Selected Topics in Decision Modeling
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Lecture - 27
Constrained NLP: KKT Conditions

So, good morning in our course Selected Topics in Decision Modeling today we are in our 27th lecture that is Constrained Non-linear Programming: KKT Conditions that is Karush-Kuhn-Tucker conditions.

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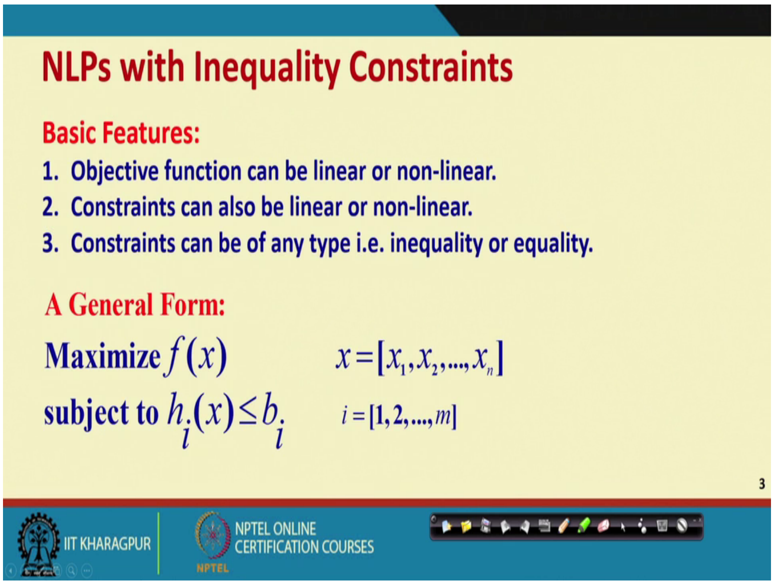

So, we have seen that, how to solve non-linear programming problem with equality constraints in our last class. There we had use Lagrange's multipliers and differentiated the function, to find out conditions for optimality. Whereas, for inequality constraints Lagrange's multiplier cannot be utilized, we need to go for a little more involved calculations with the help of KKT conditions.

So, some of the basic features the objective function can be linear or non-linear, constraints can also be linear or non-linear and constraints can be of any type that is inequality or equality. If you recall I have told in our last class that the Lagrange's multiplier is a special case of KKT conditions.

So, at some other time I will also show you that how it is really happening. So, you see any linear programming problem can be put in maximization general form. If it is not happening then we must ensure how to do it is alright. So, the first of all let me just tell its right in the beginning that all the conditions and all the discussions that we are going to have, we shall you know assume the programming problem the non-linear programming problem is available in the general form, that is maximized f which is a function of x really speaking x is having several variables x 1, x 2 to x n.

And subject to conditions h i x less than equal to b i, you know where i is 1 2 m is it alright. So, that is the general form that we are going to use for all the KKT conditions.

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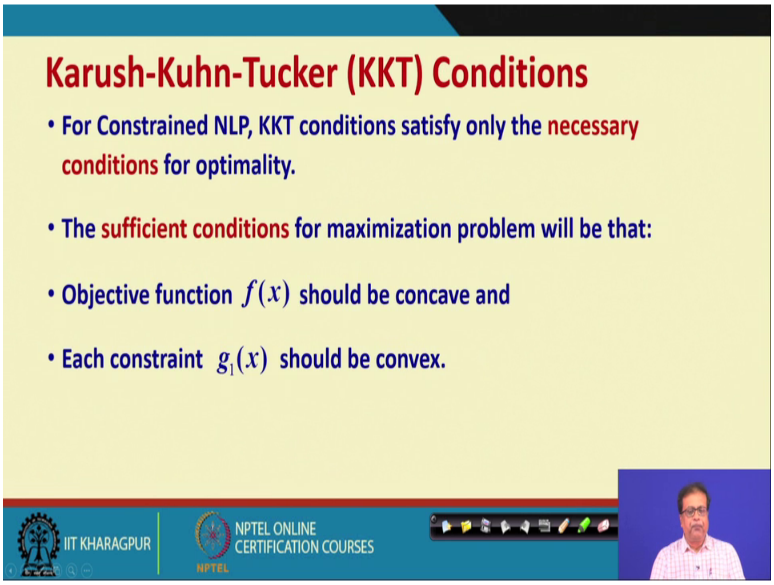

Now, Karush-Kuhn-Tucker conditions or in short KKT conditions, where you know obtained by Kuhn-Tucker, but then Karush independently obtain them actually earlier right in his thesis work. So, that is the reason why these conditions and known as Karush-Kuhn-Tucker conditions. But just remember this that these conditions they are not the only conditions required for optimality, infant there only the necessary conditions right.

So, KKT conditions are the necessary conditions for the NLP problem, the sufficiency condition is still you know the for the maximization problem the fx should be concave and each constrain should be convex is it alright. So, you just recall our discussions that for maximization, but you require the objective function should be concave and the constraints should be convex.

Now, sometimes we have linear constraints, the linear constraints are both convex and concave at the same time. So, if I have linear objective function, then automatically we can assume the objective function to be concave or if we have constraints some of the constraints a linear, than those linear constraints can be also assume to be convex. Because linear constraints or objective functions at are both concave and convex at the same time.

But if it is non-linear, then the concavity has to be seen, either by the nature of the plot or by obtaining the second differential that is you know in when multiple variables are involved, then obtaining the Hessian matrix. So, we will see that in due course of time right.

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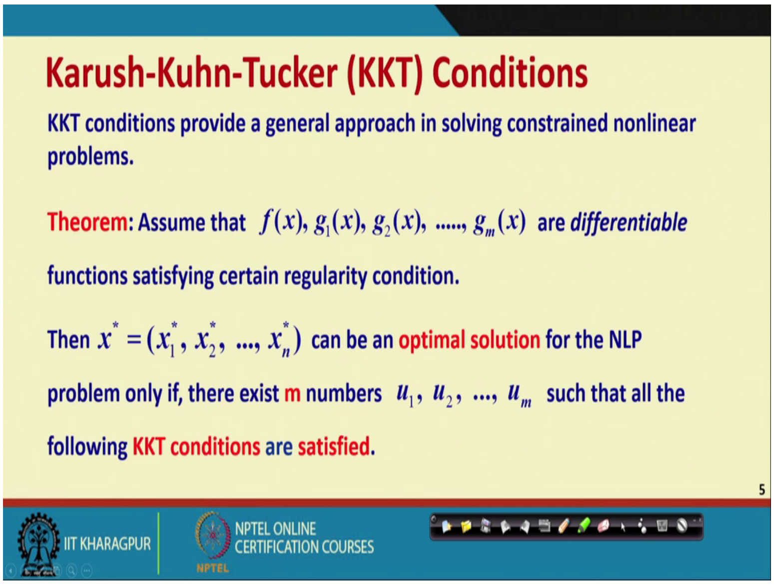

So, let us look at the KKT conditions. So, first of all when we are solving in a general approach to constrained non-linear problems, let us assume that we have certain functions that is f x gi x g 1 x g 2 x to g m x they are all differentiable functions and having certain regularity conditions as well. Then the x star which is a combination of n unknown variables or decision variables, can be an optimal solution for the NLP problem, only if they are exist m number of multipliers u 1 to u m such that a number of KKT conditions are satisfied.

So, what are the things that we said? Just imagine you know just remember that the our original problem was maximizing f x subject to number of conditions those g i x less than equal to b i. So, they all should be differentiable that is first requirement and you know they are exist certain multipliers that is u 1 to u m, you know along with them we will have those KKT conditions which must be satisfied for x star to become optimal right. So, these are the certain preconditions now let us look at the KKT conditions.

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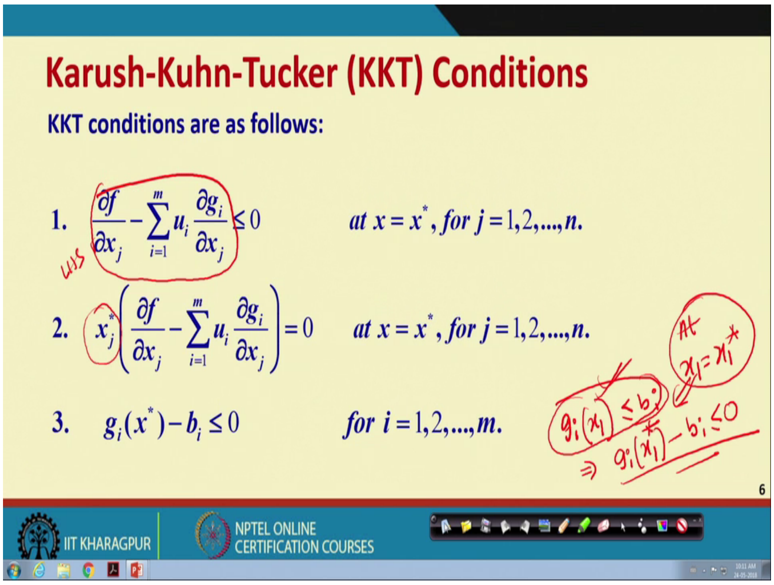

So, the first condition as you can see is del f del x j minus this sum over all the u is i mean each this sum is taken over each of the constraints u i del j i del x j less than equal to 0 that the first condition valid at the optimal point and for each of the x variables.

So, this is more like the differential condition, but the condition you see we have you know the objective function and we also have a series of constraints. So, if I multiply those constraints by our multiplier, that is the u i s. So, each safest constraint multiply by u 1, second multiply by u 2, third by u 3 etcetera etcetera and then we differentiate you know by taking appropriately that is very important the negative sign.

These negative sign of combination usually is used for maximization problem sometimes plus is used for minimization, but we can generalize with negative also we will have some discussion later. The second condition is just the first part whatever the first condition left hand side you see the same portions. So, if you see the first part, see this is the LHS of the first condition.

So, this is the LHS of the first condition. So, that multiplied by x j star is it alright. So, the LHS whatever the differential part you get for the first condition, if you multiplied by x j star that should be 0 that is the second condition. That means, either these first constraint is equality or x j equal to 0, x j star equal to 0. So, what that second condition says is that, either the first condition is an equality or the corresponding x variable that is x j star should be equal to 0 is it alright that is the second condition says. The third condition is again very straight forward because we see that you know g i x is less than equal to b i that is our specific constraint. So, it is the same constraint written in a another form. So, really the third constraint is nothing new, because the original constraint we have is g i say x 1 less than equal to b i.

So, the same constraint, we have written in this form. So, this is the constraint itself and at x 1 equal to x 1 star at x 1 equal to x 1 star these should be star. So, see this was our original constraint. So, the same original constraint has been rewritten in these form. So, that is our third constraint. So, you can understand that although they look you know rather big, but really speaking now the constraints are very simple in fact, the first one is nothing, but the differentiation, the differentiation should be less than equal to 0.

The second condition is either the LHS of the first condition that becomes 0 that means, del f del x are j should be equal to these term or the corresponding x variables should be 0 and the third one is nothing, but the constraints the corresponding constraints themselves right.

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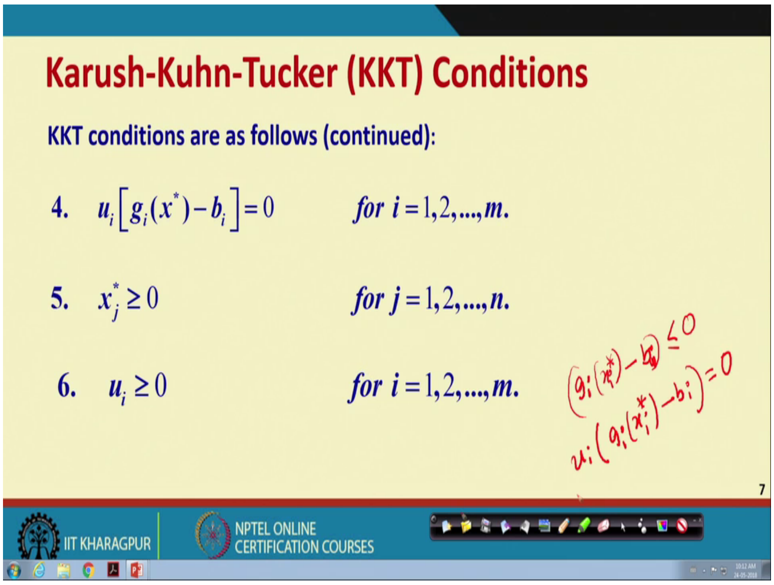

So, we have seen those first 3 conditions let us see the last few conditions. So, see like we said in the case of the constraints, the differentiation. So, our original constraints has been rewritten as gi x star minus b less than equal to 0 so, b i. So, that is that is has been written. So, again so, multiply by the corresponding multiplier multiply by the corresponding multiplier should be equal to 0. So, that is the fourth condition.

So, once again and fifth and sixth condition should be the all the multiplier should be greater than equal to 0, and all the decision variable at the optimal point should be also greater than equal to 0. So, these are our six KKT conditions. So, just look at them once again you know the if you if you if you really see, then the yeah.

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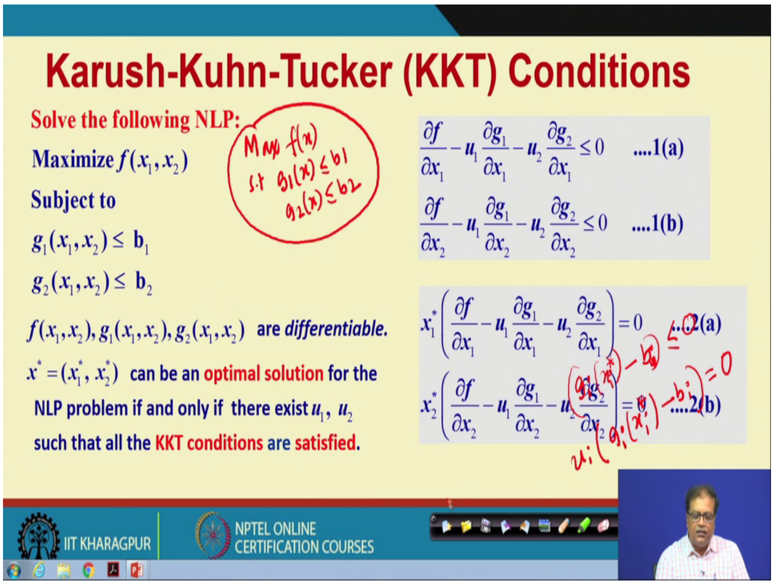

So, this is these are the KKT conditions, the first condition is the differential partial differentiation with respect to each variable. So, we had n unknown variable each variable should be taken one at a time. The first part is the objective function part, the second part is the individual constraints multiplied by the multiplier and then taking the negative of the you know the partial differentiation. The second constraint is the LHS of the first part should be either 0 or the corresponding decision variables should be 0.

The third constraint is the original you know the constraint itself written in another form, the fourth constraint is either the you know it becomes equality. The constraint become equality or the corresponding Lagrange the multiplier become equality and the fifth and sixth conditions are basically you know non-negativity constraints both for the decision variables and for the multipliers is it alright.

So, this is how we begin with the KKT conditions, these are the six KKT conditions. So, for improving our understanding let us look at a particular problem very simple one, simplified version of the KKT conditions once again. So, let us say we have a 2 variable problem maximize f x 1 x to. So, a function which is having f x 1 x 2 and subject to 2 constraints only that is g 1 x 1 x 2, in short we can also write these thing as this that maximize f x subject to g 1 x less than equal to b 1 and g 2 x less than equal to b 2. So, that is all is our problem very simple problem we take.

Now, the condition that is required is all the 3 functions that is f x, g 1 x and g 2 x should be differentiable that is the requirement. Now x star equal to x 1 star x 2 star can be an optimal solution for the NLP problem, if and only if they are exist u 1, u 2 2 multipliers such that they are actual equivalent to Lagrange's multipliers. So, such that all the KKT conditions are satisfied is it alright. So, this is the requirement.

So, let us see once again what exactly is you know this conditions are essentially. So, first condition you see suppose I write a function a let us say total you know the function the combination function.

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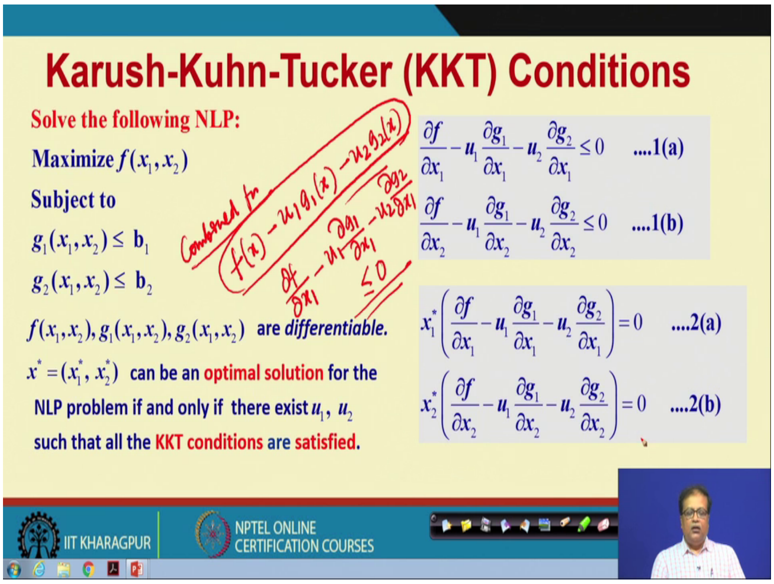

So, let us call it combined function. The combined is f x minus u 1 g 1 x minus u 2 g 2 x.

So, what is this combine function? This combined function is obtained by combining the function f x along with the 2 constraints g 1 x less than equal to b 1 and g 2 x less than equal to b 2. So, if I combined them then what you get is, f x minus u 1 g 1 x minus u 2 g 2 x. So, now you see the first constraint is nothing, but the differentiation of this function with respect to x 1 and with respect to x 2. So, please differentiate these with respect to x 1 what do you get? Del f, del x 1 minus u 1 del g 1 del x 1 minus u 2 del g 2 del x 1.

So, precisely that is what is written here g 1 is nothing, but g 1 x g 2 is nothing, but g 2 x. So, that should be less than equal to 0 that is the first condition. So, similarly the second condition is again the differentiation of the combined function, that is del f del x 2 minus u 1 del g 1 del x 2 minus u 2 del g 2 del x 2 less than equal to 0.

So, that is the second condition of the first condition second part the second condition again will be the either you know these partial differentiation of the combined function should be equal to 0 or the corresponding optimal variable that the at the optimal point. So, either the equation becomes equality or the variable corresponding variable is 0, right.

So, either these equal to 0 right this term is 0 or the corresponding decision variable equal to 0 same thing holds for x 2 also. So, I hope the first two conditions which are the most important KKT conditions you have understood clearly is it alright. So, if you have understood these two conditions clearly, let us see the remaining conditions the remaining conditions are again.

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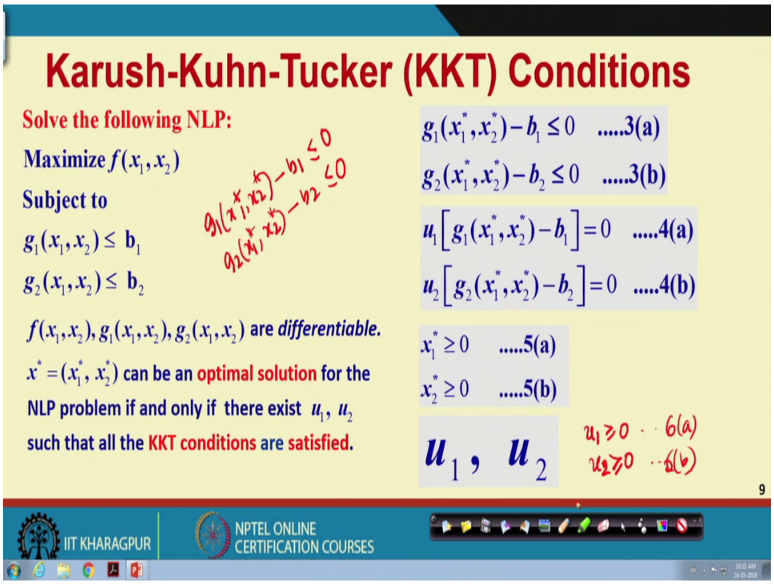

For the same problem you know how the third condition is nothing, but the constraint itself. The constraint itself that is g 1 x 1 x 2 or we can write g 1 x less than equal to 0 and g 2 x 1 x 2 minus b 2 less than equal to 0 at the optimal point.

So, the star we can write star sometimes, but to you know make simply see for simplification, we may not write star in between calculations, but we should know that we are calculating the variable at the optimal point because all these KKT conditions are true not always, but at the point of optimality is it alright. So, that is the third condition and the fourth condition is either these constraints become equality or the corresponding multiplier becomes 0 is it alright. So, that is what is essentially they say they the these are the fourth conditions.

The fifth conditions are the at the optimal point the variables should be greater than equal to 0 the sixth condition has not come properly. So, that should means u 1 should be greater than equal to 0 and u 2 should be greater than equal to 0 right.

So, these are the sixth conditions. So, this is how you can see that how the at the optimal point, the necessary conditions are that the KKT conditions must hold right. So, you understood so, once again that we are having non-linear programming problem with you know inequality constraints, which also includes the equality constraints also. In fact, equality constraints are a special case in case of equality constraints really see what happens.

These automatically it becomes 0. So, if these becomes 0 so, it is not really necessary that u 1 and u 2 to become 0 also and the first constraints then become trivial is it alright. So, we will discuss it some other time now let us look at a particular problem and see how optimal solution can be obtained for a particular problem by making use of the KKT conditions right.

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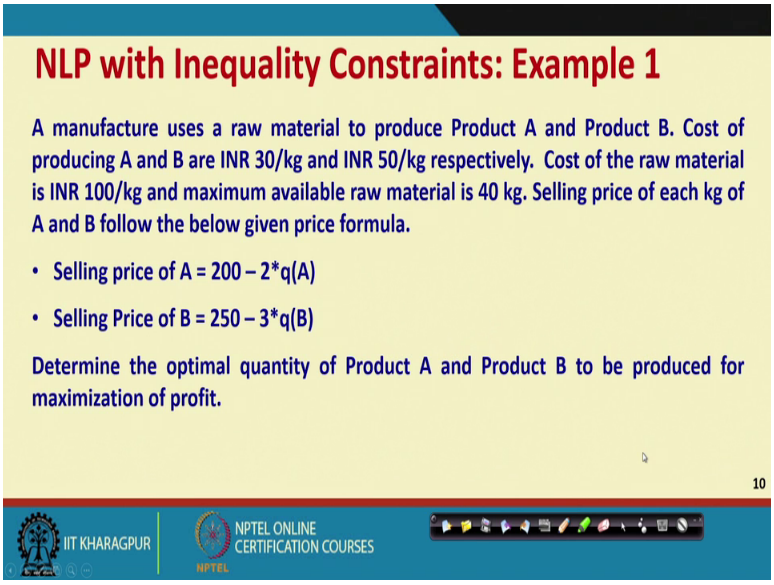

So, let us take these problem; manufacturer uses a raw material produced product A and product B right. Now for producing this product A and product B first of all there is a cost, the cost is 30 rupees per kg for A and 50 rupees per kg for B. Now the cost of raw material is rupees 100 per kg and maximum available raw material is 40 kg right.

The selling price of each kg of A and B follow the given price formula what is the price formula. The selling price of A is 200 minus 2 into quantity of A and selling price of B is 250 minus 3 into quantity of B. So, determine the optimal quantity of product A and product B to be produced for maximization of profit right.

So, these looks like a simple optimization problem, but there is a catch the selling price is not linear. I mean its linear now, but when you convert it to objective function then it will become non-linear right. So, let us see how to formulate the problem first.

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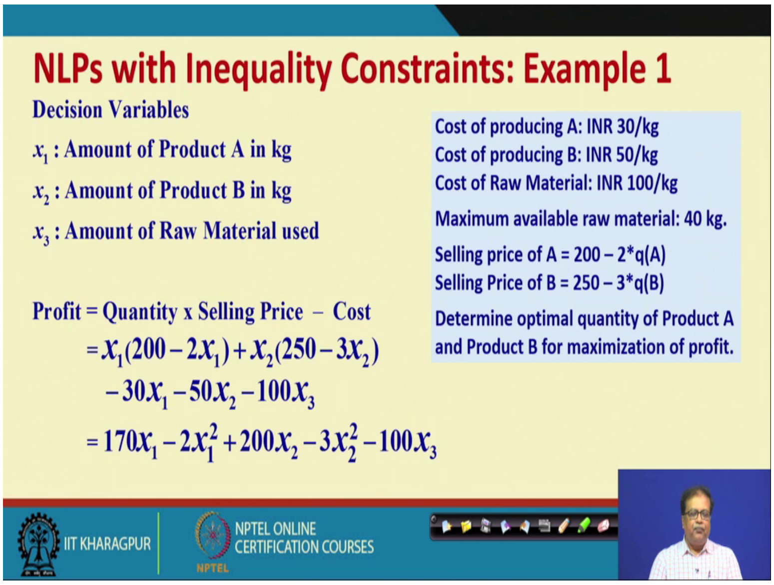

So, these are the decision variable, suppose x 1 is the amount of product A to be produced in kg, x 2 is the amount of product B to be produced in kg and x 3 is the amount of raw material used is alright. So, what is the profit equation? Profit will be quantity into selling price minus cost is alright. So, what is the quantity? Quantity is x 1 and x 2 what is the selling price? 200 minus 2 into quantity, that is 200 minus 2 x 1 and here 250 minus 3 x 2 right.

So, that will be the quantity into selling price and what is the cost? Cost is 30 50 and 100. So, 30 x 1, 50 x 2, 100 x 3 is it alright. So, that is the thing that the therefore, you can simplify. So, you see this is 200 x 1 minus 30 x 1. So, 170 x 1, 250 x 2 minus 50 x 2 so 200 x 2 and minus 2 x square minus 3 x 2 square minus 100 x 3. So, this is our objective function right. So, first part is done. So, we have obtained the profit, but there are some constraints also right

So, what is the constraint? Constraint is very clear that the 2 product x 1 and x 2 should not be the sum of these should not be more than the raw material available because that is a constraint, you cannot produce without the raw material. So, let us see what is that constraints here. So, let us see that.

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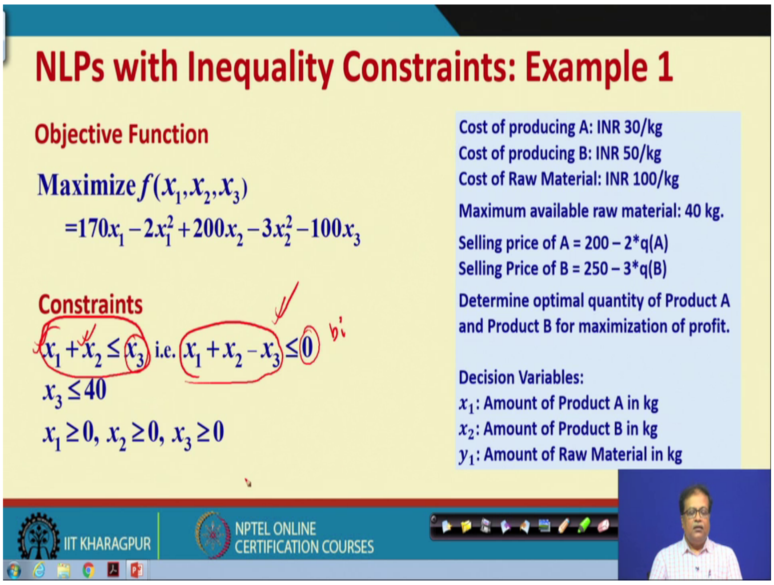

See the constraint therefore, will be x 1 plus x 2 less than equal to x x 3 why because this is the first product this is the second product. So, if you add them that should be less than the raw material that is the quantity x 3. So, these kind be reformulated as x 1 plus x 2 minus x 3 less than equal to 0.

So, see this side is the variables and this side because x 3 is not b i. So, b I has to be a constraint right cannot be a variable. So, since here we have a variable called x 3 so; obviously, should be put as x 1 plus x 2 minus x 3 less than equal to 0. So, these 0 is like bi. So, this is our b i, these are the variables right.

So, that is what we have that the first constraint. Now, the second constraint is also there that is x 3 should be less than 40, because that is the total amount if you look at the problem that is the total amount that is available and obviously, since it is a real problem x 1, x 2, x 3 all should be greater than equal to 0.

So, that is our starting point we have an objective function that is f x 1, x 2, x 3 and maximize 170 x 1 minus 2 x 1 square plus 200 x 2 minus 3 x 2 square minus 100 x 3 and these are our constraints right. So, the first question is how do we go ahead solving the problem. But before even that we should see that if we take the KKT conditions now, the KKT conditions will really provide the necessary conditions. The sufficient condition should be we have to ensure that the function is concave.

Now, incidentally we these function is concave, but then can we take the hessian matrix and find out whether the matrix is really concave I mean this objective function is really concave no problem about constraints because constraints are all linear.

So, all of them are linear constraints. So, with linear constraints automatically they are convex so, no issue there.

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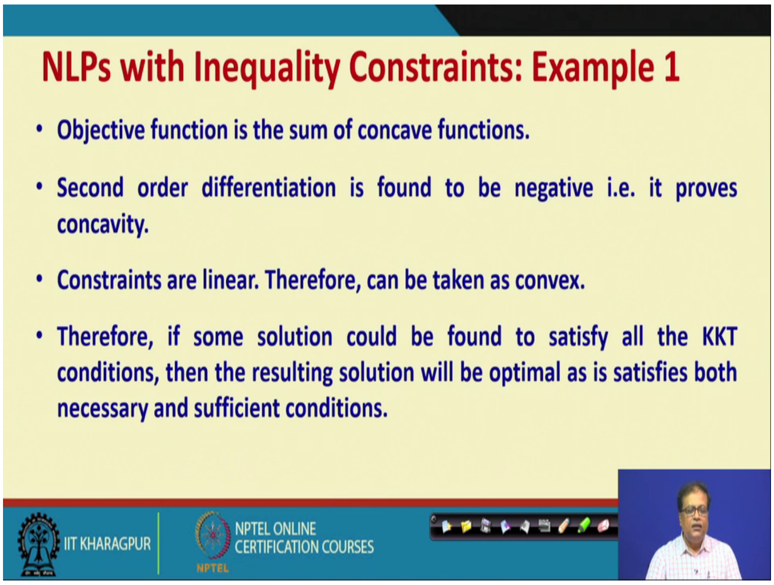

So, let us see how we can see that these are concave. So, objective function is the sum of concave functions. So, if they are automatically the second order differentiation would be negative, that is it proves concavity. Constraints are linear so, therefore, they can be taken as convex.

Therefore, if some solution could be found to satisfy all the KKT conditions in the resulting solution will be optimal, as it satisfies both necessary and sufficient conditions.

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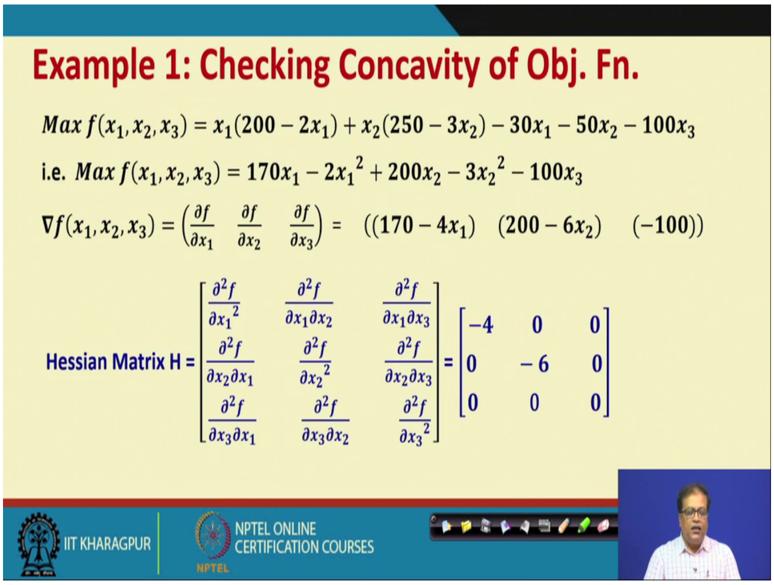

So, let us look at the Hessian matrix. So, you see this is our function right that is the total function this is the simplified the function, 170 x 1 minus 2 x square plus 200 x 2, minus 3 x 2 square minus 100 x 3. So, the gradient will be del f del x 1 del f del x 2 del f del x three. So, what should be? With respect to x 1 if you differentiate you get 170 minus 4 x 1. With respect to x 2 200 minus 6 x 2, and with respect to x 3 minus 100. So, this is our gradient. So, we got the gradient.

Next will be the hessian matrix. Now, you see since they are pure terms of x 1 x 2. So, all these terms you know they will be 0. So, all of them are 0. So, with respect to these del square f del x 1 square, that becomes minus 4. Del x d square f del x 2 square that becomes minus 6 and since there is no term this will become 0. So, this is our Hessian matrix right.

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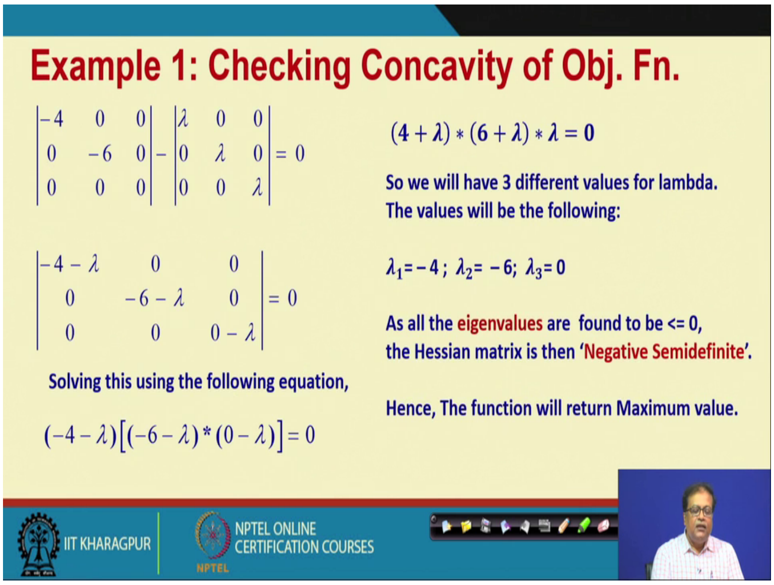

So, we obtain the Hessian matrix now, we take the eigenvalue by making use of characteristic equation and this is eigenvalue equation and very simple this one, it comes to you know these kind of equations that is 4 plus lambda plus star 6 plus lambda star lambda equal to 0.

So, 3 different values of lambda could be minus 4 minus 6 and 0. So, all the eigenvalues are less than equal to 0. So, what is our conclusion? The conclusion is the Hessian matrix is negative semidefinite because one of the term is 0. So, automatically since the eigenvalues are negative only one is 0. So, we can say the function is concave and we can have maximum value out of these is alright.

So, I will stop here in our next lecture we continue the discussion on KKT conditions specifically, the problem that we are doing we shall complete the one.