Selected Topics in Decision Modeling
Prof. Indranil Sengupta
Department of Computer Science and Engineering
Indian Institute of Technology, Kharagpur

Lecture – 29
Quadratic Programming

Right so in our course Selected Topics in Decision Modeling, we are in our 29th Quadratic Programming. So in these particular lecture you know we are going to see the how this specific types of problem that is Quadratic Programming problems can actually be solved by a combination of the (Refer Time: 00:41) conditions and the linear programming principles right.

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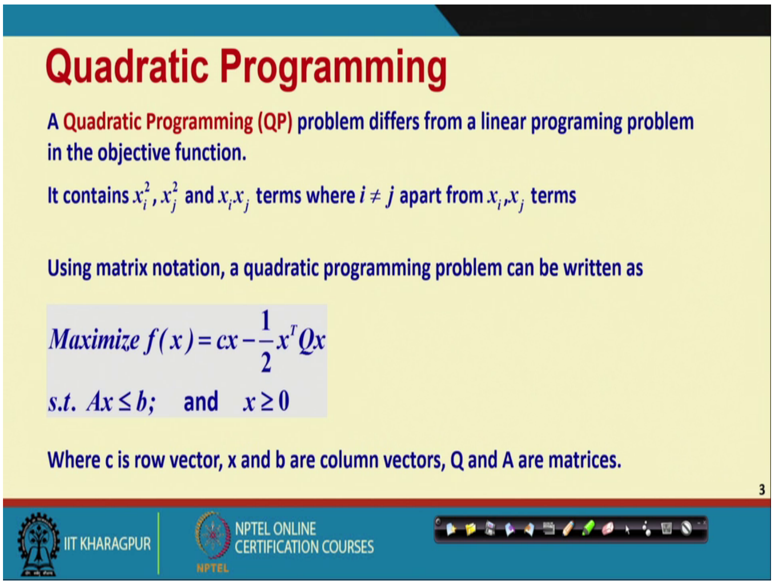

So quadratic programming essentially what is this a kind of problem which is looking very much like a linear programming problem, but with a difference; what is the difference? The difference is the objective function is quadratic in nature the quadratic in nature means that it contains x 1 square x i squarer x j square and x i x j comes where i is not equal to j right.

So the quadratic programming problems essentially therefore, you know in vector form takes these that maximize f x i equal to c x minus half x to the power T Q x this is all right. So, where c is a rho vector and x and b are column vectors so if c is a column vector also then we have to w right c transpose x as well this is all right.

So, Q and A are the matrices right; so essentially this is how the matrix form of the equations, but it will be very clear by taking an example what exactly this form means. So supposing I have a problem a quadratic problem that is maximize f x equal to 15 x 1 plus 30 x 2 minus 2 x 1 square plus 4 x 1 x 2 minus 4 x 2 whole square.

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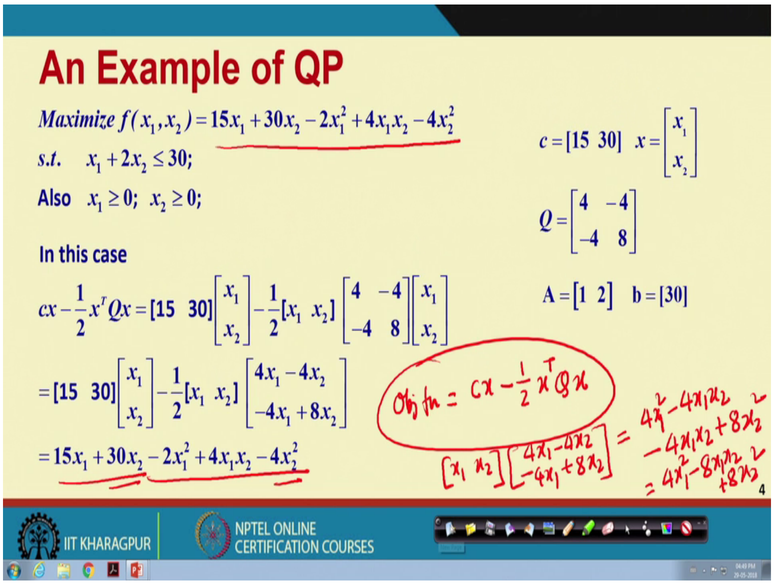

Subject to x 1 plus 2 x 2 less than equal to 30 x 1 greater than equal to 0 and x 2 greater than equal to 0. So this a quadratic programming problem; now why it a quadratic programming problem? Let us understand these; now from the nature of the problem it is clear that c is 15 30 because it was said that c is a row vector. so c is in this way so c is 15 30; x is a column vector x is 2 variables x 1 and x 2 and supposing this is the form of Q and this is an a matrix and this is a b matrix this is alright.

Ah Why A matrix is like this; because it is called row vector because there is only 1 constant if we had more constant then it would have looked more like a matrix is all right because there is only 1 constant so a is A simple form. Now why Q should be like these, so you see the objective function equal to c x minus half of x transpose Q x that is the objective function.

So, this is how you evaluate so c is 15 30 this is x 1 x 2 and half x 1 x 2; so that is the transpose see this is a column vector so we transpose it then write the Q 4 minus 4 minus 4 8 x 1 and x 2. So basically it is a kind of metrics multiplication, so what would be the matrix multiplication it should be 4 x 1 minus 4 x 2 minus 4 x 1 plus 8 x 2; so that would be this form and this would be 15 30 x 1 x 2 so 15 x 1 plus 30 x 2. Now this one now what you can do now see let us do that x 1 x 2 multiplied by 4 x 1 minus 4 x 2 minus 4 x 1 plus 8 x 2.

So, if I multiply then what do I get; first you multiply by x so 4 x 1 minus 4 x 1 x 2 that is what we get and then by x 2 so again minus 4 x 1 x 2 and plus 8 x 2 whole square. So this is what we get 4 x 1 square minus 4 x 1 x 2 minus 4 x 1 x 2 plus 8 x 2 whole square.

If I add these two then we get 4 x 1 square minus 8 x 1 x 2 plus 8 x 2 whole square so this is what we get, but then there is a half so if I take half of these term then we get 2 x 1 square that is minus because there is whole minus so plus 4 x 1 x 2 that minus so this minus plus and then minus of this plus minus of 8 half so 4 so this is the term and then this is our c x term.

So you see that the original function that is 15 x 1 plus 30 x 2 minus 2 x 1 square plus for x 1 x 2 minus 4 x 2 that would be can be written in this form. Now how to really get from here to here see these 15 and 30 is very clear, this 4 minus 4 minus 4 and 8 you can see that this is minus so this is a minus here so these 2 so that minus should be this 2 should be double of that and this 4 should be double of that and this 4 should be divided 4 and 4 here in the minus form all right.

So, it is not really very difficult to really understand that how you can put such kind of things into these formats is it all right so essentially that is how we can really put the quadratic function problems into similar formats right. So we understood that why it is called a quadratic programming problem right; so in simple terms you must understand that such problems are have being an x 1 square terms x 2 square terms and x 1 x 2 terms so and there is an inequality constant also.

Such inequality constant if this problem there is 1 in another problem there could be more than 1. So what exactly we do so if we have to solve them first you see this is a maximization problem this all the constants are less than equal to 0 and the unknowns are greater than equal to 0. So this is in the standard form so these standard form is very important that point has to be remembered that we must have the equation in standard form so that we can apply the what is known as the KKT conditions is it all right.

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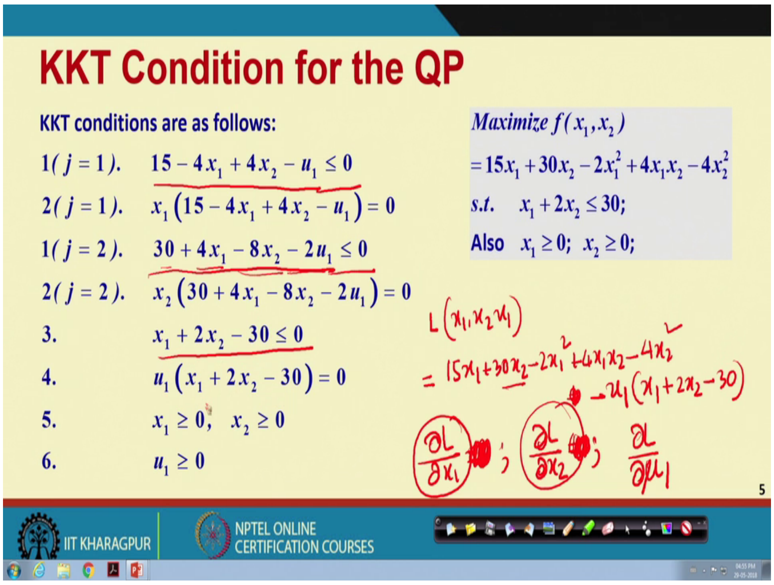

So you remember that to solve NLP problems with inequality constraint we have to first put the problem in the standard form so if it is a problem of minimization we have to make it a maximization problem how; minimize f x is actually maximize minus f x. Supposing these conditions are not less than equal to 30 suppose greater than equal to 30 then you can multiply both sides by minus and you can make it less than equal to 30.

Suppose the variable is not greater than equal to 0 the variable is like in a slightly different form again we can change them by suitably modifying the variables let us say by say suppose x 1 is less than equal to 0 then you can take another variable say y 1 which is equal to minus x 1 so if x 1 is less than equal to 0 then y 1 will be greater than equal to 0 and then replace x 1 by y 1 everywhere. So we have do some transformation to really put the problem in the standard form.

Luckily, fortunately this particular problem is in the standard form. So if we have to take KKT conditions then what we have to do we have to really write the Lagrangian function first; the Lagrangian function for these would be L x 1 x 2 and u 1 so you could have written lambda 1 also no issue; so it will be 15 x 1 plus 30 x 2 minus 2 x 1 whole square plus 4 x 1 x 2 minus 4 x 2 whole square plus sorry minus u 1 into x 1 plus 2 x 2 minus 30.

So, this will be our function so what we have to do we have then obtain del L del x 1 equal to 0 del L del x 2 equal to at least we have to obtain these del L del x naught equal to 0 really we have to obtain del L del x 2 and del L del u 1 so we have to obtain them, because they will be 0 if we have equality constraint in this case there are not equality constraints we cannot equate to 0.

So, let us take the differentiation first with respect to x 1; so what we get 15 then nothing here minus 4 x 1 plus 4 x 2 so this time is over and here minus u 1; so can you see that this will be our del L del x 1. Similarly let us take del L del x 2, so if you take with respect to x 2 then we get 30 x 2 so we get 30 then nothing here plus 4 x 1 x 2 will take differentiation so we we get 4 x 1 so this is here and then here this will be 8 x 2 minus minus 8 x 2 and here minus 2 u 1.

So you see this would be our del L del x 2 right. Lastly we have to take del L del u 1 so what will be del L del u 1 so this term nothing so it will be a x 1 plus 2 x 2 minus 30 actually minus of that so since the original condition basically we shall get right. So that x 1 plus 2 x 2 minus 30 less than equal to 0 that is the 3 terms right. So basically the KKT condition says that these terms these differentiation should be less than equal to 0 additionally x 1 multiplied by these the differentiation equal to 0 x 2 multiplied by these equal to 0 and u 1 multiplied by this del L del u 1 will be eq1ual to 0 right.

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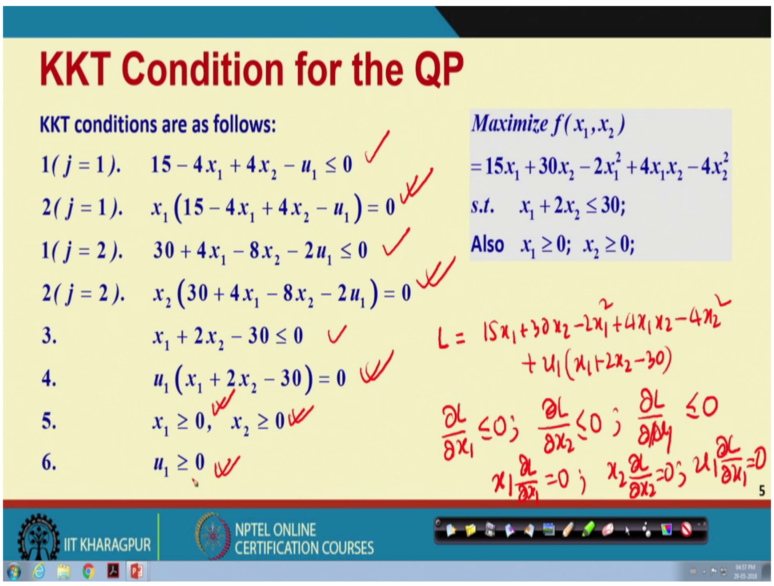

So that is essentially what we do so let us lets repeat what we did; what we did is that first we formulate L L equal to 15 x 1 plus 30 x 2 minus 2 x 1 square plus 4 x 1 x 2 minus 4 x 2 whole square plus u 1 times x 1 plus 2 x 2 minus 30 then I get del L del x 1 equal to less than equal to 0 del L del x 2 less than equal to 0 del L del lambda or del u 1 less than equal to 0 so these are our these constraints right.

And additional constant is x into x 1 into del L del x 1 equal to 0 x 2 into del L del x 2 equal to 0 and u 1 into del L del u 1 equal to 0. So those are these constraints so these are the 6 constraints that we have already seen and other then that all these terms x 1 should be greater than equal to 0, x 2 should be greater than equal to 0 and u 1 should be greater than equal to 0; so these are our KKT conditions right.

So this is gives you a quick method or how to really find the KKT conditions for all such problems basically you combine the equivalent of the Lagrangian function make it difference partial differentiation and those partial difference should be less than equal to 0 and additionally the corresponding variable into the differentiation should be 0 right. So, these are the 6 conditions apart from the non negativity criteria so we get the KKT conditions.

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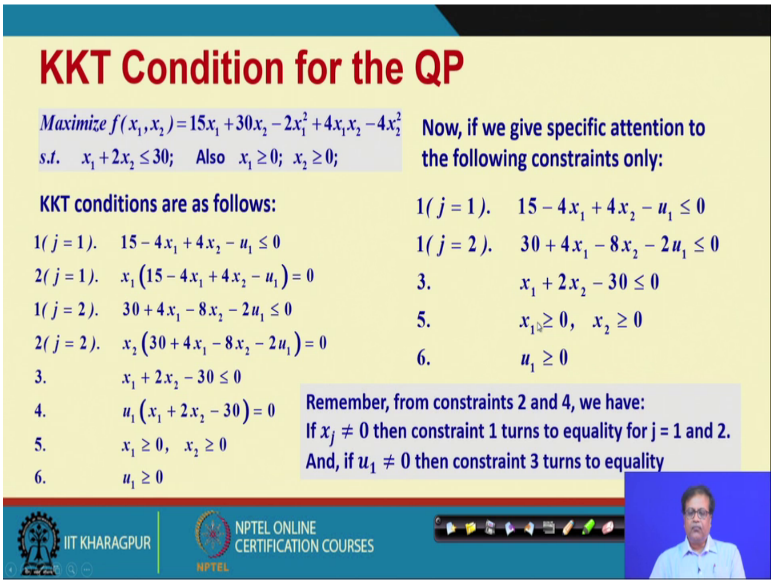

Now once we get the KKT conditions now, see very interesting thing happens. Now out of these conditions let us forget about those equality constraints for the time being right those duel constraints let us forget for the time being.

Let us only look at the less than equal to constraints and let us give only attention to those problems only. So what we get we really get then 15 minus 4 x 1 plus 4 x 2 minus u 1 less than equal to 0 then the second one 30 plus 4 x 1 minus 8 x 2 minus 2 u 1 equal to 0 x 1 plus 2 x 2 minus 30 less than equal to 0 and x 1 x 2 u 1 greater than equal to 0. So if you really look at the constraints 2 and 4 something interesting you get, that if x j is not equal to 0 then the constraint turns to equality.

Suppose if x 1 equal to not 0 then; obviously, this must be 0; that means, the constraint turns to equality is it all right so similarly if u 1 equal to 0 then the fourth condition satisfied, but if u 1 not equal to 0 then this constraint 3 turns to equality. Once again if u 1 not equal to 0, 3 turns to equality if x 1 not equal to 0 1 this first 1 turns to equality if x 2 not equal to 0 then this 1 turns to equality is it all right. So this is the short form so basically then we shall workout with only this 5 conditions is it all right; so let us see exactly what are these 5 conditions and what is their significance.

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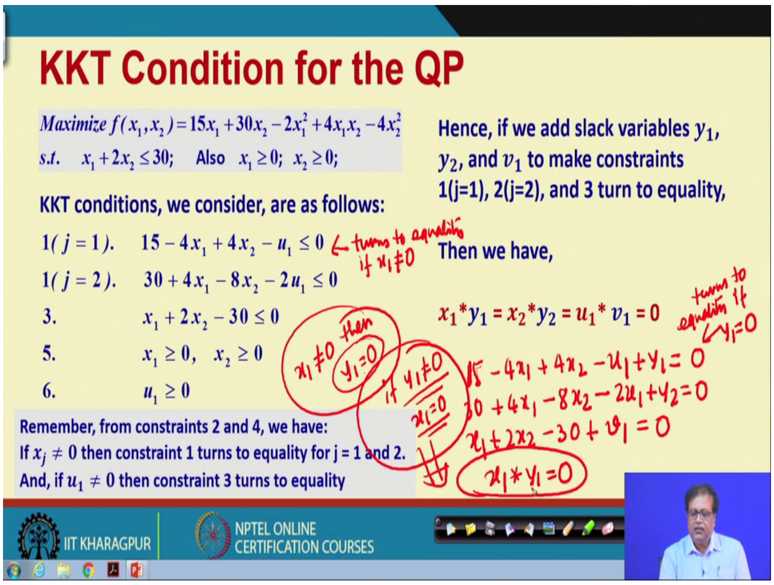

So, we find that the significance will really be like this, that is in this case you know we have this KKT conditions this are the [vocalized minus noise] our most 5 conditions so remember that from constraint 2 and 4 this is the same thing that we have again written if x j not equal to 0 constraint 1 turns to equality for j equal to 1 and two; that means, both this 1 condition turns to equality right and constraint 3 turns to equality if u 1 not equal to 0 is it all right. So these are some of the things that we actually have in this particular situation [vocalized minus noise].

So, what we can do we can now add some slack variables that is y 1 y 2 and v 1 to make those 3 constraints to equality is it all right. So let us see exactly how they look like; so you see we have this constraint so if we add some slack then we have 15 minus 4 x 1 plus 4 x 2 minus u 1 plus y 1 equal to 0 30 plus 4 x 1 minus 8 x 2 minus 2 u 1 plus y 2 equal to 0 x 1 plus 2 x 2 minus 30 plus v 1 equal to 0; so what we have we done here we have added some because they are less than equal to 0 so if we add some quantity that quantity could be our slack variables.

So, in this case we have had 3 slack variables that is y 1 y 2 and v 1 so if you add those slack variables then we shall have is equal to 0, but something very interesting that look here that these constraint turns to equality if x 1 not equal to 0 again from here turns to equality if y 1 equal to 0; so look at carefully see if y 1 equal to 0; this is turns to equality and if x 1 not equal to 0 then it turns to equality is it all right so if x 1 not equal to 0 then y 1 equal to 0 can you say this because in both the cases if y 1 equal to 0 this turns to equality if x 1 not equal to 0 then it turns to equality; so therefore, if x 1 not equal to 0 then y 1 should be 0 is it all right.

But if you look at the other way round if y 1 not equal to 0 if y 1 not equal to 0 then this equation has not turn to equality; that means, this is valid then what will happen about x 1 because you see we had a constraint you know where this part multiplied by x 1 equal to 0. So if this is not 0 then x 1 must be 0 so; that means, if y 1 not equal to 0 then x 1 must be 0 is it all right so this gives an interesting thing that you know this fact and this fact when you combine then we get x 1 star y 1 should be equal to 0 is it all right.

So, once again let us repeat exactly what we said that look here this 15 minus 4 x 1 plus 4 x 2 minus u 1 is less than equal to 0 is it all right. If I add a slack variable y 1 then this term becomes 0. Now this will turn to equality if y 1 equal to 0 and this will turn to equality if x 1 not equal to 0 that we already seen so; that means, if x 1 not equal to 0 then y 1 equal to 0, but in if y 1 not equal to 0 then this does not turn to equality, but then x 1 multiplied by this portion is 0 that was a constraint number 2; so from these 2 facts you know if we combine that is x 1 not equal to 0 then y 1 equal to 0 and if y 1 not equal to 0 then x 1 equal to 0 we get x 1 star y 1 equal to 0.

And similar logic can be given between x 2 and y 2 from the second equation and u 1 and v 1 for the third equation; so that is x 1 star y 1 equal to x 2 star y 2 equal to u 1 star v 1 equal to 0 is it all right. So this is exactly what we get.

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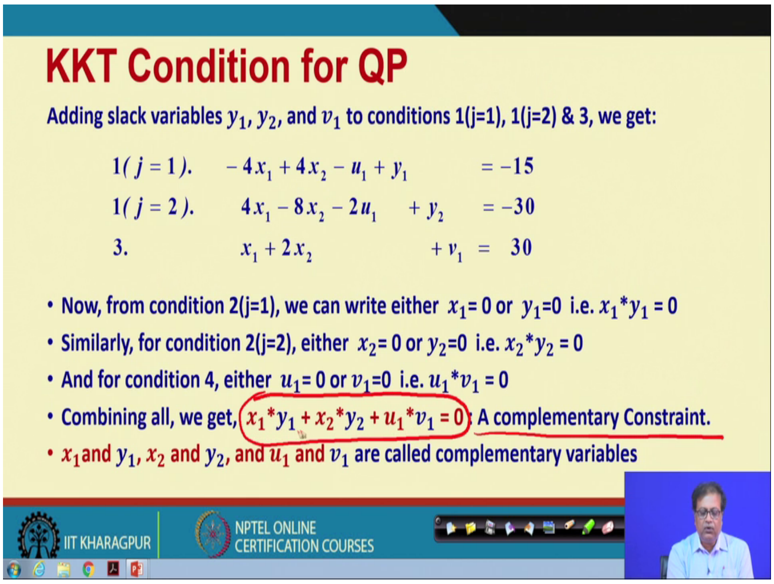

And let us see how it combines; so the KKT conditions for quadratic programming by adding slack variables y 1 y 2 and v 1 the equations then become you know what we did is we have taken these 15 on the other side and then minus 4 x 1 plus 4 x 2 minus u 1 plus y 1 that y 1 is a slack variable equal to minus 15 and 4 x 1 minus 8 x 2 minus 2 u 1 plus y 2 equal to minus 30 and x 1 plus 2 x 2 plus new 1 equal to 30.

So, now from the conditions 2 we can write x 1 equal to 0 or y 1 equal to 0 already we have seen the logic; so x 1 star y 1 equal to 0. Similarly for condition to either x 2 equal to 0 or y 2 equal to 0 so x 2 star y 2 equal to 0 and for condition 4 either u 1 equal to 0 or v 1 equal to 0, so u 1 star v 1 equal to 0. We can very interestingly combine them all and you know we can get what is known as a complementary constant constraint in this way that is x 1 star y 1 plus x 2 star y 2 plus u 1 star v 1 equal to 0.

So, these can be called as complementary constraint right why it is happening look here that either x 1 equal to 0 or y 1 equal to 0 either x 2 equal to 0 or y 2 equal to 0 either u 1 equal to 0 or v 1 equal to 0 so either of them will be 0 so; that means, if you know that whole total of these has to be 0 right. So you know that this particular constant with then ensure that you know this would actually happen so x 1 and y 1, x 2 and y 2 and u 1 and v 1 are called complementary variables is it alright.

So, precisely that is how we can really create a simplex table out of this situation let us see how; so we can again slightly reformulate this that you know this is this are your original equations; so was just remember that this first two we had negative value we know for simplex algorithm you must have positive values everywhere so you can again make minus equal to plus and then we can re write them as 4 x 1 minus 4 x 2 plus u 1 minus y 1 equal to 15 4 x 1 minus 4 x 1 plus 8 x 2 plus 2 u 1 minus y 2 equal to 30 and rest of the constraints as they are apart and adding the complementary condition.

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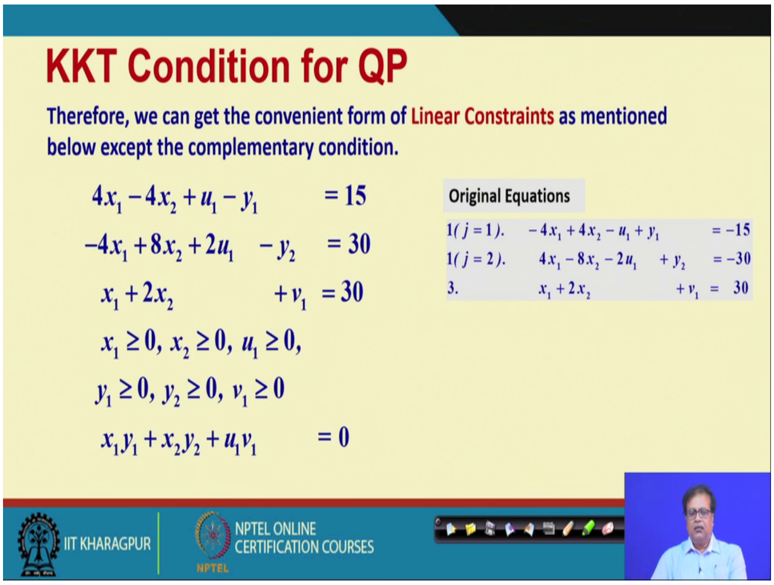

So, this is how we can actually get a set of linear constraints that we have to solve right. Now really how to solve them is we let us take two you know artificial variables z 1 and z 2 really to get what is known as the canonical form so this is how an lp can be constructed that is minimize z equal to z 1 plus z 2 subject to 4 x 1 minus 4 x 2 plus u 1 minus y 1 plus z 1 equal to 15 and these plus z 2 equal to 30 and this will be therefore, 30 and all these variables along with this complementary condition.

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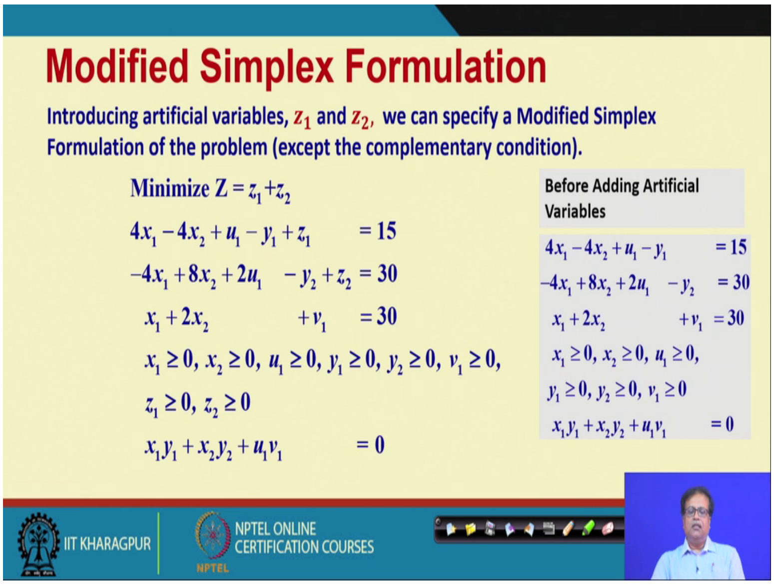

So, this is nothing but a linear programming problem and can be solved as an usual minimization problem; so moment you can throughout the artificial variables from the bases the solution can be obtained right. So solution of this is really not in our interest because we know how to solve linear programming problems the essential idea is that how the quadratic problems with the help of KKT conditions can actually be translated into a set of linear programming problems because linear programming problems are easy to solve; so quadratic programming problems can actually be having a modified simplex formulation and that can be easily solved right.

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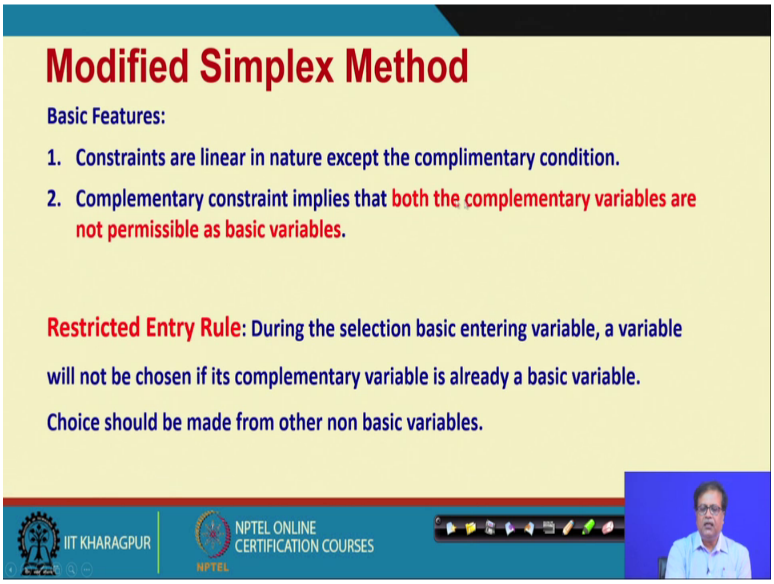

This particular problem is solved in the Hillier and Lieberman book; so what is really happening we now have linear constants along with those complementary variables so the idea is both cannot be coming to basic variable so we have a restricted entry rule during the selection of basic entering variable and variable will be not be chosen if it is complementary variable is already a basic variable. So, I should be made from other non basic variables.

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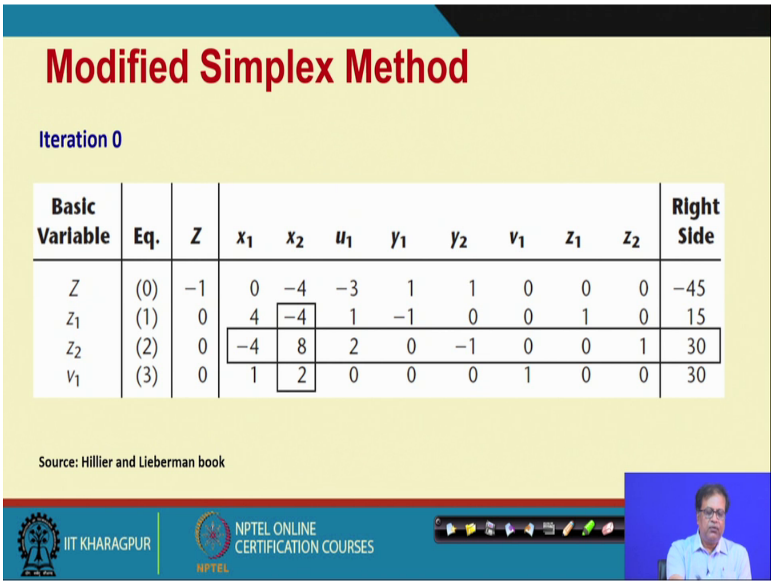

So we have from Hillier Lieberman book this is how we have the modified simplex method formulation and usual simplex rules can be used we are not going deep into it, but you know if you form really follow the basic simplex procedure so this is stable 0 then table 1 then we have table 2.

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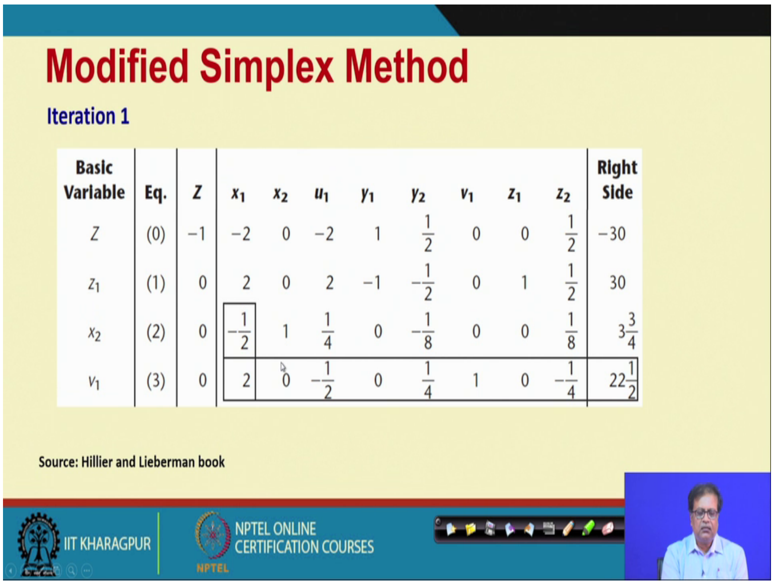

Basic Variable	Eq.	Z	x_1	x_2	u_1	y_1	y_2	v_1	z_1	z_2	Right Side
Z	(0)	-1	-2	0	-2	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	-30
z_1	(1)	0	2	0	2	-1	$-\frac{1}{2}$	0	1	$\frac{1}{2}$	30
x_2	(2)	0	$-\frac{1}{2}$	1	$\frac{1}{4}$	0	$-\frac{1}{8}$	0	0	$\frac{1}{8}$	$3\frac{3}{4}$
v_1	(3)	0	2	0	$-\frac{1}{2}$	0	$\frac{1}{4}$	1	0	$-\frac{1}{4}$	$22\frac{1}{2}$

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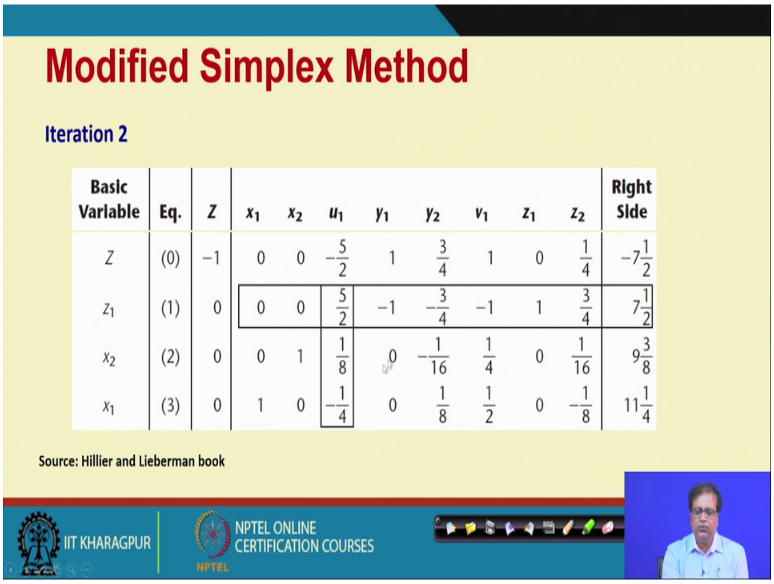

Basic Variable	Eq.	Z	x_1	x_2	u_1	y_1	y_2	v_1	z_1	z_2	Right Side
Z	(0)	-1	0	0	$-\frac{5}{2}$	1	$\frac{3}{4}$	1	0	$\frac{1}{4}$	$-7\frac{1}{2}$
z_1	(1)	0	0	0	$\frac{5}{2}$	-1	$-\frac{3}{4}$	-1	1	$\frac{3}{4}$	$7\frac{1}{2}$
x_2	(2)	0	0	1	$\frac{1}{8}$	0	$-\frac{1}{16}$	$\frac{1}{4}$	0	$\frac{1}{16}$	$9\frac{3}{8}$
x_1	(3)	0	1	0	$-\frac{1}{4}$	0	$\frac{1}{8}$	$\frac{1}{2}$	0	$-\frac{1}{8}$	$11\frac{1}{4}$

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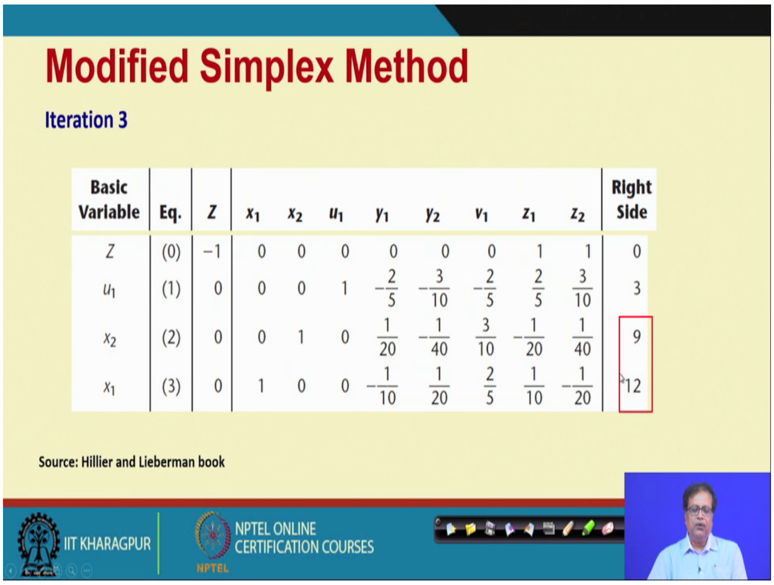

And at the 3rd table the solution is found the artificial variables are thrown out and we have x 1 equal to 12 and x 2 equal to 9 as the optimal solution right. So we stop here and in our next lecture we shall see some real examples of Lagrange's Lagrangian multipliers and KKT conditions right so

Thank you very much.