Selected Topics in Decision Modeling
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Lecture - 22
Single-Variable Unconstrained Optimization

So, in over course Selected Topics in Decision Modeling, now we are into our 22 lecture, that is Single-Variable Unconstrained Optimization problems these essentially the non-linear programming problems we are discussing. So, within that we are now discussing the single variable unconstrained optimization problems.

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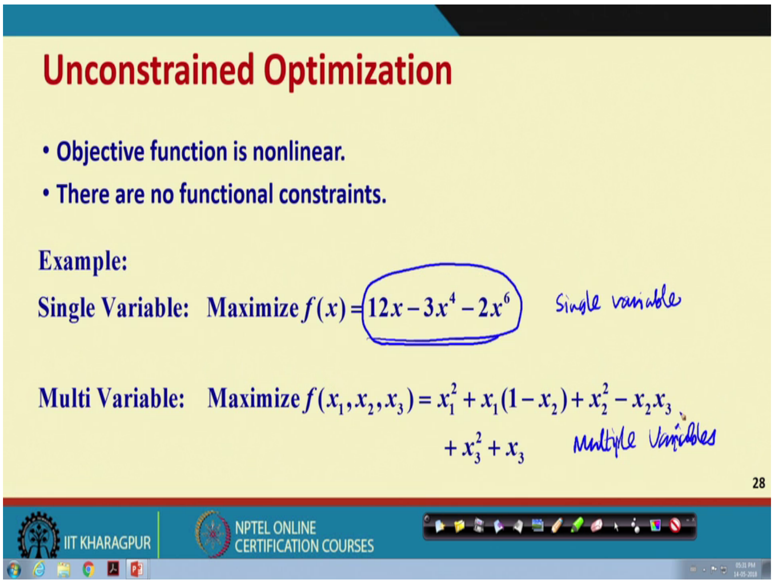

Now, a unconstrained optimization problem in the context of non-linear programming we might have the objective functions is non-linear and there are no functional constraints. So, that is the essential definition of unconstrained optimization that there is you know non-linear objective function, but there is no functional constraint.

So, it could be single variable or it could be multiple variable so, you can see in the first case we have a case where we have maximize f x equal to 12 x minus 3 x to the power 4 minus 2 x to the power 6. So, we have only one variable so, since there is only one variable we call it the single variable problem. On the other hand the second example you can see the objective function is x 1 square plus x 1 into 1 minus x 2 plus x 2 square minus x 2 x 3 plus x 3 square plus x 3.

So, you can easily see that there are multiple variables. Now, multiple variables problems are a little a more complex then the single variable problems. In this particular lecture we are mainly discussing the single variable problems only right.

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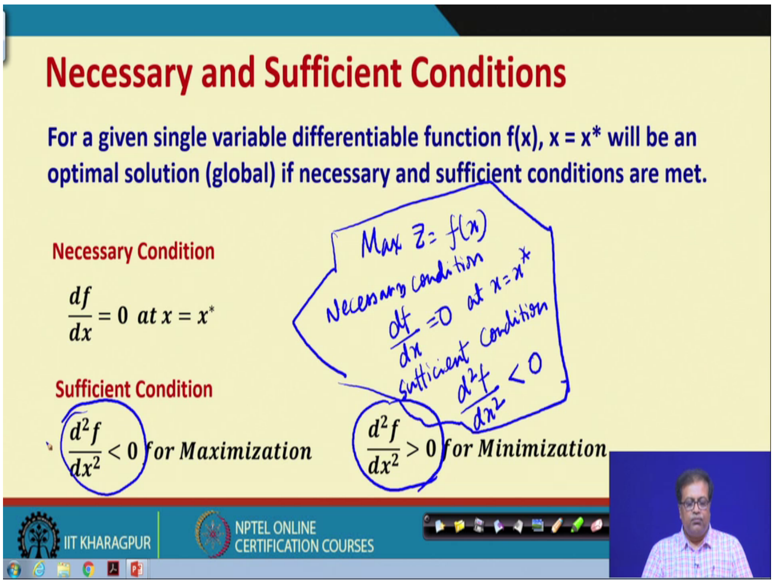

So, in the single variable problems for a given single variable differentiable functions f x x equal to x star will be an optimal solution if necessary and sufficient conditions are met right.

So, that is what we are discussing here that is, we have let us see a function maximize z equal to f x all right. So, what is the necessary condition necessary condition will be df dx equal to 0 at x equal to x star, but we cannot call x star as optimal until and unless it also satisfies the sufficient condition; that means, the condition of sufficiency that the second derivative should be negative right the second derivative should be negative. So, this is what happens in case of maximization problem.

What will be there for the minimization problem, the minimization problem everything else is similar the necessary conditions is again the first derivative should be 0 at the point of optimality, but additionally there should be the second derivative should be greater than equal to 0. So, second derivative less than equal to 0 for maximization second derivative greater than equal to 0 for minimization right so, this is was we have to do right, we have to find out.

The problem really comes when we might be able to find out the derivative values, but from the derivative values which may not be possible to find the value of the variable decision variable very easily, because higher order equations are really involved so, that is where the problem comes.

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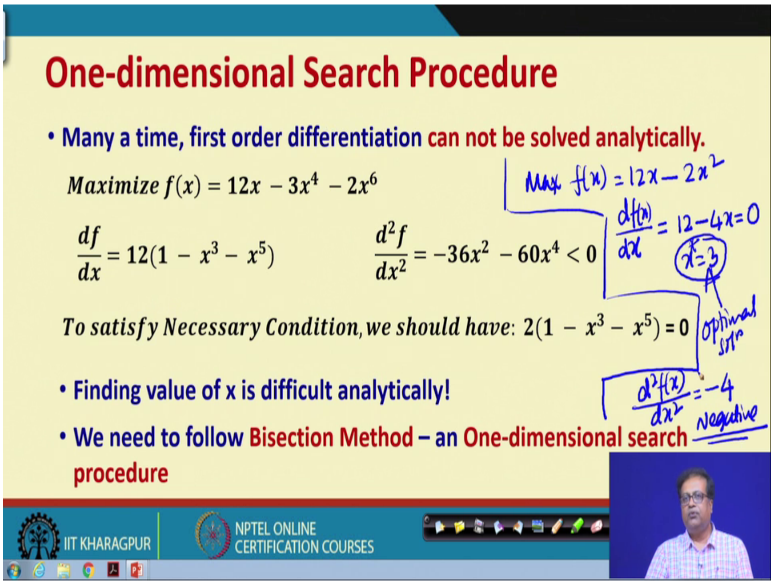

So, simple problems are easy, but difficult problems let us look at some of these things before we solve these problem. Let us take a very simple problem suppose I have only these problems to solve that is let us say maximize supposing I have this problem to solve that is maximize f x equal to 12 minus 2 x square right. So, only this problem to solve now we find what is it is say 12 x.

So, maximize f x equal to 12 x minus 2 x square only this much nothing else. So, if I am solving only this much and not the problem that is given here then if I find the first derivative df x dx then we find is equal to 12 minus 4 x. So, when I put equal to 0 I can easily find x equal to 3, what is the second derivative second derivative d square f x by dx square equal to minus 4 we can see that this is negative.

So, since it is negative so; that means, the original function is concave objective function was concave and therefore, the x equal to 3 is our optimal solution alright. So, once again look what we have done we have a problem maximize f x equal to 12 x minus 2 x square we took the first derivative and we got 12 minus 4 x we put the equal to 0 and we got the x star equal to 3.

So, when we got x star equal to 3 that is over optimal solution for maximization because the second derivative is equal to minus 4 which is negative there is no difficulty here alright. So, this simple example is very easy which are solve and there is no con confusion what is waiver, but what happens in this problem, let us see now see maximize f x equals to 12 x minus 3 x to the power 4 minus 2 x to the power 6.

So, if we take first derivative we see the first derivative is 12 minus 36 x cube minus 24 you know x 5 right. So, we see sorry 12 into yeah. So, minus 2 x 6 so, minus 12 x 5 so, this can we shown as df dx equal to the first derivative 12 into 1 minus x cube minus x 5.

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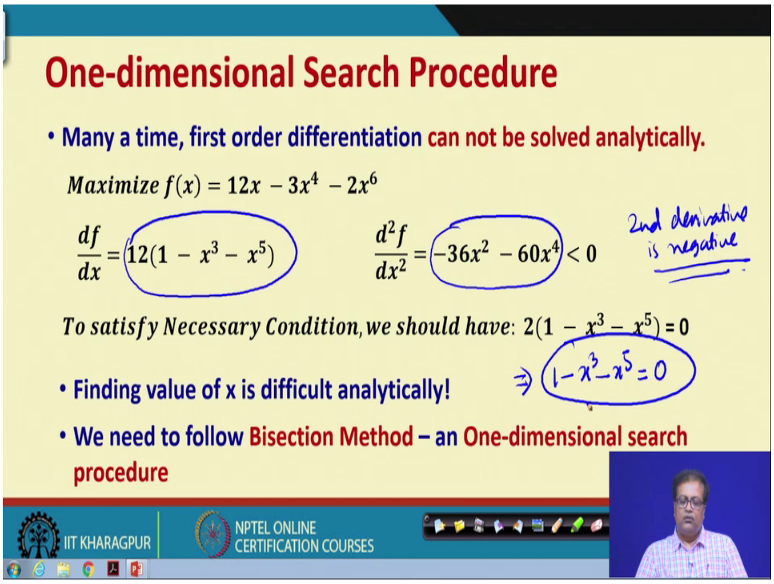

So, this is very easy to verify that that is our first derivative, if you take the second derivative that we minus 36 x square minus 60 x to the power 4 alright. Now, second derivative is less than equal to 0 y because put any value of x, so, x square will be positive term and x to the power 4 will be also a positive term. So, this is multiplied by a negative number and here also multiplied by a negative number. So, whole total must be negative so; that means, the second derivative is negative.

So, it is conclusively seen that the second derivative is negative. So, we have no problem here the second derivative is negative, but what about the first derivative we have to put the first derivative equal to 0 that is our necessary conditions. So, to satisfy the necessary condition we should have the first derivative equal to 0.

So, if you put that first derivative equal to 0 which essentially means that 1 minus x cube minus x 5 equal to 0, now this is not going to be easy to solve this because you would not be able to factorize it. So, easily this particular problem may not be so difficult because you know equal to 1 equal to x cube plus x 5. So, you can take x cube common and all those things you can try, but then let us not the usual thing because you may have different co-efficients for another problem we should really find method where such problems can be solved all the time not for a given problem.

So, one can see that finding value of x is very difficulty here in an analytical manner. So, therefore, there are such an procedure which can be called as one dimensional search procedure several methods are there we discuss one method that is bi section method. So, how do we find out the solution for such problems in it is simple manner by an iterative procedure.

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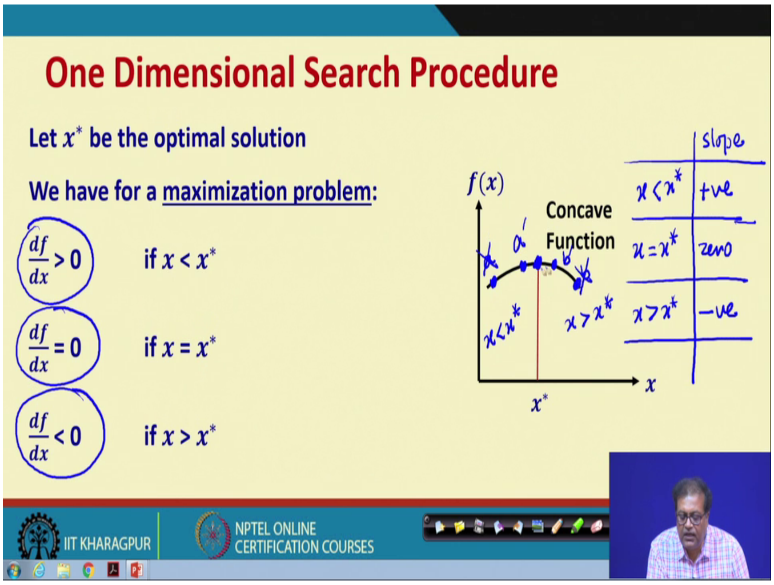

So, what we can do you see this is how we gain, supposing a x star is optimum solution and we are really solving a maximization problem, for minimization problem it will be slightly different, for a maximization problem you can see the function is concave. So, if the function is concave you can see that this is where the, you know is our optimal point.

So, this is the optimal point so, the before the optimal point so, this side is x is less than x star and this side is x is greater than x star is it know. So, what is your basic concept about suppose I make a small chart x less than x star x equal to x star and x greater than x star. So, these are the 3 regions and now what is your comment on the slope so, this side it is increasing slope.

So, we can we say that is slope is positive this is the point which is the settled point at the between point so, slope is 0 and here it is falling. So, slope is negative now we know that df by dx is nothing, but slope alright. So, same thing is noted here you know this if x is less than equal to x star then the slope is positive, if x is equal to x star slope is 0 if x is greater than x star slope is less than equal to 0. So, this is very easy to understand that essentially that is what we have written here right.

So, what we do is basically, suppose we do not know the function, but we take 2.0 suppose we take 1.0 here and 1.0 here is alright and then you know this where from we get these 2.0 it is not that these 2 points has to be exactly on the both sides, but we should be able to find 2 points which we know very very much with certainty that one is on this side, other on the other side that we should know, if we know then what we do we take the average supposing the average is here right.

Now, for that average we find out this slope or the gradient, then we imply checked that is this slope negative or if the slope positive supposing we know the slope is negative suppose this is our original point a and this is the original point b, then if is see the slope is still positive then we discard this a and called this as a is alright then again we take average and supposing we get another point suppose here.

So, then that we called b and we discard this original b so, like this you know what us happened, again we take average in by is by section method we are essentially nearing the point when we are coming to very close to this optimal point we may say yes optimal point is reached right. So, this is how the one dimensional search problem actually shall proceed.

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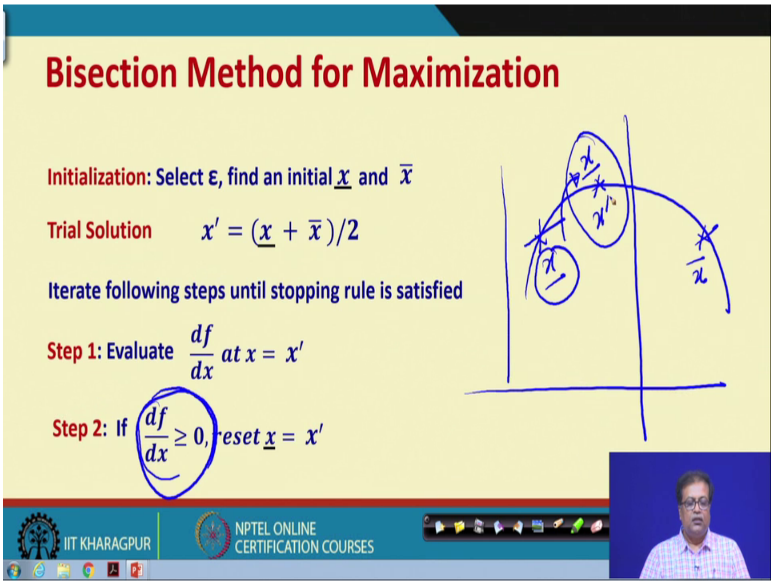

So, let us see that how it happens. So, initialization select a epsilon find an initial x bar and you know x underline and x bar and then the trail solution is x underline plus x bar divided by 2. So, this x underline is on one side and x bar are on the other side. So, supposing we have a function, this is our function so, this is our function this is suppose to be optimal.

So, this is your x bar then the other side is your x underline and x bar, now the question is evaluate df dx; that means, the gradient at the point supposing this is our point so, this is our x dash. Now look here supposing the x dash is on the upward slope if it is see this is the upward slope; that means, greater than equal to 0.

So, if it is in the upward slope then these points so, should really come here right. So, this point goes here; that means, x dash now will come x underline right. So, if df dx is greater than equal to 0 reset x bar with x dash which is the trail solution which is the average solution. So, basically this is how we proceed and similarly now same thing we might do with the next case.

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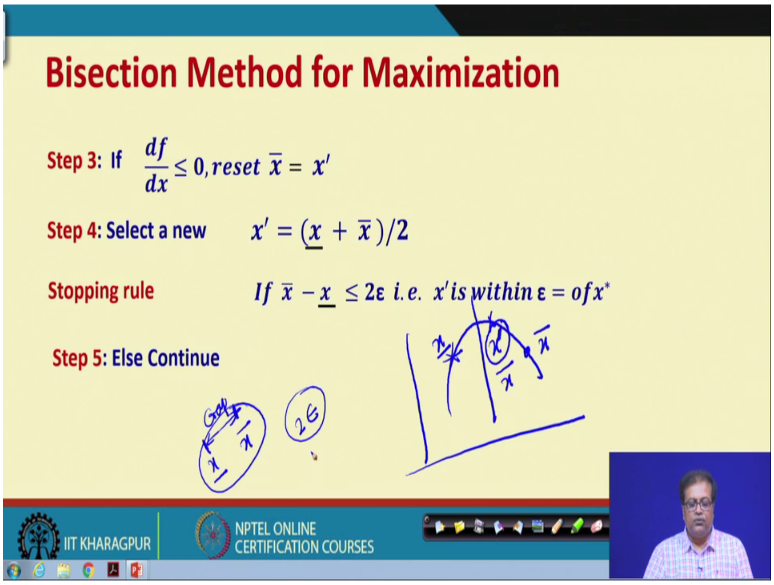

Then if the slope is on the other side then take x bar equal to x dash right. So, how it is done once again so, this is our plot, this is our curve. So, this is our let say x underline and this is another point which is x bar. So, this is our x dash so, if x dash has come in such a way the slope is actually negative which is falling under that. So, these x dash now should be taken as our nu x bar right. So, that should be reset.

So, that is how you do and then select a new x dash and again follow an if the difference between the x bar and x underline you know the difference between them if they are within 2 epsilon then we should stop; that means, x dash is within epsilon because x dash is in the middle.

So, if the gap this is the gap this is the gap so, if the gap x is somewhere in the middle. So, if the gap is less than the 2 epsilon we may say that x x bar is within epsilon of x. So, we can stop otherwise continued so, that is how the method is so, very simple method let see how it is operated.

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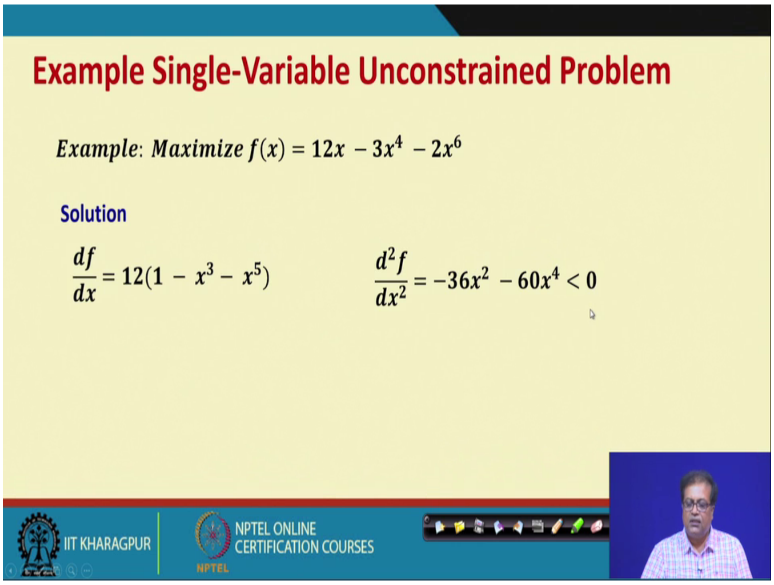

Let us took take the example once again. So, the example was maximize f x equal to 12 x minus 3 x to the power 4 minus 2 x to the power 6. So, we took the 2 derivative df dx once again it should be 12 minus 12 x 3 minus 12 x 5. So, 12 is common 1 minus x 3 minus x 5 and the second derivative is minus 36 x square minus 60 x 4 which is less than 0.

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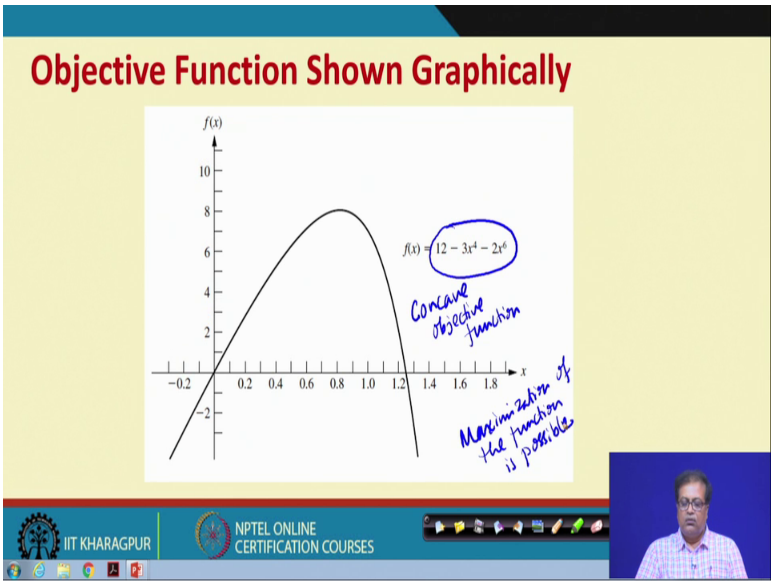

So, that is the first take then after that this is the plot you know 12 minus 3 x to the power 4 minus 2 x to the power of 6, that is our objective function to really show that the objective function is concave right. So, these shows the objective function is concave and we have also shown the objective function is concave by taking the second derivative and we saw that the second derivative is negative.

So, since it is a concave objective function the maximization of the function is possible right maximization of the function is possible because if you would have had a concave objective function non I mean convex objective function then there was no you know global maxima that was not possible to find. So, that is very important consideration that has to be seem first right, now what we do that 2 inspection it can be seen.

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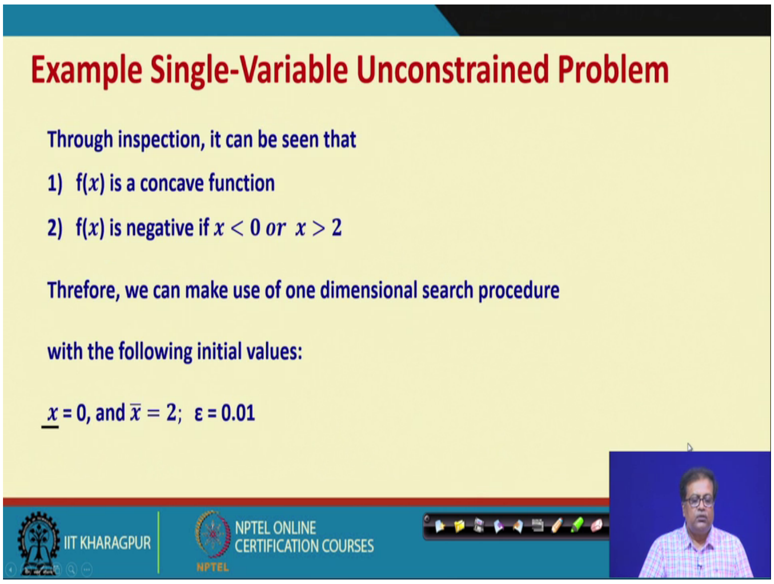

So, how, we see that inspection.

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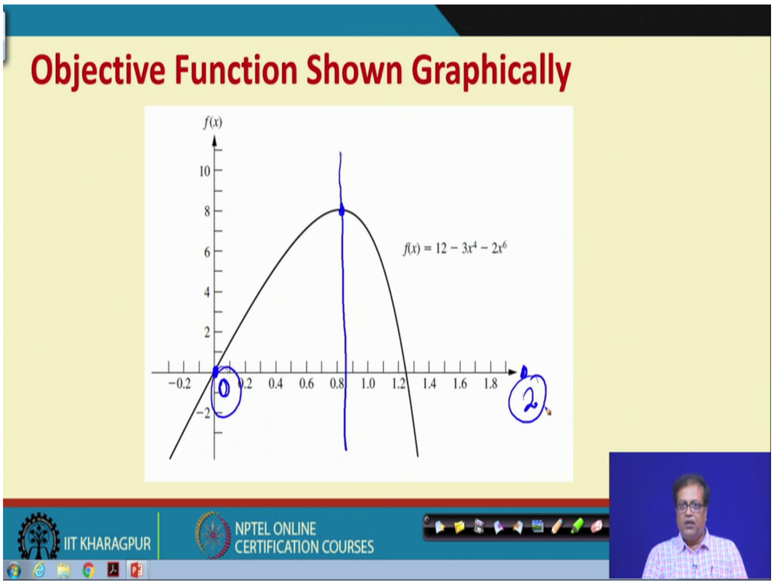

So, you can see that these functions is going through you know certain points; obviously, it goes to 0 and this side is another point let us a 2. So, defiantly you know it this has to be between 0 and 2. So, 0 is the supposing where is the objective function. So, supposing this is our maxima so, maxima will be somewhere here now it is known that these maxima is definitely form a 0, 0 this is 0 and somewhere here that is 0.2 right 0 will be one side, 2 on the other side.

So, if you carefully see the graph at least one can work out that there should be 0 or 2 right. So, which should be within 0 and 2 so, these 2 limits are required. So, we must have those 2 limits first. So, when we have ascertained that those are our limits then form those limits you know we can begin.

So; that means, we can say that first of all things you have to seen that f x is concave f x is negative if x is less than 0 or x is greater than 2 alright. So, therefore, we can make use of one dimensional search procedure that we can make with the following initial values that is x underline equal to 0 and x bar equal to 2 and let us take a epsilon equals to 0.01 so, with this values we begin.

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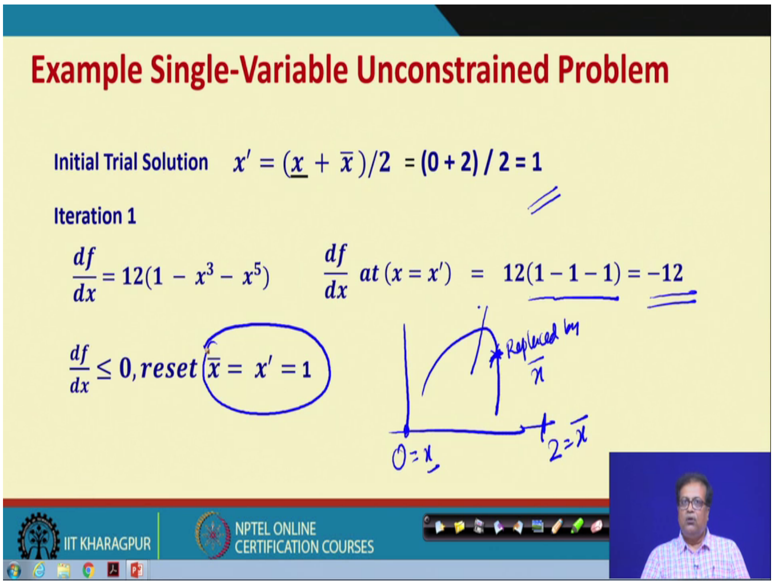

So, now we can first take x dash equal to the average of the 2 so, 0 plus 2 by 2 equal to 1 right. So, 1 will be our new value, now we check what happens at 0.1 you recall that, that was our first derivative df dx is 12 into 1 minus x cube minus x 5. So, when you put that value 1 we find 12 into 1 minus 1 minus 1 equal to minus 12. So, we get a negative since we get negative; that means, which side it will fall we call this is our card. So, this is negative so, this point must have fall in this side so, this side means. So, this is 0 side and this side is 2 side right.

So, 0 was x underline 2 was x bar; that means, this is new point should be replaced by x bar x bar. So, new value for x bar should be therefore, x dash equal to 1. So, that should be our new x bar right so, that is the first iteration, in the first iteration therefore, we find the new value of x bar.

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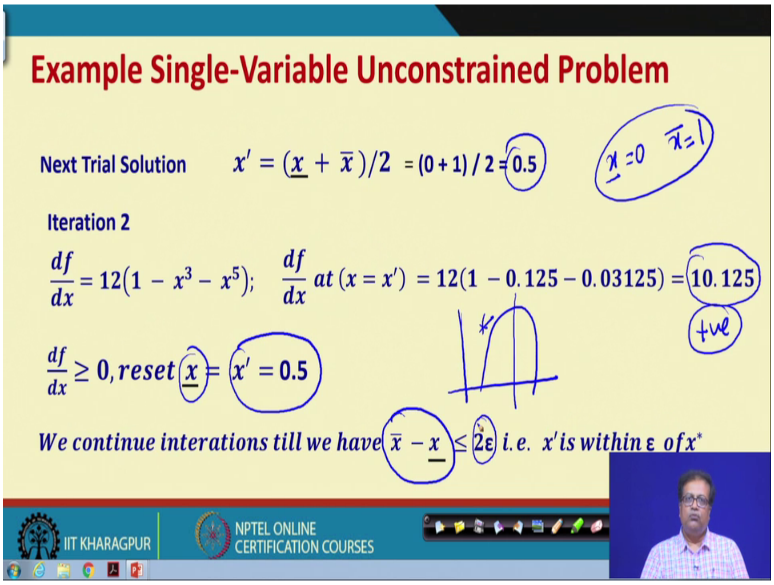

So, once we find the new value of x bar, now what happens again we find that what was the value please now remember once again x underline was 0 x bar has become 1 right. So, those are the values so, using those values we can now find next trial solution that is x dash and then that you find 0 plus 1 by 2 equal to 0. 5. So, this is going to be over new trail solution so, once again derivative 12 into 1 minus x cube minus x to the power 5.

So, at x is equal to x dash these values comes out to be 10.125 right. So, what is the 10.125, this is a positive value. So, since it is a positive value now it will fall on the other side that is 0 side. So, these values somewhere here right. So, since it is that side now x bar should be reset so, what should be the new value of x bar, it should be reset to x dash that is 0.5 right.

So, if you continue our iterations so, with that again we can have you know further iterations and when you stop when we see the difference between the x bar and x underline is within 2 epsilon right; that means, x dash will be within epsilon of x star that is a optimal right. So, let us how we continue the process so, all of these can be shown nicely in the form of a table so, that table let us look at the table.

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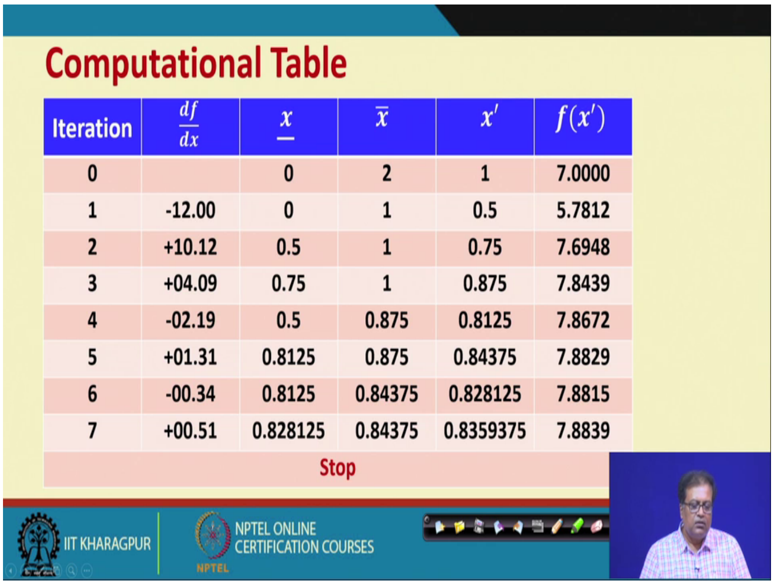

Computational Table

Iteration	$\frac{df}{dx}$	$\underline{x}$	$\overline{x}$	x'	$f(x')$
0		0	2	1	7.0000
1	-12.00	0	1	0.5	5.7812
2	+10.12	0.5	1	0.75	7.6948
3	+04.09	0.75	1	0.875	7.8439
4	-02.19	0.5	0.875	0.8125	7.8672
5	+01.31	0.8125	0.875	0.84375	7.8829
6	-00.34	0.8125	0.84375	0.828125	7.8815
7	+00.51	0.828125	0.84375	0.8359375	7.8839
Stop					

So, this is how the computation goes on that at the first iteration between 0 and 2 we find 1 look how the f x dash is increasing. So, then after that the new value of x dash becomes 0.5, then again at the next 0.5 to 1 it becomes 0.75, then 0.75 to 1 the value; see now slowly the value is increasing and then between point say 0.5 to 0.875, then 0.8125 like that, it goes and finally, you know at the 7-th iteration we find that these values become 0.828125 and 0.84375 and the gap is within the 2 epsilon right.

So, we have this kind of value that is 7.8839 right so, then we stop.

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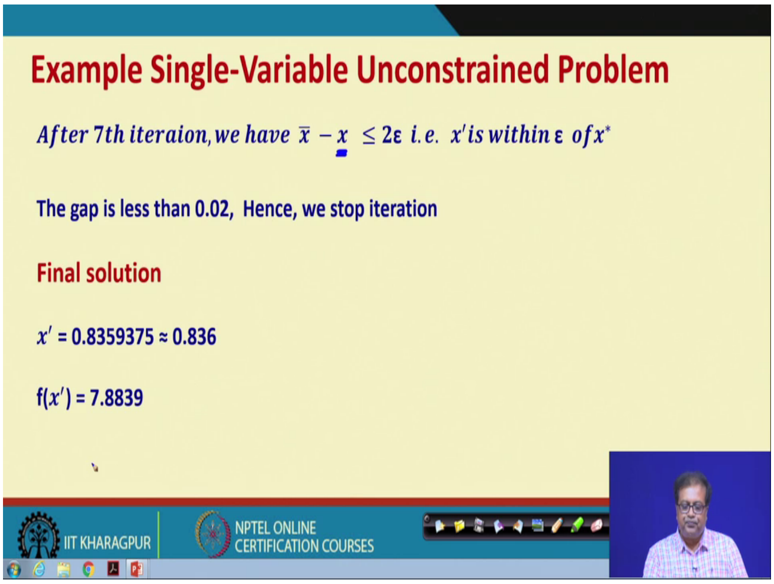

So, after 7-th iteration we have the gap has fall in within 2 epsilon. So, that is how we can actually continue. So, this x bar minus x underline is within 2 epsilon the gap is less than 0.02 we stop our iteration and final solution they are becomes 0. 8359375.

So, all these 2 x dash and x x underline we take again average and we get the x dash is 0. 833 and the function value become 7.8839. So, this is how we can through an iterative procedure obtain the optimal solution right. So, let us take one more, quick example.

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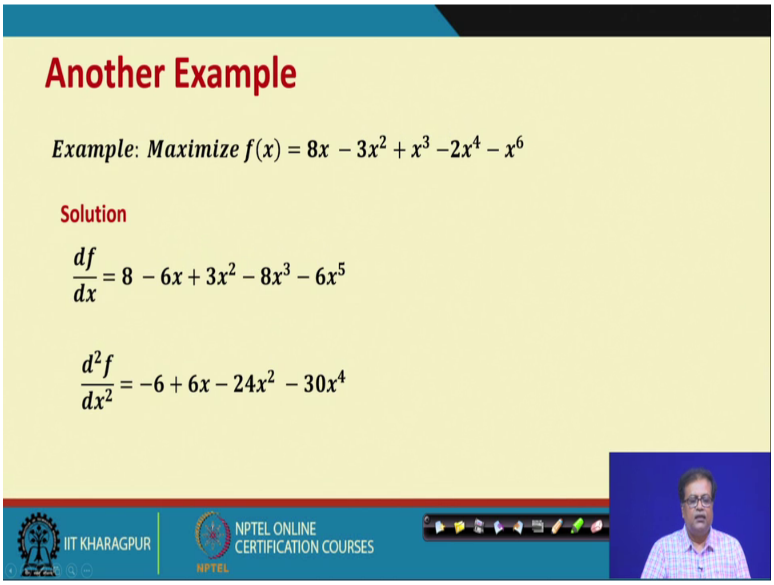

So, that we understand our process very easily so, maximize f x is equal to 8 x minus 3 x square plus x cube minus 2 x to the power 4 minus x to the power 6. So, we take the first derivative that is 8 minus 6 x plus 3 x square minus 8 x cube minus 6 x 5 and second derivative minus 6 plus 6 x minus 24 x square minus 30 x 4 right. So, those are the first and the second derivative now this is the function.

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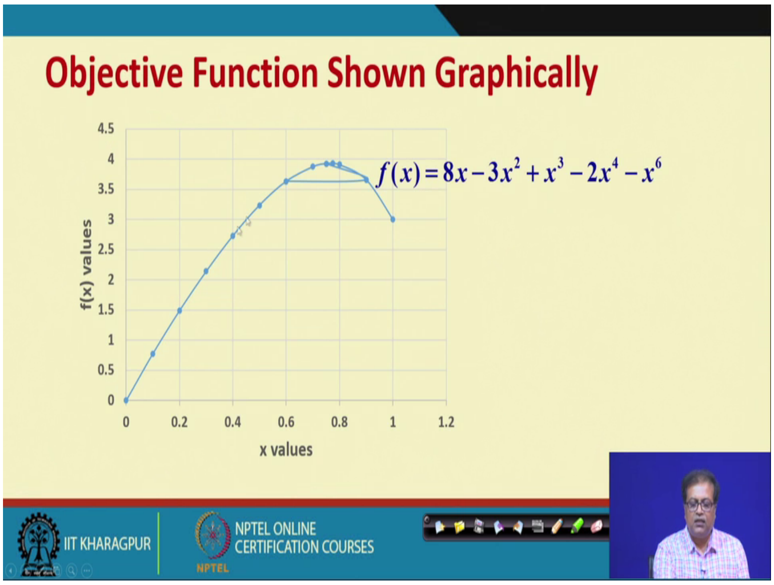

The function you can see it is between 0 and 1.2 there on the 2 directions. So, through inspection it can be seen that f x is a concave function, it is a concave function that is the function a nature is like this and the second derivative is also for certain values we have to see explore if further and f x is negative.

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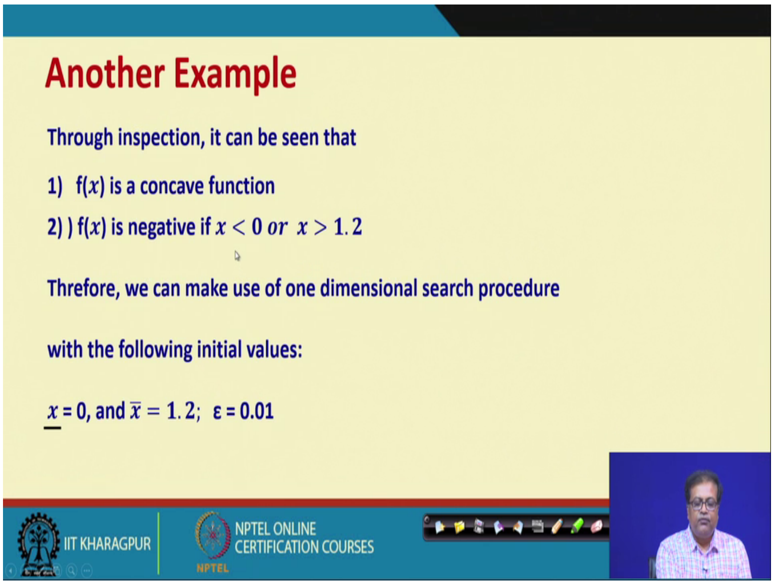

If x is between x is less than 0 or x is greater than 1.2 so; that means, x underline you can take as 0 and x bar 1.2 let us take epsilon is 0.01.

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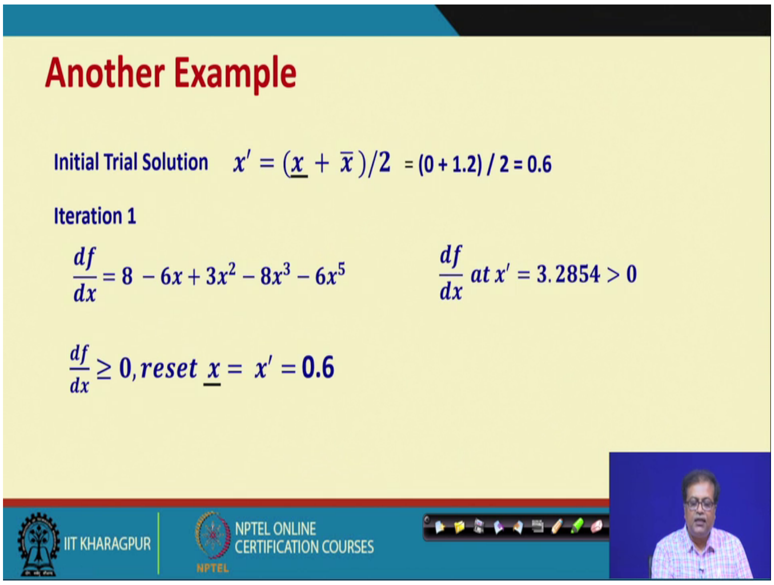

So, at the first detraction x dash comes to 0.6 and if we evaluate df dx you know at x dash it comes to 3.2854 that is greater than 0; that means, this is an upward slope; that means, we reset lower values the lower value we then become 0.6 right. So, again we take average and we take get 0.9.

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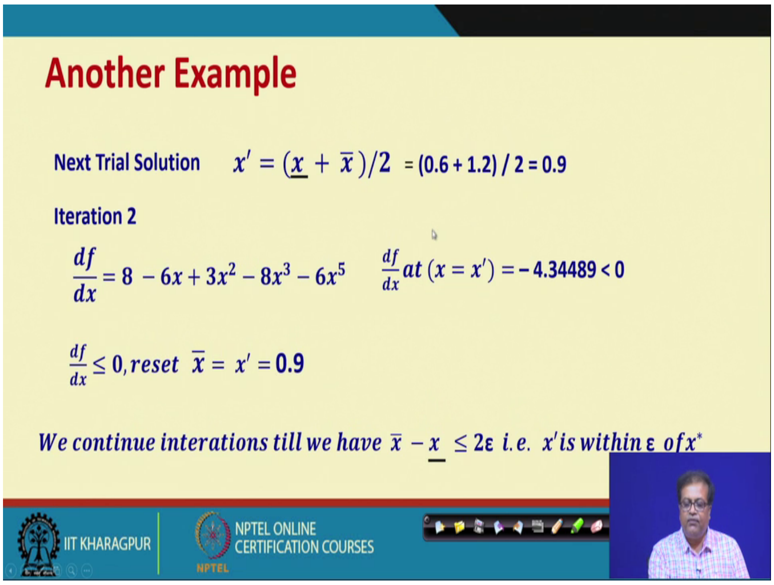

And again if we take the gradient we find this time the gradient is negative; that means, it is on the other side; that means, reset the x bar now that is on the upper limit to x dash equal to 0.9 like this we continued. So, here is the computation table.

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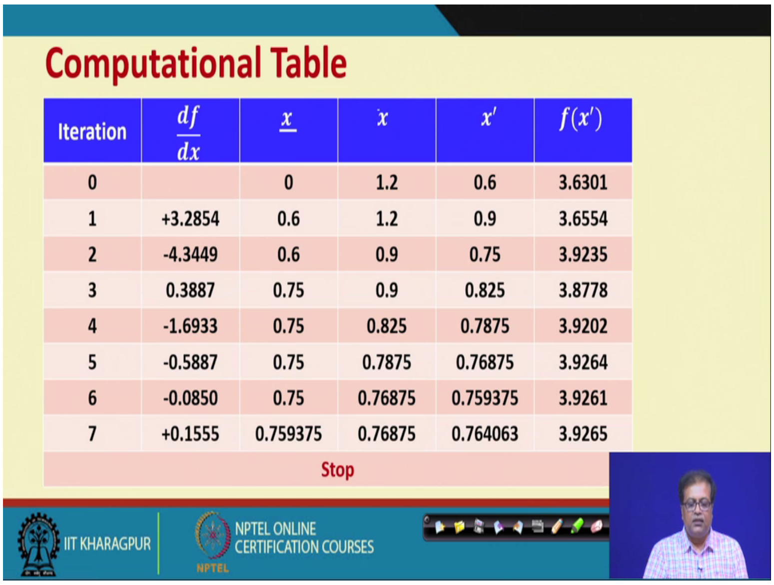

Like this we continue and as we continue we finally, after the 7-th iteration we get reasonably good value.

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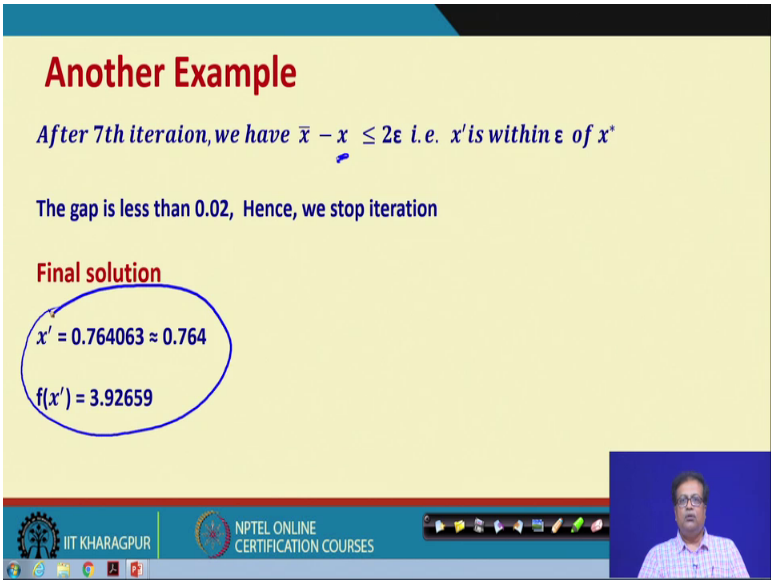

So, we have the x bar minus x underline equal to we stop and the gap is less than this and these are our final solution right. So, this is how we can solve the single variable NLP problems by first derivative and the second derivative right. So, these are on the easier side, next class we shall see how multiple variables are to be considered, right.

Thank you very much.