius ratios so that the box is having minimum surface area. Look here the volume of right circular cylinder is pi r square h and the surface area is you know 2 pi r square plus 2 pi r h; 2 pi r h you know this where r is the radius of the cylinder and h is the height of the cylinder is it alright.

So, we know that this part is because the right circular cylinder really look like this that this is how which should look like right. So, if this is r then this perimeter will be 2 pi r and this height is h. So, 2 pi r h is this portion this curve portion and this one is pi r square and this one is also pi r square.

So, if you add then this is 2 pi r h and this pi r square plus pi r square that is 2 pi r square plus 2 pi r h; that is the total surface area; what we have to do is we have to minimize the surface area right. So, what will the function? Minimize fx equal to 2 pi r square plus 2 pi r h; subject to pi r square h equal to 60 and r greater than equal to 0 h greater than equal to 0.

So, that would be our problem formulation is it not this fair and simple? So, question is that how do we solve this problem by making use of Lagrangian multipliers?

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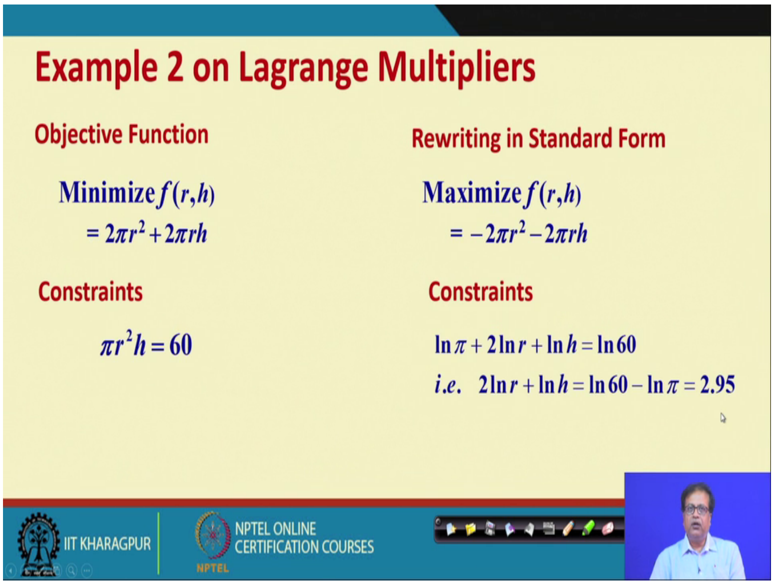

So if you write this is the objective function minimize 2 pi r square plus 2 pi r h and constraints is pi r square h equal to 60.

So, if you rewrite in the standard form then we get maximize f r h is minus 2 pi r square minus 2 pi r h the negative form. And see this constraint we can make it a linear constraint by making l n pi plus 2 l n r because that is how the l n works plus l n h equal to l n 60; so, just taken l n of this. So, then 2 l n r plus 2 l n h is l n 60 minus l n pi and see l n 60 and l n pi can be can be obtained from calculators and we can get a value of 2.95.

So, basically the problem then becomes maximize minus 2 pi r square minus 2 pi h pi r h subject to 2 l n r plus l n h equal to 2.95, right. So that is the problem. So, how it should be solved?

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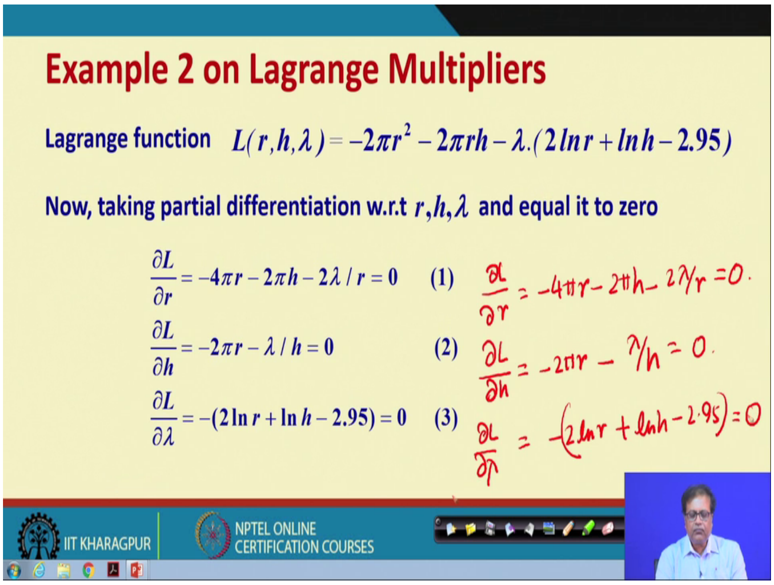

So, first we define the Lagrangian function l r h lambda equal to minus 2 pi r square minus 2 pi r h minus lambda into 2 l n r plus 2 l n h minus 2.95 right then we should take the differentials.

So, please find out these differentials del l, del r, del l, del h and del l dl lambda and they should be equated to 0; the first one del l del r; so minus 4 pi r minus 2 pi h and here minus 2 lambda by r equal to 0; why 2 lambda by r? Because 2 lambda and l n r if you differentiate 1 by r, so that is how. And the second case del l del h; the first term nothing then minus 2 pi r and this should be minus lambda by h equal to 0 and third is straight forward minus 2 l n r plus l n h minus 2.95 equal to 0 right.

So, these are going to be our 3 equality conditions that we get from the Lagrangian multipliers.

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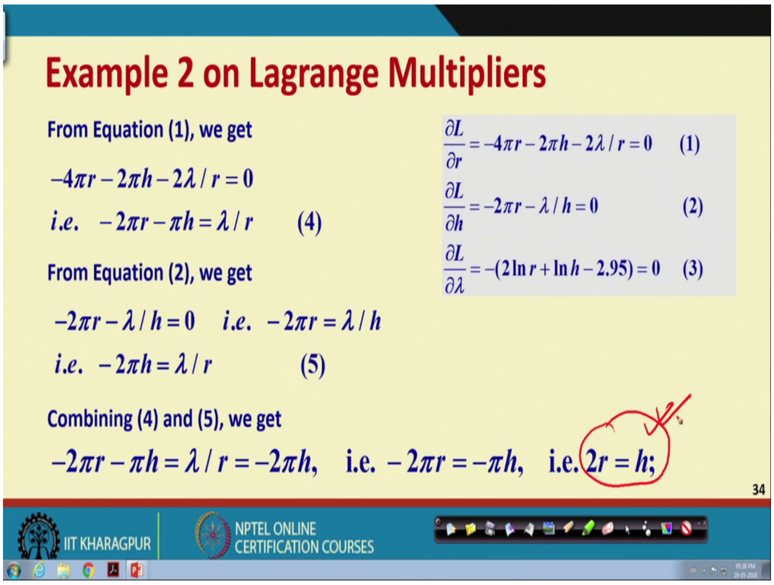

So, having obtained that now we can tried to solve. So, if we try to solve then we find that the first conditions I know if take this portion that side and divided by 2. So, minus 2 pi r h minus 2 pi r minus pi h equal to lambda by r and here minus 2 pi r equal to lambda by h by slight re orientation minus 2 pi h is also lambda by r.

So, 4 and 5 both are each is lambda by r; that means, l h is can be equated. So, minus 2 pi r minus pi h equal to lambda by r equal 2 pi h; so, little reorientation gives minus 2 pi r equal to minus pi h. So, if you cancel pi from both sides and equal I mean minus sign also from both sides we get 2 r equal to h right. So, that is precisely the condition that we have got h equal to 2 r right.

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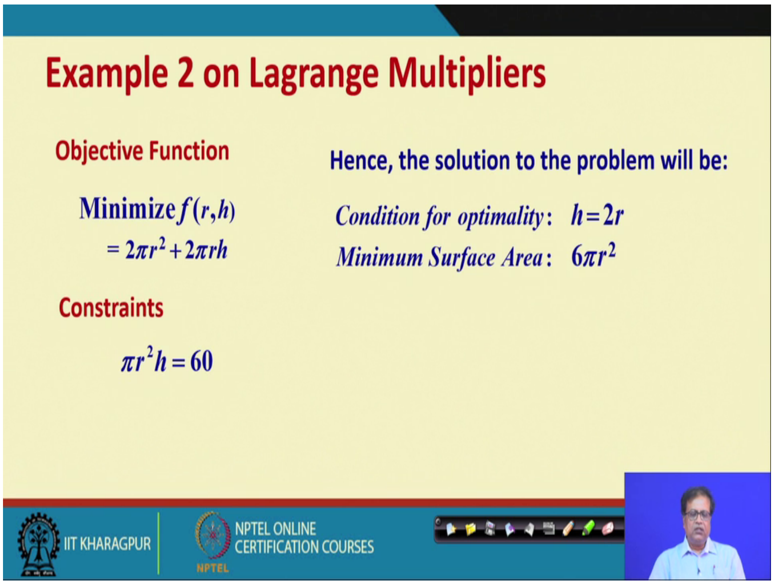

So, that is the thing that h is really 2 r 2 r; that means the height should be the equal to diameter right. If, r is the radius then height should be equal to diameter and minimum surface area is 6 pi r square, but additional thing has to be obtained that since these problem is one minimization; that means, this function should be really convex that should be ensured also.

So, we saw a problem with Lagrangian multipliers and as a just a quick overview let us we take look one another problem on the KKT conditions.

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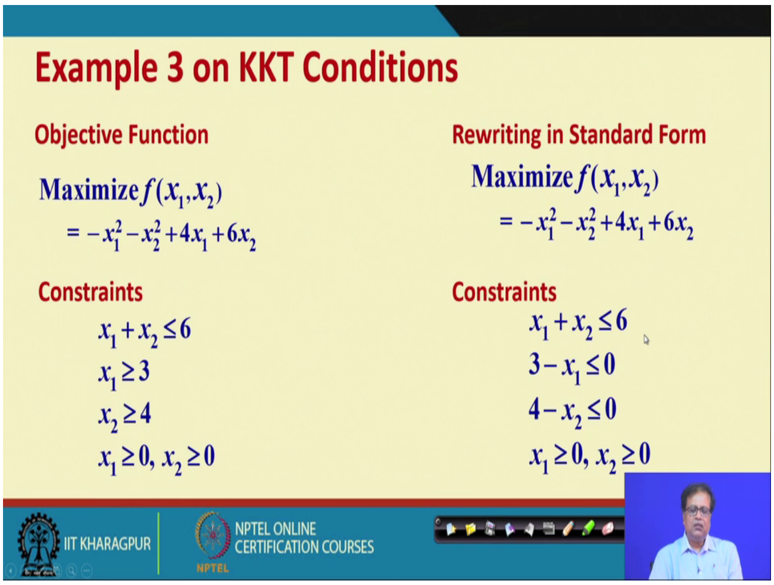

Just an overview so that our idea becomes clear; so, here is a problem that let us say maximize a function a minus x 1 square minus x 2 square plus 4 x 1 plus 6 x 2 and constraints are x 1 plus x 2 less than equal to 6 x 1 greater than equal to 3, x 2 greater than equal to 4 and x 1 x 2 greater than equal to 0 right.

So, if we try to put it in the standard form then these conditions has to be re written as 3 minus x 1 less than equal to 0 and 4 minus x 2 less than equal to 0; rest are the same. So, this is how we have got the original problem. So, what we do then?

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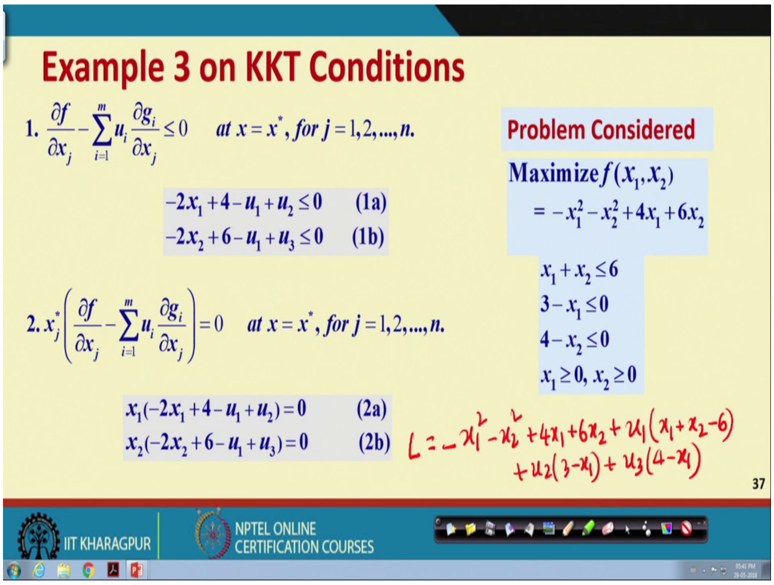

We this is our problem; so, these problem now see if you right take the Lagrangian method, then what will be the Lagrangian function? It should be minus x 1 square minus x 2 square plus 4 x 1 plus 6 x 2 plus u 1 into x 1 plus x 2 minus 6 plus u 2 3 minus x 1 plus u 3 4 minus x 1.

So, that will be the Lagrangian and we have to then differentiate with respect to x 1, x 2, u 1, u 2 and u 3 is alright all of them. So, if we if we differentiate with respective x 1, we get this condition; if we differentiate with respective x 2 we get this condition is alright. And additional constraint is that these part multiplied by x 1 should be 0, this part multiplied by x 2 should be 0 right.

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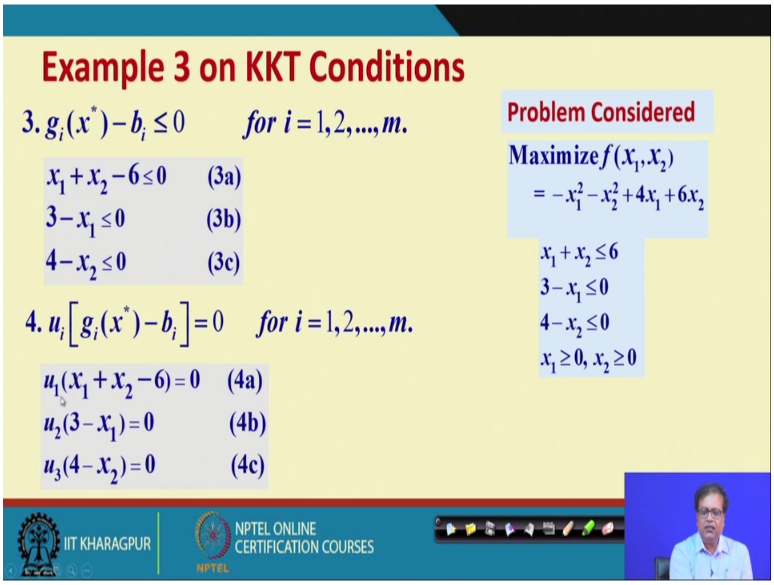

So, these are the first 4 conditions; now the constraints themselves are the third that is with respect u 1, u 2 and u 3. And; obviously, the first constraints multiplied by u 1, second constraint multiplied by u 2, third constraint multiplied by u 3 should be equal to 0 right and apart from the greater than equal to conditions.

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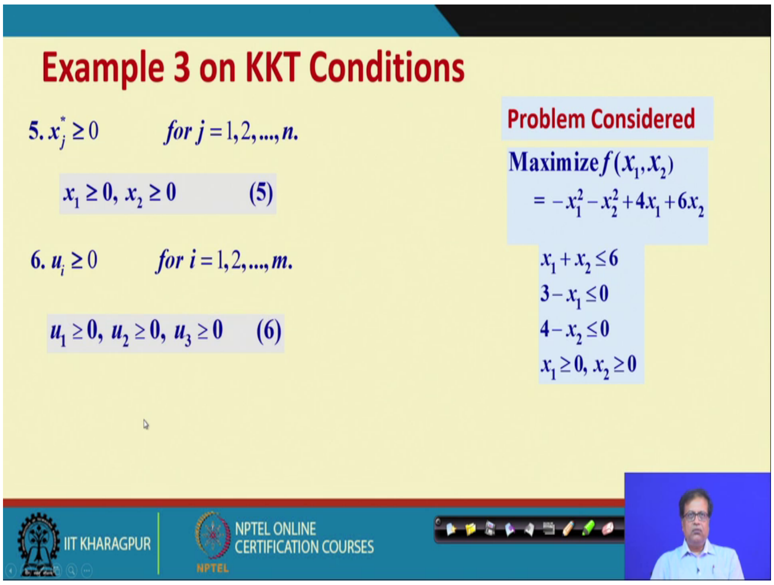

So, these are all the KKT conditions.

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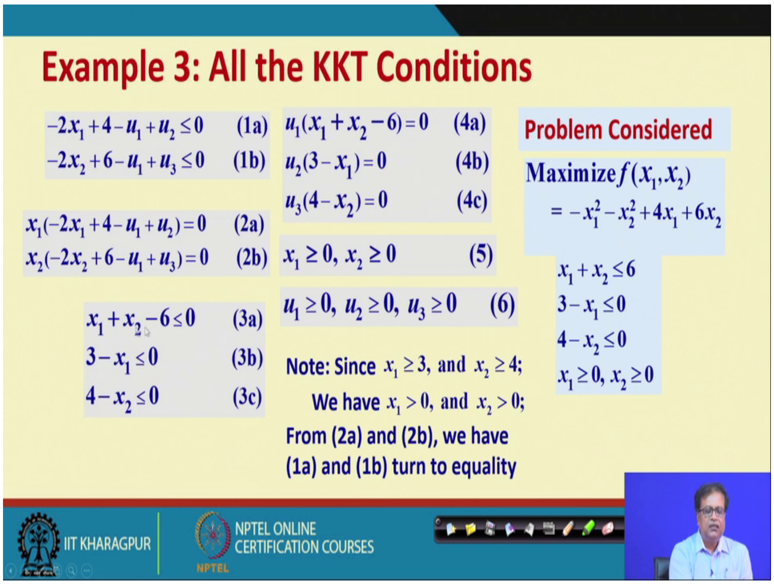

So, all the KKT conditions are written here, but you know one interesting thing you can see that x 1 is greater than equal to 3 and x 2 is greater than equal to 4. So, what you can say that both x 1 and x 2 are greater than equal to 0. So, if x 1 and x 2 are greater than equal to 0; obviously, these terms equal to 0. So, this is immediately we can see that we have these terms are equal to 0.

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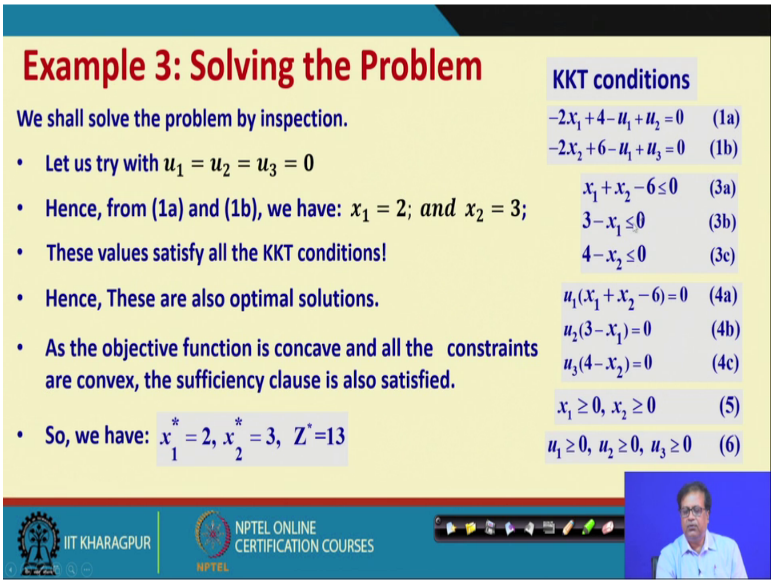

So, what we get then from the KKT conditions that these are equal to 0, these are another set of condition and these are another set of conditions and these are another greater than equal to conditions. So, we can first try with u 1 equal to u 2 equal to u 3 equal to 0. So, if these terms are equal to 0 then by putting here we can solve and you can see x 1 equal to 2 and x 2 equal to 3 really takes care of them. And you see there can be obtained from these 1 a and 1 b and if we put them into the condition 3; all of them as satisfied and rest of them a satisfied also is alright.

So, we see that x 1 star equal to 2, x 2 star equal to 3 and z star equal to 13 will be our optimal solutions right. But the additional thing that you must also ensure is that the original functions should also be a concave function because the constraints are all convex because they are all equal to. So, they can be I mean they are all linear; so, they can be both convex and concave, but the objective function must be concave for a maximization problem is it alright. So, that must be ensured also, right.

So, in this particular lecture we have seen different examples of constraint NLP problems. And I hope by this time you have got sufficient expertise on unconstrained and constraint NLP problems. And I hope you will be able to solve different NLP problems without problems is it, alright.

Thank you very much.