Selected Topics in Decision Modeling
Prof. Biswajit Mahanty
Department of Industrial and Systems Engineering
Indian Institute of Technology, Kharagpur

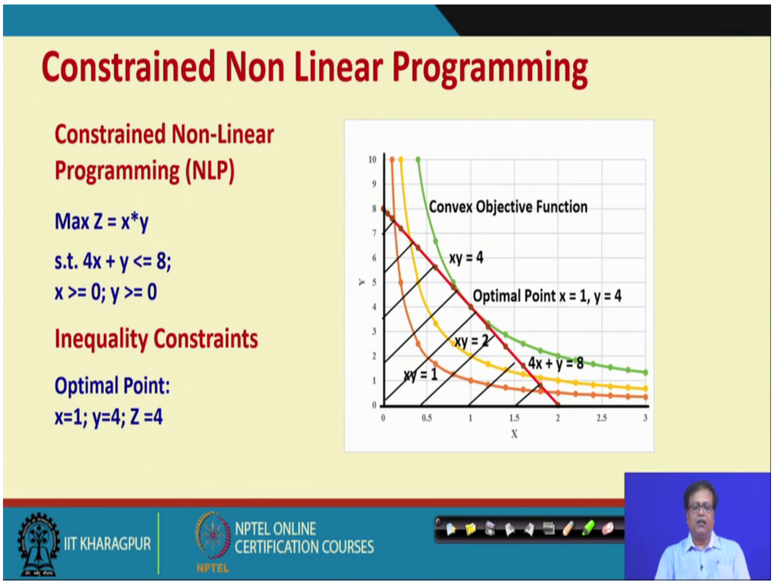
Lecture - 26
Constrained NLP: Lagrange Multipliers

So, in our course Selected Topics in Decision Modeling, now we are in our 26 lecture that is Constrained non-linear programming Lagrange's multipliers right.

(Refer Slide Time: 00:30)

So, in the last few classes lectures we have seen how to handle non-linear programming, specifically when we have unconstrained problems.

(Refer Slide Time: 00:41)



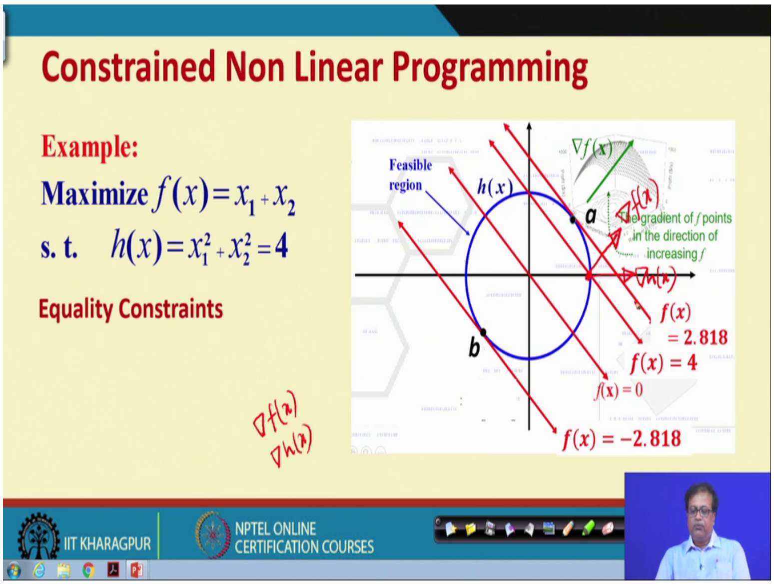
See the basic difference between unconstrained and the constrained problems are that, in unconstrained problems we have a function and we have to find the extrema that is the maxima, minima values, but then there are no constraints. The constraints can be equality constraints or inequality constraints.

So, here is the example of an inequality constraint problem as you can see these example we have seen earlier that is supposing we have to maximize Z equal to x y, subject to 4 x plus y less than equal to 8 and x and y both greater than equal to 0. So, what we basically do here, we have the axis line x equal to that is this one is x equal to 0 and this is y equal to 0 and we also plot you know the 4 x plus y equal to 8.

So, you know these between these 3 constrains lines these you know the shaded area that is our solutions space, is it alright. Now, x y on the other hand that is maximize z equal to x y so, you know as x y increases you know with every different curve here x y equal to 1, x y equal to 2, x y equal to 4.

So, what happens these line x y equal to 4 you know just touches the optimal point these constant line at the optimal point is it all right. So, there you see the optimal point is x equal to 1 and y equal to 4. So, this is only a graphical explanation of the thing.

(Refer Slide Time: 02:43)

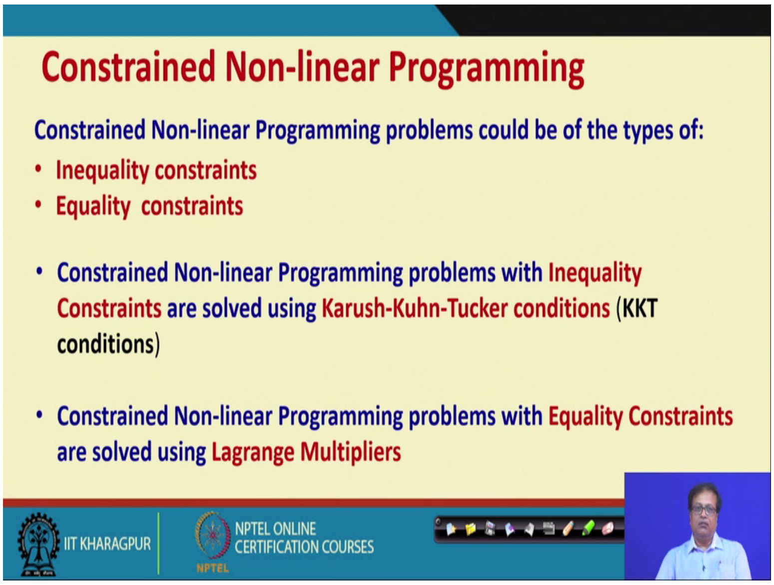


On the other hand if we have kind of equality constraints. So, you can see here that supposing a function f x is x 1 plus x 2. So, actually these red line you know x y f x equal to x 1 plus x 2 function and supposing there is a constraint that is h x equal to x 1 square plus x 2 square equal to 4 an non-linear constraint. So, these particular circle is essentially is our x 1 square plus x 2 square equal to 4 constraint line that is your h x.

So, you see as the profit line moves you know it just touches the constraint line you know in two points, one is b at the lower end and a on the other end is it alright and interesting thing note also that the gradient of these function f x that is del f x will be perpendicular to the line. An interesting question would be what would be the gradient of the constraint line, see say for example, if I think of a point let us see here can you tell me, what would be the you know the direction of del h x.

So, we have seen the direction of del f x so, one is del f x the gradient of the function f x and the other could be the gradient of the function h x let us say in a point here. So, you see at this point the del f x direction will be still this way because this is the line, but then del h x will be in this direction right. So, this point we might remember which will be useful later right.

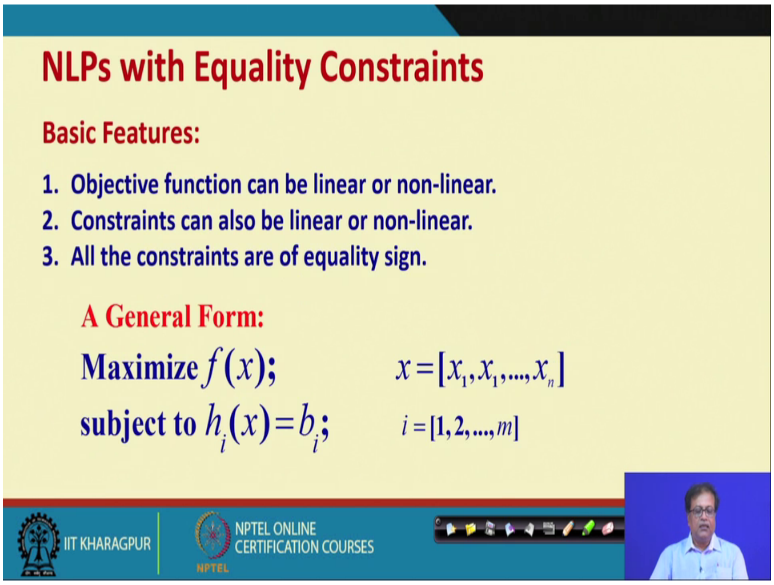
(Refer Slide Time: 05:03)



So, if we understood these, then let us move ahead the constrained non non-linear programming problems again are of two types the inequality constraints and the equality constraints right. The constraint non-linear programming problems with inequality constraints could be solved by using so, called KKT conditions or Karush -Kuhn -Tucker conditions.

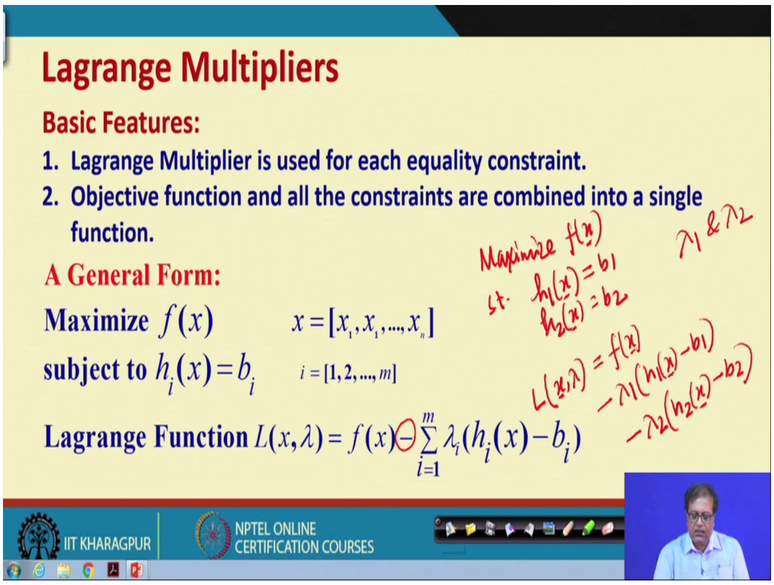
We shall discuss Karush- Kuhn- Tucker conditions in due course of time in our next set of lectures, but in these lecture we shall see how we solve constraint non-linear programming problems with equality constraints by making use of Lagrange's multipliers is it alright.

(Refer Slide Time: 05:47)



So, what exactly is these Lagrange multiplier let us look at these. So, here we have an NLP with equality constraints the objective function could be linear or non-linear. Constraints can also be linear or non-linear, but the constrains are equality type is it alright. So, here is a general form maximize f x, x could be x 1 to x n subject to h i x is equal to b i, i could be 1 to m.

(Refer Slide Time: 06:26)



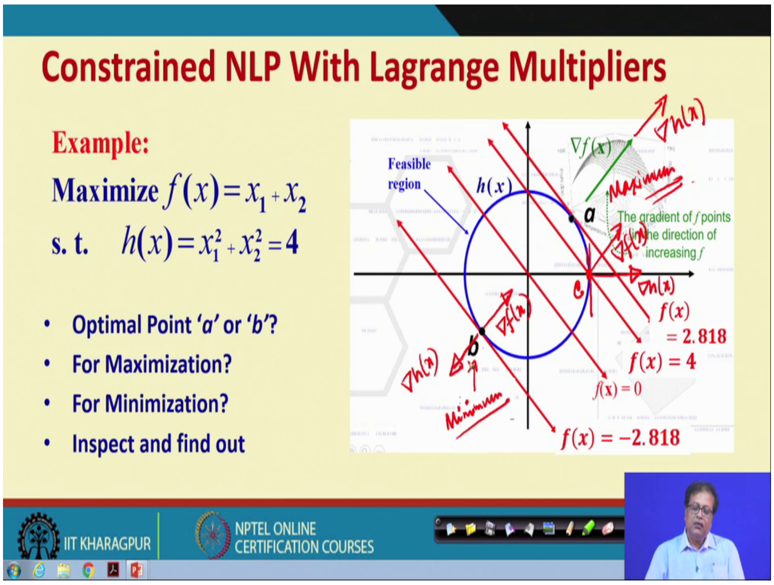
So, then we define a Lagrange function which is L x lambda is f x minus sum over h i h i x minus b i is it alright. So, that is the general form of Lagrange functions. So, to explain it little bit more supposing I have an objective function say maximize f x subject to say 2 constraints h 1 x say equal to b 1 and h 2 x equal to b 2. So, what would be our Lagrangian function in this case? So, you see we need 2 multipliers lambda 1 and lambda 2, lambda 1 for the first constraints and lambda 2 for the second constraint. So, the Lagrangian functions will be x lambda equal to f x minus lambda 1 h 1 x minus b 1 minus lambda 2 h 2 x minus b 2 is it alright. So, that would be the Lagrangian function.

Now here the x is basically a vector you know it has could be a number of variables is a function of x right. So, x could be x 1, x 2, x 3 it is a function of several x variables. So, this is how we define Lagrangian function, now one interesting point to note here about this negative sign out here you might find in some texts they use a positive sign is it alright.

Actually speaking you see what really happens that if I use a plus sign, the lambda value will become then just the opposite of what we get if you take minus and since lambda is a scalar quantity therefore, you know it will only affect the value of lambda, but it will not affect the value of the unknowns that is or the decision variables that is x.

So, we get the same answers with regard to the unknowns, but with regard to the lambda value we might get slightly deferent quantity. In the discussions that will follow in all such problems we shall going to be using the negative sign in all such situations right. So, that is how we define the Lagrangian multipliers now let us go come back to the problem once again.

(Refer Slide Time: 09:34)



We had maximize f x is equal to x 1 plus x 2 subject to h x is equal to x 1 plus x 2 square equal to 4. So, I have already told earlier that you know this is our h x and these are our profit lines. So, you can see 2 points a and b, where the h x line see this is our maximization function f x and this is h x. So, f x is just a tangent to the constraint line is it alright.

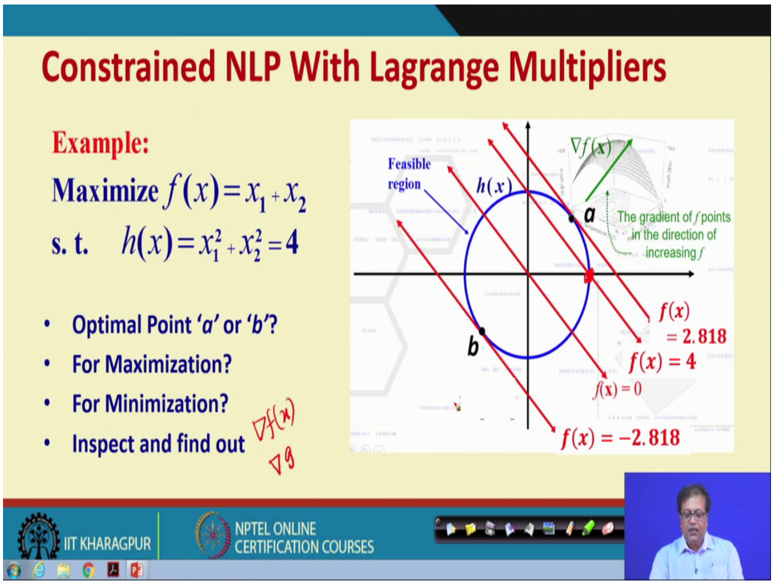
So, there are two such points a and b so, can you tell which one is a optimal point for maximization and which one is the optimal point for minimization in this particular problem right. If you look at it a little more carefully you can easily understand that point b actually corresponds to your minimum value and point a, correspond to maximum value is it alright.

Now, another very interesting thing about the gradient now see there are two gradients, one is the gradient of h x that is the constraint line, the other one is the gradient of the profit line f x see what is happening supposing I take another point c this is the point I was explaining earlier.

So, if I take a point c then at this point you see this is the tangent to the constraint line. So, these will be the direction of del h x, but then these would be the direction of del f x. So, you see the del f x and del h x they are not in the same direction. So, if you have to add them you have to do vector addition, but look at point b in point b the del f x is in these direction why because this is f x line this is where the gradient is increasing in this, but then exactly in the opposite direction can you see. So, if this is a tangent at this point for h x function would be the direction of del h x. So, del f x and del h x are just opposite you know just in the opposite direction.

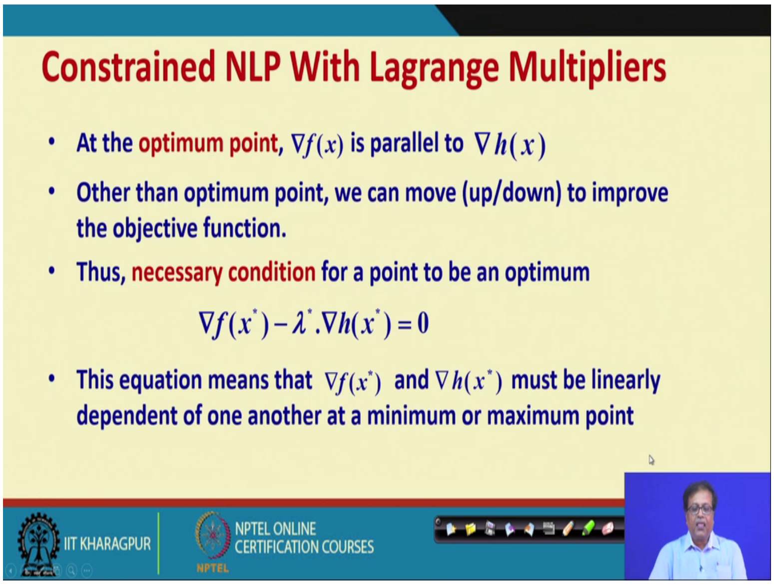
Whereas, at this point a del f x is already shown the del h x is also in the same direction why because this is h x function this is tangent so, outwards will be the del h x direction. So, if you really combine this facts you can find out that at the optimal point the del f x and del h x that is the gradient of the function and the gradient of your the constraint line they are parallel to each other right and they are actually you know can be added or subtracted easily right. In fact, if I multiply by a suitable scalar then del f x and del h x you know they will cancel each other.

(Refer Slide Time: 13:30)



So, exactly these factors have been retained in the next slide. So, I will come back to the slide.

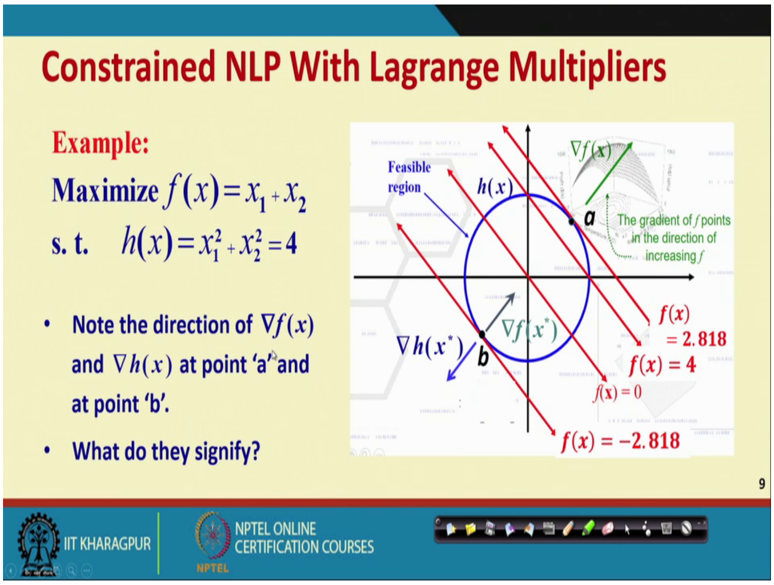
(Refer Slide Time: 13:40)



So, you see here at the optimum point del f x is parallel to del h x; that means, the gradient of f x and the gradient of h x the objective function line and the constraint line they are parallel to each other and other than the optimum point we can move up and down to improve the objective function and equation you know these equation del f x star at the optimum point minus lambda star del h x star equal to 0 because you see del f x and del h x are parallel and they are linearly dependent on one another right.

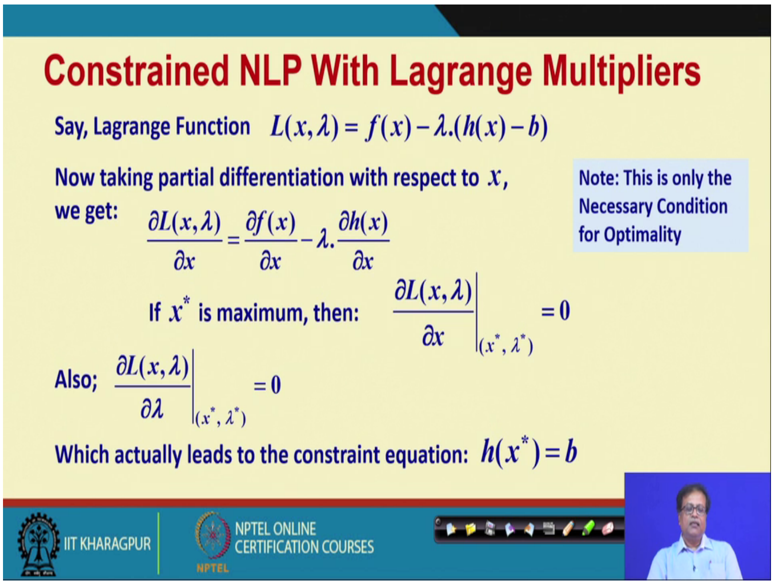
So, they can actually be added or subtracted as the case maybe so, the with a suitable value of lambda, lambda being a scalar could be positive or negative we can really construct the equation that the gradient of the function f x objective function and gradient of the constraint line together they should be equal to 0 that is del f x star minus lambda star del h x star equal to 0 right. So, this is the essential idea of the Lagrange's multiplier where lambda is a Lagrangian multiplier.

(Refer Slide Time: 15:00)



So, if we go back to that slide once again you can see once again that in the optimal point that del h x and del f x that in the opposite direction at point b and they are in the same direction at point a is it alright. So, this is a very important fact and we can make use of them in our subsequent discussion.

(Refer Slide Time: 15:27)



So, already we have seen the Lagrangian function is L x lambda is f x minus lambda h x minus b. So, if we take a partial differentiation then you know we get actually you know del f x del x minus lambda del h x del x.

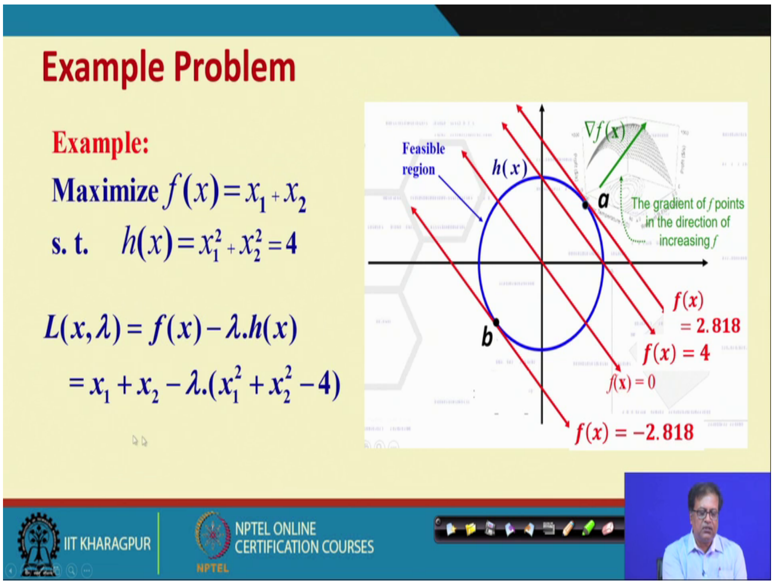
So, you know compare these with these del equation. So, del del x of f x minus del del x of h x into lambda you know these would be equal to 0 is it alright. So, if extra reason optimum point then this would be 0 and also the del L del lambda should be 0 also because if you take del L del lambda then nothing will come out of f x and the here you actually get back h x minus b equal to 0 or h x star equal to b which is the constraint line itself alright.

So, essentially therefore the Lagrangian multiplier method is that if I construct Lagrangian function L x lambda equal to f x minus lambda h x minus b and we differentiated with regard to the x variable a partial differential and also with regard to lambda and put them equal to 0 we get a set of equations and if we solve them the value that you get of x out of them is going to be our optimum value and these would form a set of necessary conditions for non-linear programming problems with equality constraint is it alright.

As a special case you know I mean you should also remember that these are only necessary conditions. So, special cases are those where you can also tell about convexity or concavity of the function as the case maybe because you know if you really want a maximum the function should be concave is it not the objective function should be concave so, that has to be really ensured that has to be really ensured.

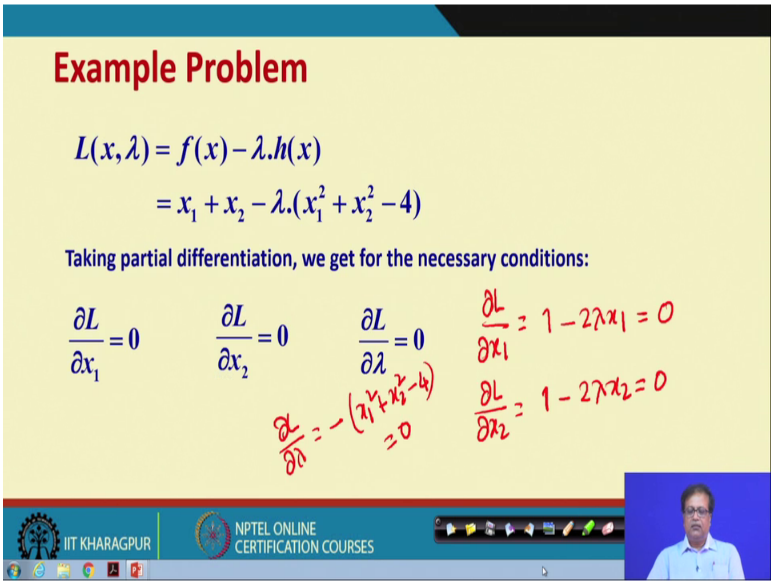
So, in this particular case in the f x it is a x 1 equal to x 2 is a linear function and these objective function is really both convection concave function. So, both maximum and minimum could be found out that should be remembered. So, the Lagrangian's multiplier by taking partial differentiation equating to 0 that would really give us the necessary conditions the sufficiency conditions have to be separately evaluated as well right so, these much has to be remembered.

(Refer Slide Time: 18:19)



Now, let us look at the problem once again so, this is our example this is the Lagrangian function f x minus lambda h x is equal to 0 that is x 1 plus x 2 minus lambda x 1 square plus x 2 square minus 4 not 0 that is the Lagrangian function.

(Refer Slide Time: 18:40)

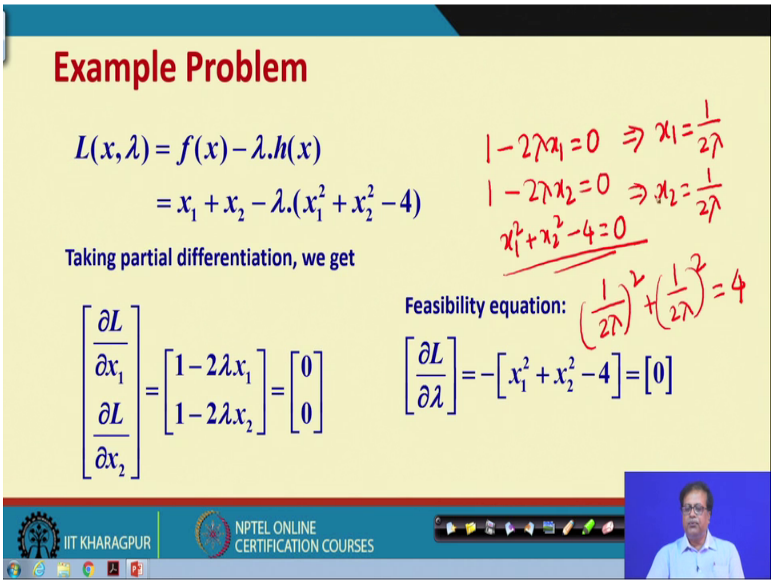


So, now, we equate to 0 the partial differentiation. So, what is the partial differentiation of these particular function can you workout, who will work out. Now, therefore, these values what will be the value, say del L del x 1 what will be del L del x 2 and what would be del L del lambda so, this 3 values can you workout.

The first one you know if you differentiate with respect to x 1 then what we get, we get the first term we get 1 and the second term nothing, third term we get minus 2 lambda x 1 and nothing from the rest. So, that should be equal to 0, similarly here we get 1 minus 2 lambda x 2 equal to 0 and the third term if I differentiate we simply get minus x 1 square plus x 2 square minus 4 equal to 0 right.

So, these are simple partial differentiation and through this partial differentiation we can get these 3 sets of equations. So, let us look at it once again that is these you know these is how we find out the partial differentiations.

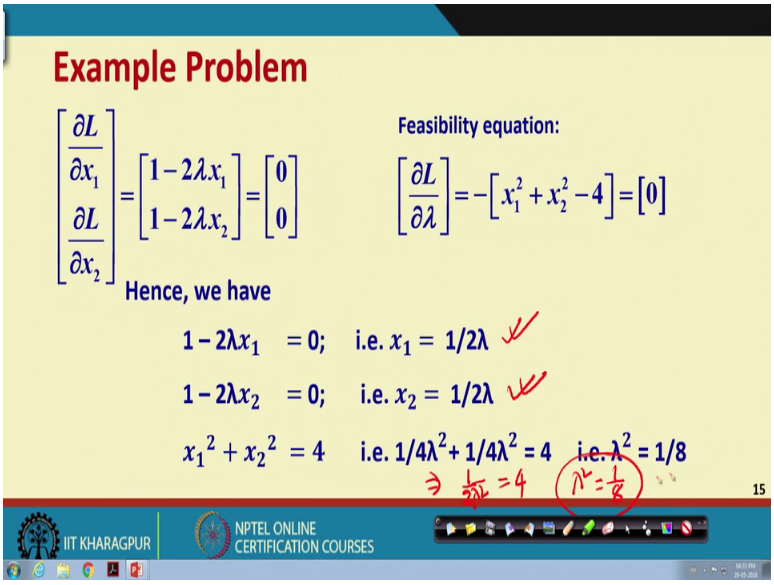
(Refer Slide Time: 20:14)



So, the partial differentiations would be these so, del L del x 1 1 minus 2 lambda x 1 del L del x 2 1 minus 2 lambda x 2 is put to 0 and del L del lambda is minus x 1 square plus x 2 square minus 4 equal to 0. So, if I combine them what are the equations that we get 1 minus 2 lambda x 1 equal to 0 1 minus 2 lambda x 2 equal to 0 and x 1 square plus x 2 square minus 4 equal to 0 is it alright. So, if we have to solve look here we have 3 unknowns and we have 3 equations.

So, easily we can solve in fact, you can little bit of you know if you really look then from first equation we get x 1 equal to 1 by 2 lambda and from here we get x 2 equal to 1 by 2 lambda. So, what we can do, we can put them into these equation and then 1 by 2 lambda whole square plus 1 by 2 lambda whole square we shall put equal to 4 right. So, this is if you solve them we get the value of lambda and then you can put them in x 1 and x 2 and we can get a value of x 1 and x 2.

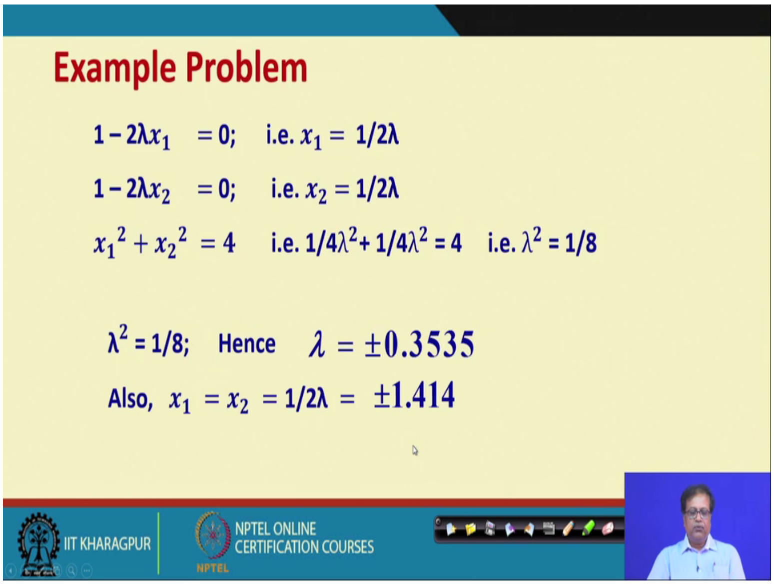
(Refer Slide Time: 22:00)



So, let us look at them so, this is how we solve. So, these equal to 0 so, x 1 equal to 1 minus 2 lambda x 2 equal to 1 by 2 lambda and by putting those things here we get 1 by 4 lambda whole square plus 1 by 4 lambda whole square is equal to 4 so; that means, lambda square equal to 1 by 8 because this will give half 1 by 2 lambda whole square equal to 4.

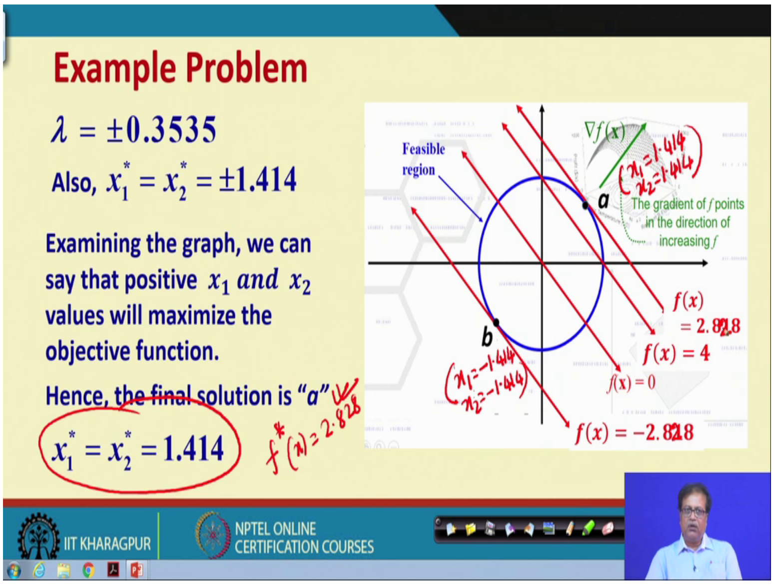
So; that means, you know lambda square you take that side and 4 you take this side is 1 by is it alright. So, we get lambda square equal to 1 by 8 so, this exactly that is what exactly we have got lambda square equal to 1 by 8.

(Refer Slide Time: 22:56)



So, if we try to solve them further then we get that 1 minus 2 lambda is 1 equal to 0 and lambda square equal to 1 by 8. So, these would be the possible value of lambda plus minus 0.3535 putting that equal to x 1 and x 2 we get plus minus 1.414. So, we have been able to solve the value of x 1 and x 2 in this particular problem.

(Refer Slide Time: 23:23)



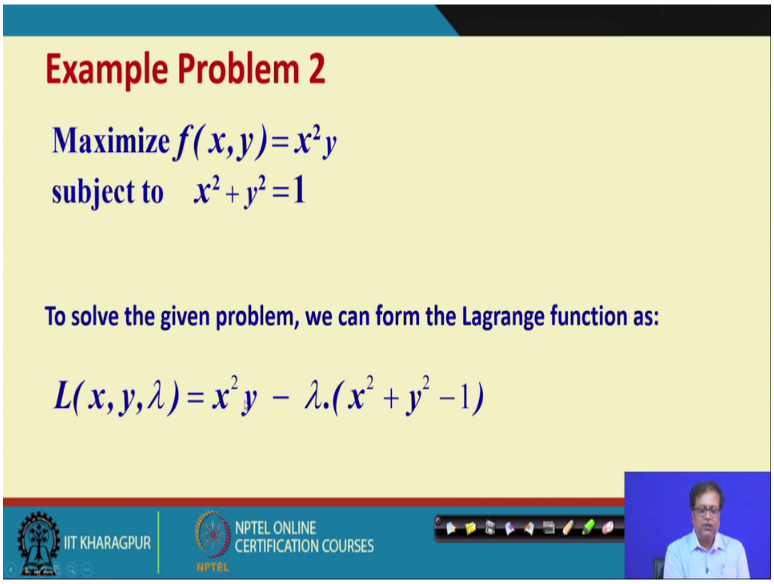
And if we examine the graph, we can say that you know the really speaking if you really look at this graph carefully, then you see the point b actually x 1 equal to minus 1.414 and x 2 equal to minus 1.414. So, that f x is these value and at point a x 1 equal to 1.414

and x 2 equal to 1.414 is it alright. So, obviously, it is a original problem was one of maximization. So, the maximum value would be obtained that point a and that would be our optimal solution with f star x equal to 2.818 right is it alright. So, that would be the value 2.828 that will be the functional value right.

So, that is how we can solve those problems. Once again remember that we form the Lagrangian function with make partial differentiation, solve them. But then you know since the Lagrangian multiplier that differentiation really gives unnecessary conditions only; we also have to remember about the maximum or minimum value of the function is ensured by either the second derivative conditions or if we know the pattern of the graph or if you know the concavity or convexity of the graph. Then from that knowledge we have to really ascertain the sufficiency part of the problem right.

So, once we understood to in order to make our knowledge more clear let us look at another problem.

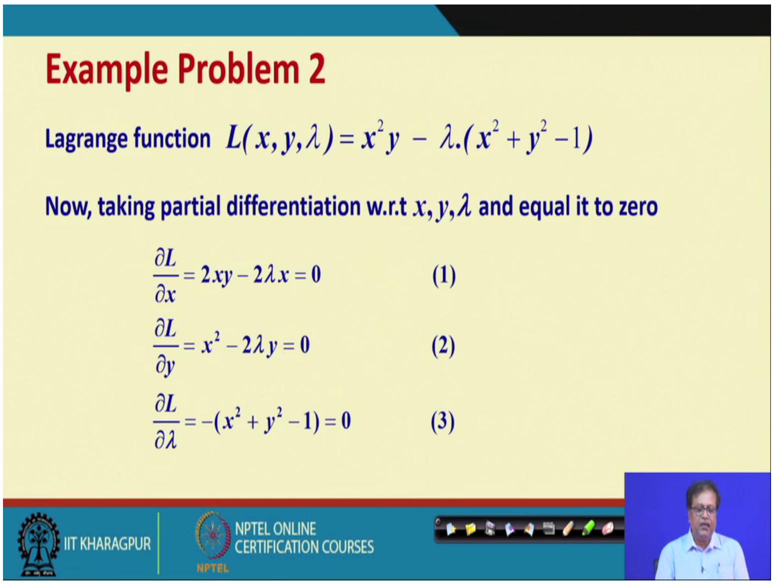
(Refer Slide Time: 26:34)



So, this particular problem we have maximize a function of x y equal to x square y subject to x square plus y square equal to 1. So, you see both the, your constraint and the objective function they are actually non-linear right. So, really speaking the objective function here is a function x square y you can see; obviously, you can you have to prove it otherwise, but we see that from the nature of the function that this is a concave function right. So, because it is a concave function the maximization should be possible.

So, we first form the Lagrangian function with the 3 variables x y and lambda that is x square y minus lambda x square plus y square minus 1 is it alright that is the first task.

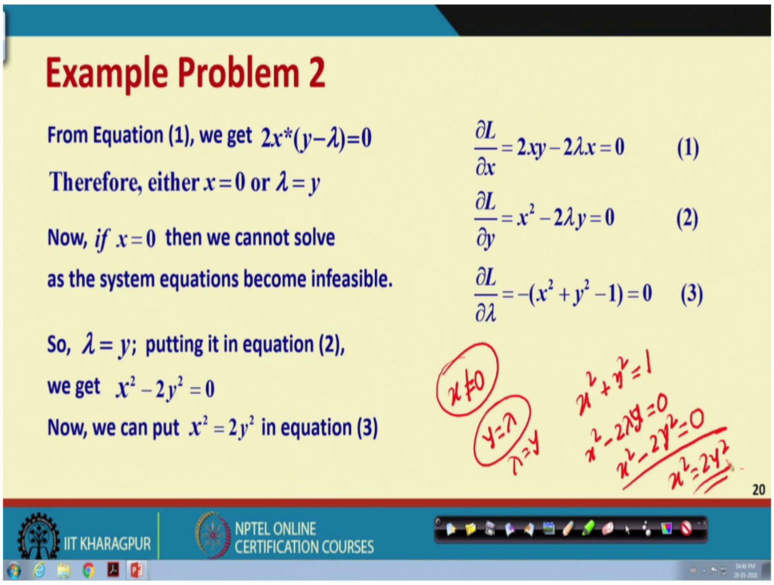
(Refer Slide Time: 26:33)



Then we make those 3 differentiation with respect to x with respect to y and with respect to lambda the with respect to x what do we get, 2 x y minus 2 lambda x equal to 0 with respect to y we get x square minus 2 lambda y equal to 0 and with respect to lambda we get minus of the constraint line x square plus y square minus 1 equal to 0.

So, these are the 3 equality constraints we have to solve to find the values of the unknowns and the Lagrangian multiplier.

(Refer Slide Time: 27:16)

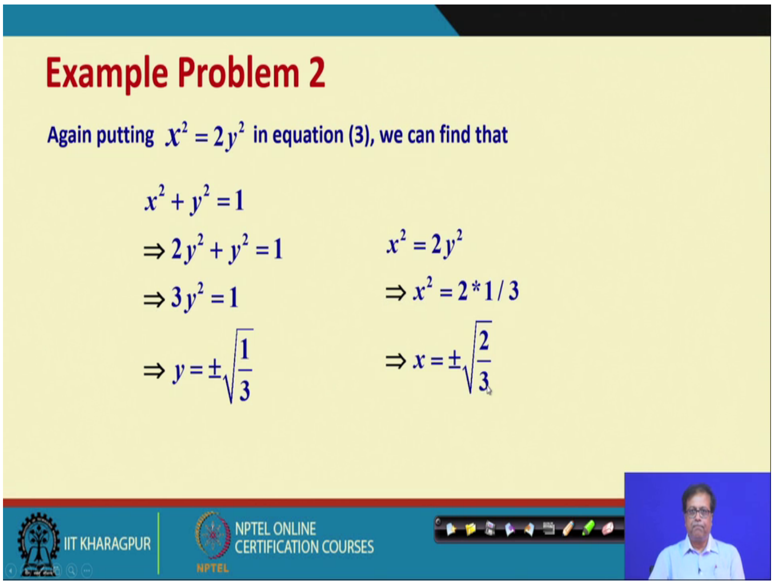


So, what we can do essentially from these 3 constraint lines, that is since 2 x by taking common y y minus lambda equal to 0. So, it automatically means that either x equal to 0 or lambda equal to y for the time being assume x equal to 0. So, if x equal to 0 then you know from the second equation you can see the 2 lambda y equal to 0 is it and since lambda equal to y then both lambda and y will be 0 so, everything becomes 0.

If everything is 0 then how can these 0 square plus 0 square minus 1 will never be 0. So, x cannot be 0 is it alright. So, essentially what we have just now seen that x cannot be equal to 0 because if x equal to 0 then we cannot solve the system equations. So, if x not equal to zero; that means, y equal to lambda right so, this second part will come. So, from the second part we can put that in the third constraint that is x square plus y square equal to 1. So, we get basically that x square you know or lambda equal to y.

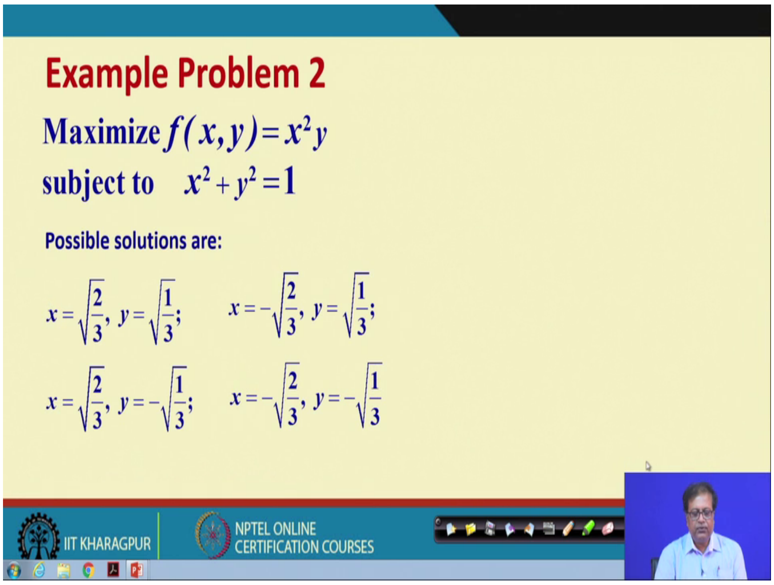
So, you know we put not in this we put in x square minus 2 lambda y equal to 0. So, if I put lambda equal to y then I get x square minus 2 y square equal to 0 so, that is x square equal to 2 y square right. So, that is what we get now as you get these we can put it here.

(Refer Slide Time: 29:15)



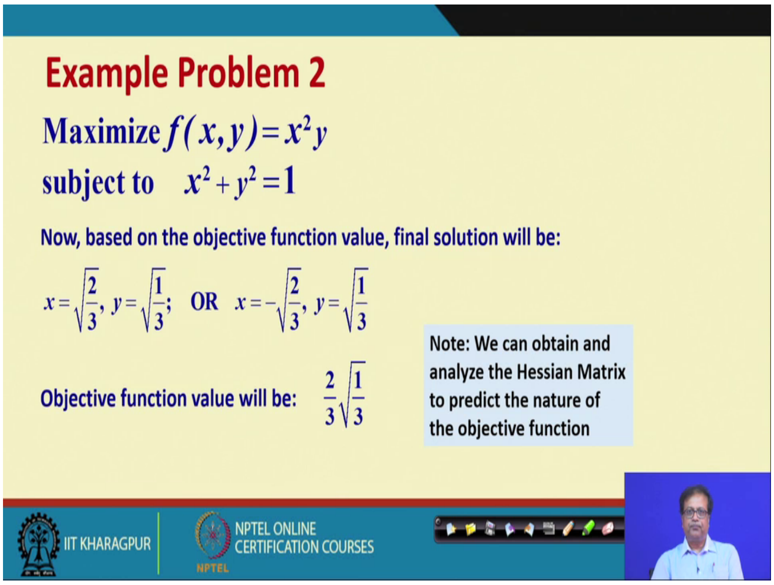
Then we get 2 y square plus y square equal to 1 or 3 y square equal to 1. So, y will become plus minus 1 by 3 root over and from here x will be plus minus root over 2 by 3. So, this is very simple really so, once we get these then we can.

(Refer Slide Time: 29:34)



See this is our original problem so, these are the 4 possible solutions what are they plus plus minus plus plus minus and minus minus right. Now, since the problem is one of maximization it is and we have an x square. So, which combination should we take is very clear that we should take the first two combination that is these or these because x square will be then 2 by 3 whether you take these or these and y is root over 1 by 3 that would give the maximum possible functional value.

(Refer Slide Time: 30:22)



So, therefore, we see that this is our solution and the objective function value is 2 by 3 root over 1 by 3, right. We can also obtain the Hessian Matrix to predict the nature of the objective function then.

So, thank you very much and in our next class we shall see how these KKT conditions are used for inequality constraints.