Selected Topics in Decision Modeling
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Lecture – 03
An Investment Problem

Welcome to all of you. We have started the Selected Topics in Decision Modeling and specifically we are discussing the Dynamic Programming part. And today, let us take a new topic that is An Investment Problem.

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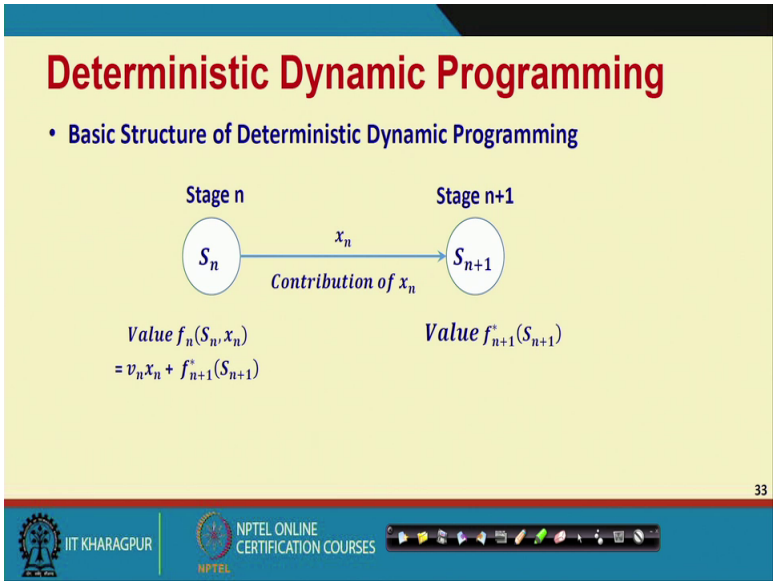

Now, the investment problem that we shall consider will be a part of what is known as the Deterministic Dynamic Programming. The basic structure of the Deterministic Dynamic Programming will be as you can see in the diagram. So, what we have is you know if there are multiple stages of the particular problem, now from that multiple stage from a given stage n to the given stage n plus 1, you know there will be decision variable x n, and the contribution of x n can be included which can be taken as the figure here that is v n x n right.

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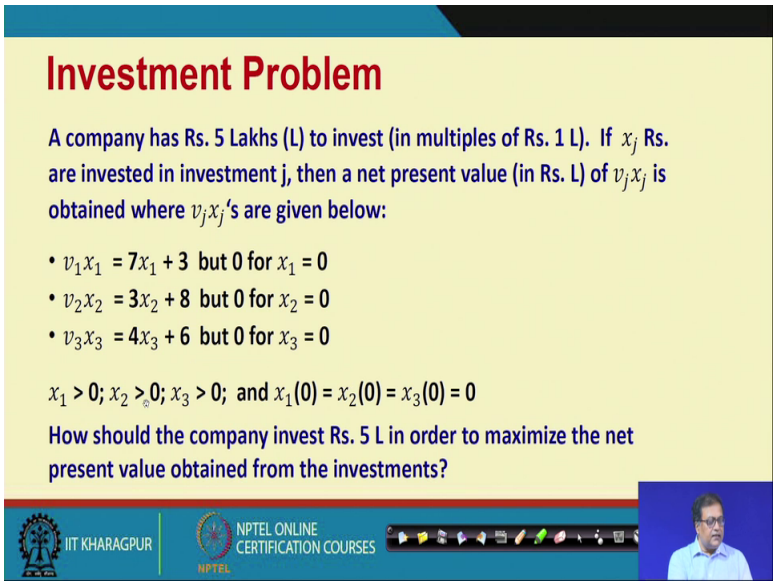

So, the v n x n is you know the contribution of the x n and suppose the optimal value that we have got at our next stage that is stage n plus 1 obviously; we are talking about backward dynamic programming. Suppose that value is f star n plus 1 which is a function of S n plus 1 which may be a particular state variable. Then, the value at stage n could be computed as f n S n x n which is the function of x n as well will be v n x n plus f star n plus 1 S n plus 1 right. So, basically now for different values of x n, the optimized value of f n x S n x n will be what we shall be looking for. So, essentially that is the idea and over and above there will be what is known as a recursive relationship that will determine the specific problem that we are considering at a given stage.

Now, let us look at the an investment problem that we are talking about. Let us say a company has rupees 5 lakhs which we shall denote by L to invest in multiples of rupees 1 lakh. Is it all right? Now, if x j rupees are invested in investment j, then the net present value in rupees in lakhs of v j x j is obtained where v j x j's are given by three formula which can be given here. That means, in the first investment option is v 1 x 1 is 7 x 1 plus 3, but if x 1 equal to 0, obviously if you do not invest anything, then there is no return, the return will be 0. The net present value or n p v you know very well that is a kind of you know pattern or kind of a parameter by which the investment is evaluated. So, like this we have the three investment options and you know it is said that x 1, x 2, x 3 you know they should if they are bigger than 0 only then these equations are valid, but

if the value is 0 that means, you do not invest a particular stage then, the value obtained in that particular stage will be 0.

So, the question is how should the company invest that is rupees 5 lakhs in order to maximize the net present value obtained from the investment. So, how do we go about this problem? In the last you know few lectures; we have seen that Bellman's optimality principle. And essentially the idea is that if we can divide the problem into multiple stages and define the problem in terms of states and the decision variables, then the optimal at any particular stage will depend on that particular stage you know whatever is remaining and we can make use of the optimal values that we have already got; that is what we have seen in the principle that I discussed in the previous slide.

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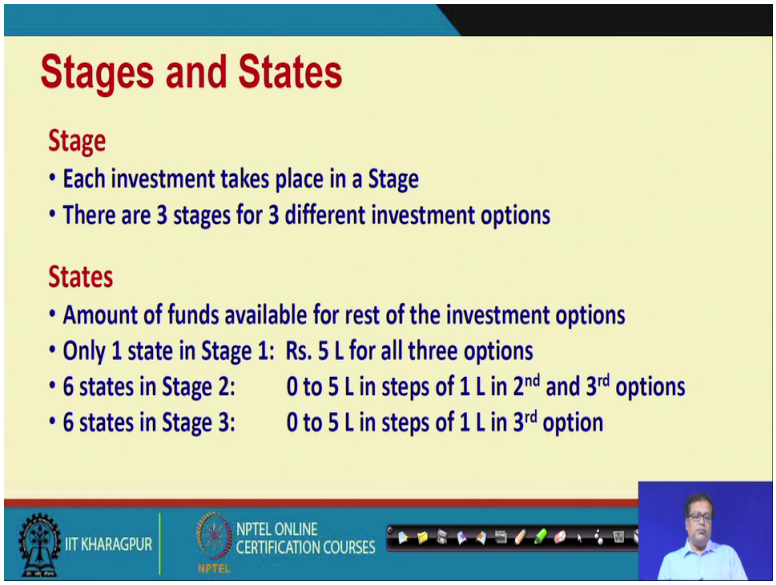

So, what could be some stages and the states in this given problem? So as you know the each investment takes place in a particular stage, and there are three stages for the three different investment options. So, we have three investment options, and each investment options will basically contribute to one stage. What could be some states the amount of fund available for rest of the investment options. So, this is slightly tricky. So, supposing the only one state in stage one, so at the very beginning in the first stage that is only one thing that is we have 5 lakhs remaining for all three options. So that is what is the state. What is state, how much fund is available for the rest of the investment options; is it ok?

Now, in stage 2, what when you go to stage 2 then already first option is exhausted. So, let us say in first option we have spent a some amount of money, the remaining amount of money would be available for the remaining stages or remaining options. So, let us the first option could be 0 as well. So, assuming the first option could is 0 then, even entire 5 L is available for rest of the options, but if it is not 0, 1, 2 or anything, the rest of the money will be available.

So, what happens in the stage 2? We have 0 to 5 lakhs in steps of 1 lakh is available in 2nd or 3rd options. And finally, when it comes to stage 3 again depending on optimality whatever remaining portion, we can spend or we can invest that is 0 to 5 L in steps of 1 lakh that is the 3rd option. So, these are the stages and the states.

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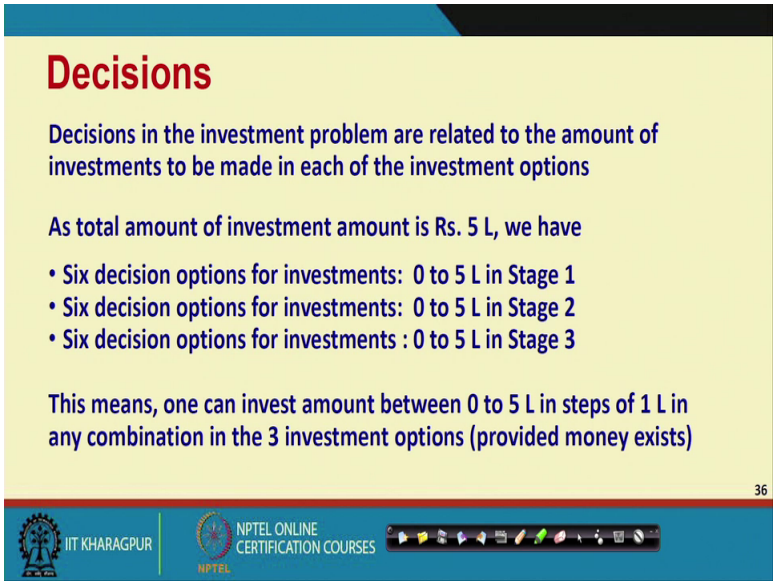

Now, once we know the stages and the states, so what are the decision that we take? So, in this case, the decision variables at each stage which can be your x n, decisions in the investment problem are related to the amount of investments to be made in each of the investment options. So, as total amount of investment amounts is rupees 5 lakhs. So, we have 6 decision options that is 0 to 5 L in stage 1; 6 decision options that is again 0 to 5 L in stage 2 and again 6 decision options that is 0 to 5 L in stage 3. This means one can invest amount between 0 to 5 L in steps of 1 lakh in any combination in the 3 investment options. Obviously, provided money exists; that means if you have spent certain amount in stage 1, only the remaining amount is available in the stage 2 and stage 3. Since, 0 is

also included, one can easily think that entire 0 to 5 could be available in the stage 2 or in the stage 3. So that is how we start that means we define the states which are you know the amount of money that is available that is our previous slide.

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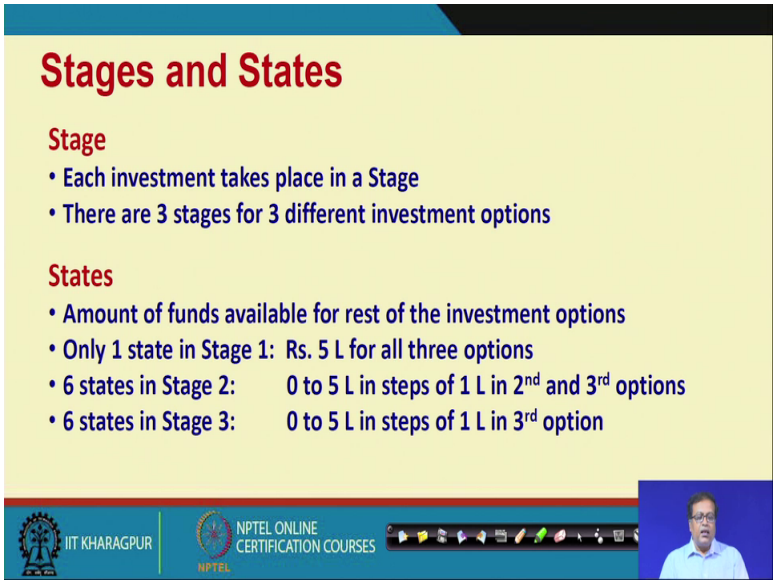

The amount of fund available for rest of the investment options; there are three stages; there are you know the three investment options.

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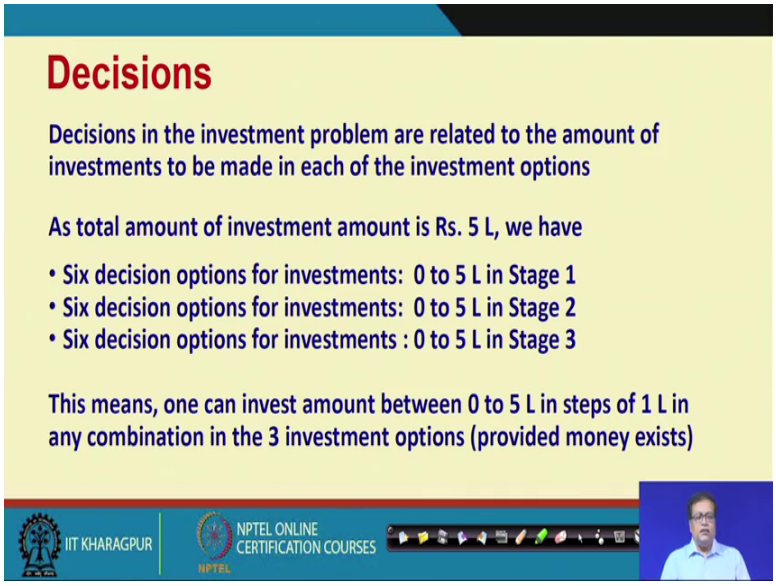

And the decisions is how much you spent in a particular investment it a given you know choice that is given investment option. So, having said that these are the states, these are the stages, and these are the decisions.

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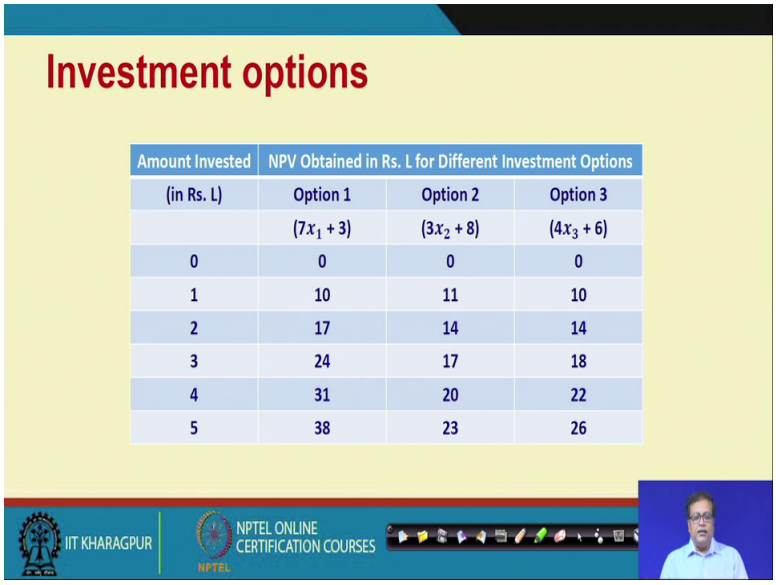

Let us see how they combine. What we have done here, we have computed based on the formula that is given to us that is the amount invested that is amount could be 0, 1 lakh, 2 lakh, 3 lakh, 4 lakh or 5 lakh depending on that what we have done? And for the three options what are going to be the net present value in rupees lakhs for the different investment options that we have already obtained beforehand. As an example for the option 1 the NPV is you know 7 x 1 plus 3. So obviously, this would be your 0, 10, 17, 24, 31 and 38. So, you know you can think of this as basically the x n. You know this amount invested is nothing but the x n. So, how much amount you have actually invested in a particular investment options, so that is the different options.

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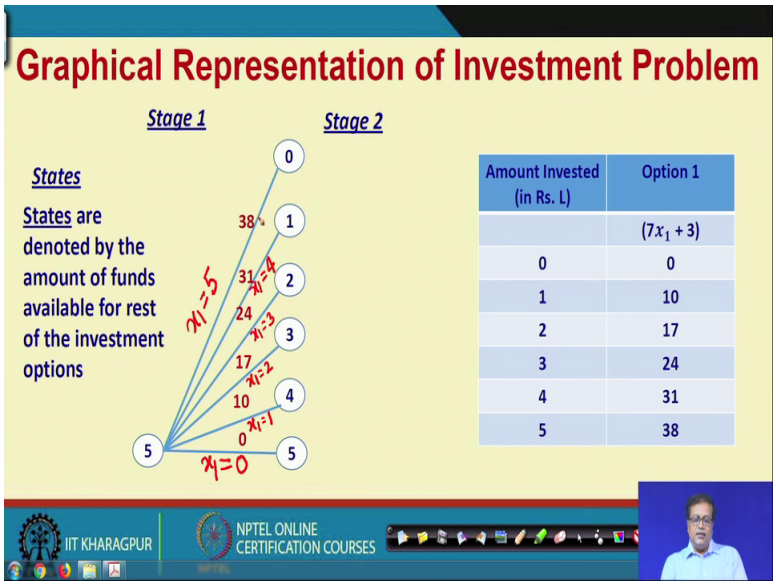

Now, let us see how these things the stages and the states how do they combine. And let us look how we can actually graphically represent the whole problems. So, once again if you see on this side of the screen, what we have done we have put the amount invested versus the options. So, option 1, these are the 0, 10, 17, 24, 31, and 38. Now, this side the states, the states are denoted by the amount of fund available for rest of the investment options. Now, we said that stage 1, there is only one investment state. What is that? That 5 lakhs. What is this 5 lakhs? This 5 lakhs is the amount remaining for investing in all the different investment options, is it all right.

Now, from these investment options in stage 1 to the investment option in stage 2, what could be the amount that is available? Now, one can see that it could be anything from 0 to 1 to 2 to 3 to 4 to 5. How? You know if depending on whether what is our value of x n. So, if suppose sorry if we have our x n, so our x n value could be you know here let us write suppose here can you write down the different x n values for these option.

What is the x n value at stage 1? If we have here that is the state 5 to state 0, if we move from stage 1 to stage 2. This should be how much? It should be 5. Why, it is 5? Because it is very clear that from 5 we have spent 5 here n equal to 1. So, this x n you can even say so you know you can even say that x 1 equal to 5, because at this stage your x n is nothing but x 1 because if you invest 5 lakhs in the first investment option which gives

you a return of 7 x 1 plus 3, then you know you can you have reached from a state 5 to state 0 in the stage 2, is it all right.

And look at this formula that is 7 x 1 plus 3, so that gives us that a total profit or net present value the return that can be return will be 38. So, what is this 38 that if we invest 5 lakh rupees in the first option investment option then, we get 38 lakhs as the net present value. And in dynamic programming representation that we move from state 5 to state 0, because state is the amount of money that is remaining for investment because if we invest 5, then remaining money will be 0. So, we move from 5 to 0 with an investment of 5 and the value that has really gone in will be 38. So, I hope this is clear. Now, if this is clear then rest of the things should be clear also.

Then, can you tell me tell us what will be the x 1 value, if we move from 5 to 5, what will be the value of x 1 in this case? This value should be equal to 0. Why it should be equal to 0? Because we had 5 lakhs at stage 1, we also have 5 lakhs at stage 2 that is the amount remaining for investments; that means you have not invested anything in the first option, therefore this must be 0. So, if this is 0, look at this chart we have got NPV that is return of 0 only right. So, if we move from 5 to 5, obviously, x 1 equal to 0 and our net gain is 0 only. So, obviously, now you can understand the second one should be x 1 equal to 1, third one will be x 1 equal to 2, next one will be x 1 equal to 3, next one will be x 1 equal to 4, and returns will be 0, 10, 17, 24, 31 and 38 right. So, that is how you move from stage 1 to stage 3. Having said that next one that is the graphical representation of the investment problem that is already we have seen the stage 1 to stage 2 we have come.

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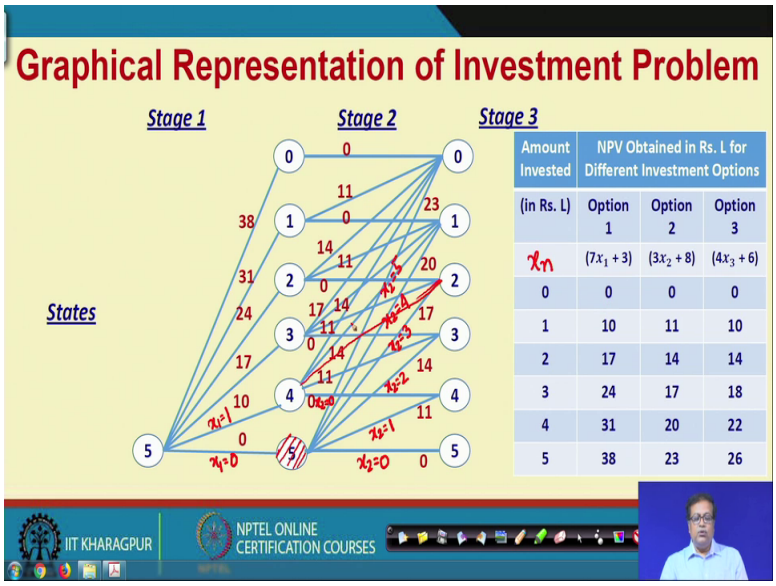

So, what happens in the next stage? You see now of this graph looks slightly more complicated, but let us not get you know too much overwhelm by these. Let us simply understand that the first portion of the graph that means, from stage 1 to stage 2, where the transition is having from 5 to 0, 1, 2, 3, 4, 5 these states; and this 0, 10, 17, 24, 31, 38 this we already understand. So that part is already taken care of. Now, look at what is happening at the option 2. In the option 2, this is our guiding formula that is 3 x 2 plus 8. And if we invest 0, then we get 0. See please be very much you know careful about knowing that what is the difference between x n and these states. The states are not x n they can be called S n. So, these S n is the amount that is available to invest in all the remaining investment; and x n is that you are investing at a given investment option right, so that is called decision.

The another thing is the stage which are the three investment options, the states which are the amount of money that is available. So, once you are very clear about this, now let us look at option 2. So, depending on how much you invest in option 2, whether it is 0 or 1 or 2 or 3 or 4 or 5, based on that the option 2 will be having a return of that is net present value of either 0 or 11 or 14 or 17 or 20 or 23. So, how these things are actually can be coupled in these particular diagram?

So, let us look at supposing for example, suppose in stage 1, you have not invested anything; that means, we are in stage 5 that means, this is where we are when it comes to

the second stage. Now, from here basically at the stage 3 that is at the last stage, you can have one of the six states either you have entire 5 lakhs available or 4 lakhs available or 3 lakhs available or 2 lakhs available or 1 lakh available or nothing available. So, what is this one, So, how much you have invested at the, so here you have invested x 1 equal to 0. So, what is this one? This must be then x 2 equal to 0 that means, you have not invested anything, the decision here is 0, and therefore from 5 you have move to 5.

So, next one therefore, must be x 2 equal to 1. The next one must be x 2 equal to 2, the next one must be x 2 equal to 3, next one x 2 equal to 4 and finally, it should be x 2 equal to 5. So, you see there are too many lines; so you know please understand the specific lines that this one is 0, this is 1, this is 2, this is 3, this is 4 and this is 5. And what are the corresponding returns, the returns are 0, 11, 14, 17, 20 and 23. So, obviously, this will be our returns that is 0, 11, 14, 17, 20 and 23. I hope it is clear to you that for different investments at the second investment options will lead to different returns at that particular stage.

Now, assuming x 1 is not 0 right rather x 1 is equal to 1. So, if x 1 equal to 1, then at stage 2, we have reached a value of 4 right. Now, if we have 4, now depending on that then x 2 will have several options, not so many options slightly less number of options. So, here if x 1 equal to 1, then this will be our x 2 equal to 0 and that option will lead to a return of 0 at the second stage right. So, similarly x 2 equal to 1, x 2 equal to 2, x 2 equal to 3 right 0, 1, 2, 3 these are the different options that will be we shall have. There is one line missing here. So, this one is 0, this is 1, there will be one line which is missing. So, this line will be actually x 2 equal to 2 anyhow. So and then the corresponding profits that is 0, 11, 14 and all those things that would actually return; so like these all these lines can be drawn and you know you have then completed what is known as the stage 2 level of investment.

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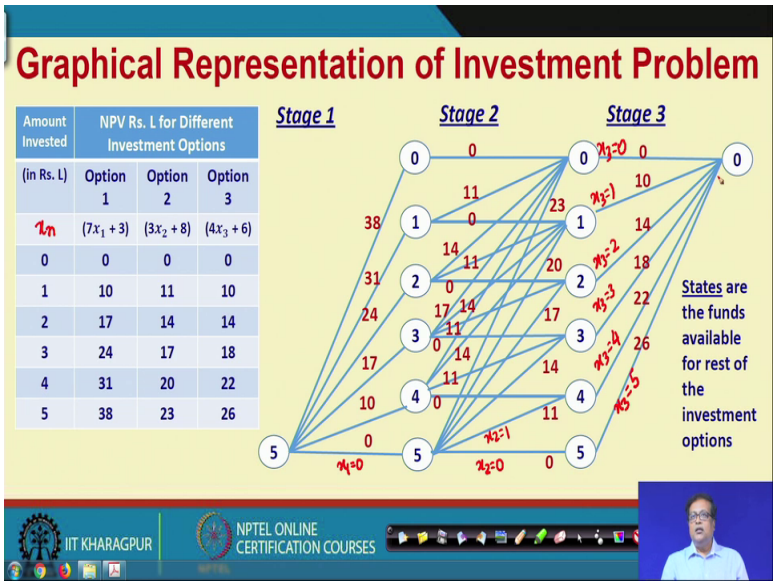

Now, having done that, let us now go to the third stage. So, now, we have gone to the third stage. So, now, you see at the third stage obviously, now you know what we have got is after having come to 0, 1, 2, 3, 4, 5, now what are or everything is not shown, let us assume that if we invest the remaining amount at the stage 3. Then, if we invest the remaining amount only that part is shown in the graph; if you do not invest then, what will happen obviously, then here also we could show all these things. But to simplify matters, here only some of the things are shown.

So, let us see what happens in the stage 3. Now, stage 3 assuming that we have not invested anything that is x 1 equal to 0, x 2 equal to 0; then obviously, we have entire 5 amount remaining after stage 2. So, if we invest the entire amount at the third stage, then we shall have x 3 equal to 5, but if we had x 2 equal to 1; that means, at the third stage we can invest maximum x 3 equal to 4. By similar logic this could be x 3 equal to 3, this could be x 3 equal to 2, this could be x 3 equal to 1, and this could be x 3 equal to 0. So, these are the different options that can be invested in the third option.

Now, you see the third option here. In the third option, depending on what you invest, so let us say x n suppose, if n equal to 3, now if we invest nothing then you get 0; if we invest 1, we get 10; if you invest 5, we get 26. So, those 0, 10, 14, 18, 22 and 26 they are all return in the third option. So, now, you must be clear that how this entire diagram is completed. Now, once the diagram is completed, does this diagram represent or you

know look like some diagram that we already know of. Please think over we have already completed the stagecoach problem.

What was the stagecoach problem? The stagecoach problem is we are moving from a city to another city. But while moving, we had certain costs and what is how we can move from a city to another city such that we do it at the least possible cost right or the insurance premium that you give that is what we have already done. Here the problem is slightly different. Here basically the purpose is what is that path? What should give us the maximum possible return right? That is the difference. So, it is like a stagecoach problem with a small difference that here our purpose is not to compute the cost or the least cost path, but maximum return path.

So, what is that path which gives us the maximum return? You know if you examine obviously that is not how we shall do that is not the dynamic programming algorithm, but in order to know the answer you know please look if I take this path then that means, I spent 5 in the first itself then I get 38. If I take the next path, then I get 42, but here is a path with us that is 24, 14, and 0 which gives us how much 38, but look at this path 17, 11 and 10 again this gives 38. So, there must be some path for example, 17, 11 and 14, this is gives us 42; this one 17, 11 and 18. So, there are different paths which gives no not this one this is 17, 0 and 18, so that is 35. So, anyhow there are different paths which gives us different kinds of values. So, essential idea is basically it is like a stagecoach problem, you have to find that path which gives us the maximum possible return or the NPV value out of the three investments right.

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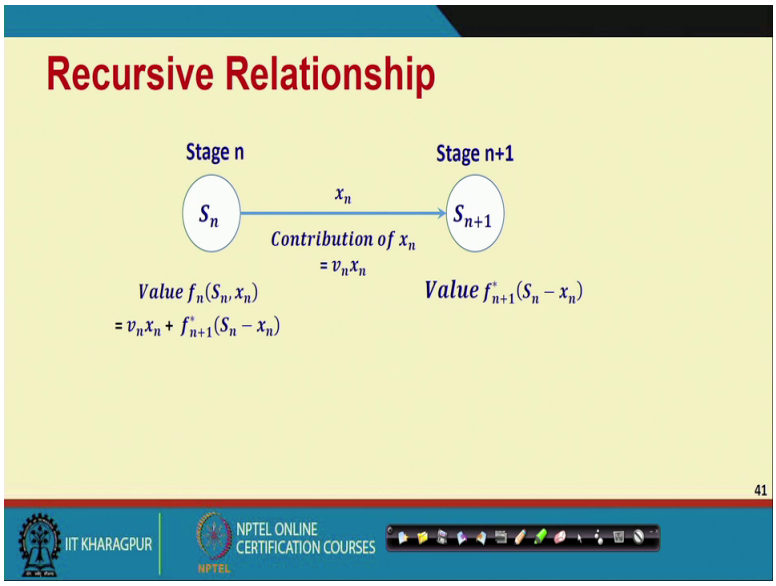

So, now, having known that what is it, let us see that now. The recursive relationship now at this stage now, we have drawn the recursive relationship for such problems in general right in the beginning; but here we are writing it in a very specific manner. So, what has happened? Now at stage n, the value is f n S n x n that is the function. And at stage n plus 1 that is S n minus x n, because x n is the that is the money that is invested that is the decision and its contribution is v n x n. So, if that S n minus x n that is our value that is f star n plus 1, the optimum value at stage n plus 1 is f star n plus 1 S n minus x n and the recursive relationship therefore should be f n S n x n equal to v n x n plus f n star n plus 1 S n minus x n is it all right.

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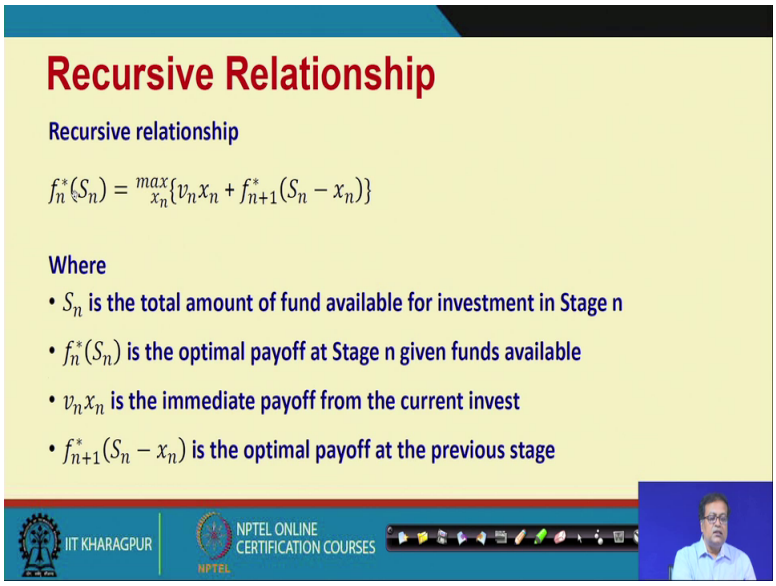

So, let us look at how does it look? It looks like f n S n equal to maximum of so in the previous slide we have computed f n x n S n x n, here we are writing f n star that is the optimum value is the maximum value of these expression for different x n values, where S n is the total amount of fund available for investment in stage n; f n star x n is the optimal payoff at stage n; v n x n is the immediate payoff from the current investment; and f n plus 1 star S n minus x n is optimal payoff at the previous stage right. So, we have to make use of these recursive relationship.

So, what exactly we have done in this particular lecture is for an investment problem, we have identified the states, we have identified the stages, we have identified the decisions, and we have identified the recursive relationship; which includes those states and combineS them from one state one stage to the other.

And in the next lecture, we shall see how we can make use of this recursive relationship and really solve the problem by less number of iterations then brute search or an exhaustive search right.

Thank you very much.