Selected Topics In Decision Modeling
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Lecture – 21
Non-Linear Programming: Introduction

Right for our course Selected Topics in Decision Modeling; today we are going to begin the Non-Linear Programming and in this particular topic it is non-linear programming today the first lecture that is Introduction.

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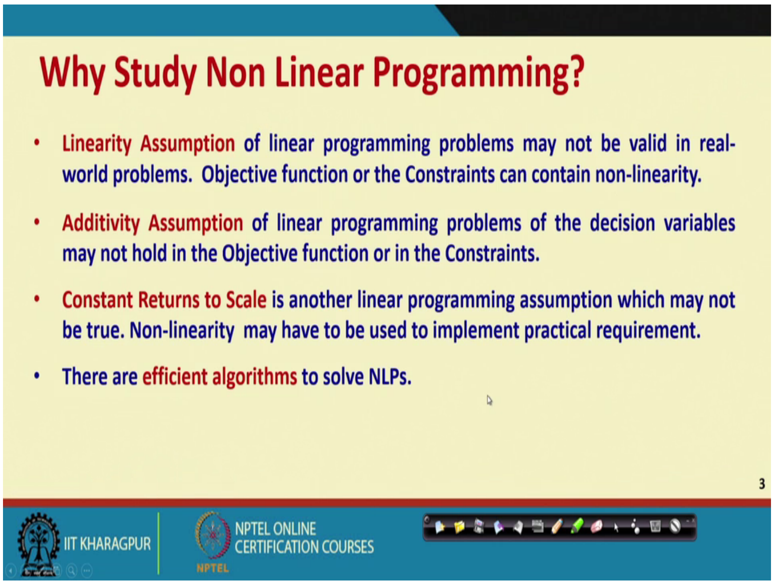

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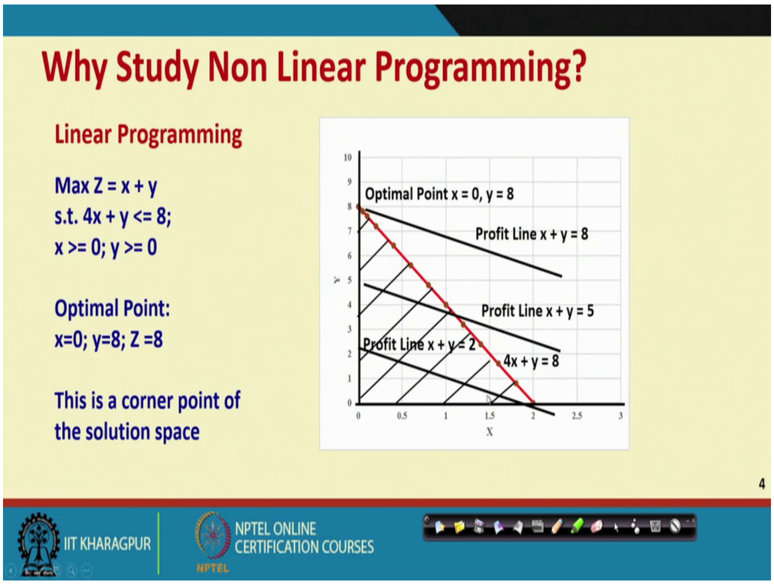

So in the introduction is the first lecture that we shall have and basically you see the non-linear programming first of all; how it is different from the linear programming problem that is a very basic issue that we should understand; there are certain assumption if you really recall about the linear programming.

What are they the linearity assumption, then the additivity assumption there was a constant returns to scale and certain other assumptions. Out of all the assumptions the first assumption, that is the linearity assumption will may not be valid for the non-linear programming problem is it alright; so the objective function or the constraints can actually may contain non-linearity. What exactly it means? That they are not linear functions; that means, they cannot be represented by straight lines there are not first order constraints, but the objective function are the constraints they can be higher order functions is not may not be first order.

Then there is this additivity assumption;, which essentially says the decision variables may not hold you know in the objective function that or in the constraints. What happens in additivity assumption suppose we have two variables that is x 1 and x 2 is it alright. So what we are saying that x 1 and x 2 they are using a say common resource is it alright; so we have may be the common resource is a 40 and we say that 2 units are taken by the first and 3 units are taken by the second per unit so therefore, we might a put a constraint 2 x 1 plus 3 x 2 less than equal to say 40.

Now while we write this constraint there is an inherent assumption that we can add 2 x 1 and 3 x 2 right; that means, they really consume the same resource and in that proportion, but supposing they just cannot be added in that manner ah; that means, the additivity assumption is not holding is alright; so that is could be another the assumption that may not work.

Then there could be constant returns to scale, so when we write objective function we write that maximize z equal to say x 1 plus x 2 inherent assumption here that the profit per unit you know is x 1; so profit of 5 units will be 5 suppose 1 unit is 1 then 5 unit should be 5, but then sometimes we see that profit is something for 1 unit, but it might exponentially raise for more number of units.

So, in that case x square is a better function then x the that brings non-linearity as well there could be other reason of studying NLP that there are efficient algorithms are available for the NLP problems right. So having down this let us look at another very vital thing about linear programming.

Let us take a very simple problem maximize z equal to x plus y subject to you know here the x plus y subject to 4 x plus y less than equal to 8; x and y both greater than equal to 0. So what have we done here look here we have drawn a line that is 4 x y x plus y equal to 8 and you know this side is 4 x plus y less than equal to 8 and we have a solution space given by the this and the profit line which is x plus y so you can see that actually x and y scales are different x is 0 to 3 y is 0 to 10 that is why the lines are different.

So, x plus y equal to 2 this is the line x plus y equal to 5 this is the line and x plus y equal to 8 this is the line. So you can see that these line will cut at you know this point so which is your, you know the optimal point x equal to 0 and y equal to 8 that becomes optimal.

In other words the basics feasible solutions are essentially the corner points of the solution space here there are three corner points so 3 will be 3 basic feasible solutions and the optimal solution is one of the corner points since the optimal solution lies at one of the corner points in the linear programming problem; you know that is the advantage the basic advantage of linear programming problem is that the solution lies at one of the corner points right.

So, we have to search finite number of points once we get the solution space you know there is no need really to worry anymore and we just explore those you know corner points the feasible corner points or sometimes called basic feasible solutions and we know one of them is going to be our optimal solution. That was a very basic advantage of linear programming let us see what happens in non-linear programming.

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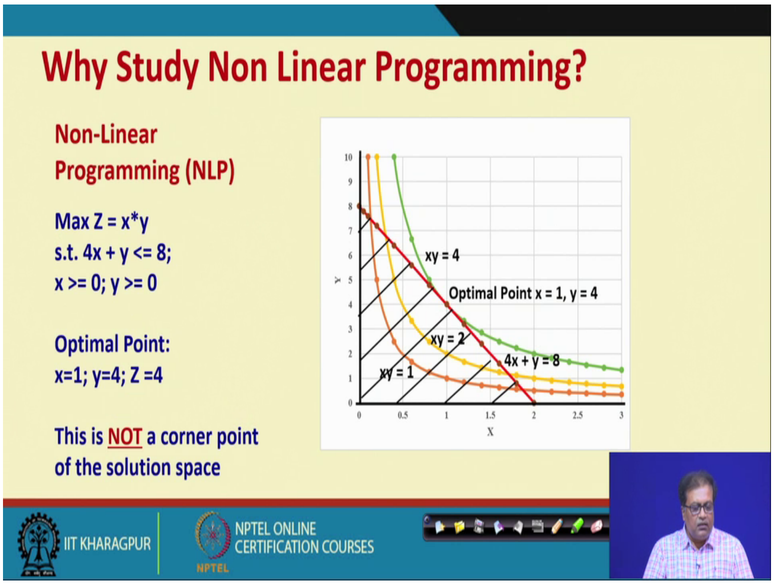

The problem is same, but objective function is different earlier we had x plus y, now here if you see the objective function is x star y right not x plus y subject to 4 x plus y less than equal to 8 and x and y greater than equal to 0; the problem is same since the problem the constraints are same the solution space is also same.

So, it is a same solution space between 2 2 8 the line that is 4 x plus y equal to 8 and the objective function x y see this is the x y equal to 1 line, this is the x y equal to 2 line and this is x y equal to 4 line. So if you look that the x y equal to 4 line just touches you know actually these line 4 x plus y and these x y equal to 8 x y equal to 4 just touches at a point and you know that is our optimal point, because if I take any lower value of x y maybe x y less than equal to 4 less than 4 then; obviously, it will be the lower objective function value, but any higher x y value will be you know not touch the solution space at all. So, therefore, that is our optimal point the optimal point is then x equal to 1 and y equal to 4.

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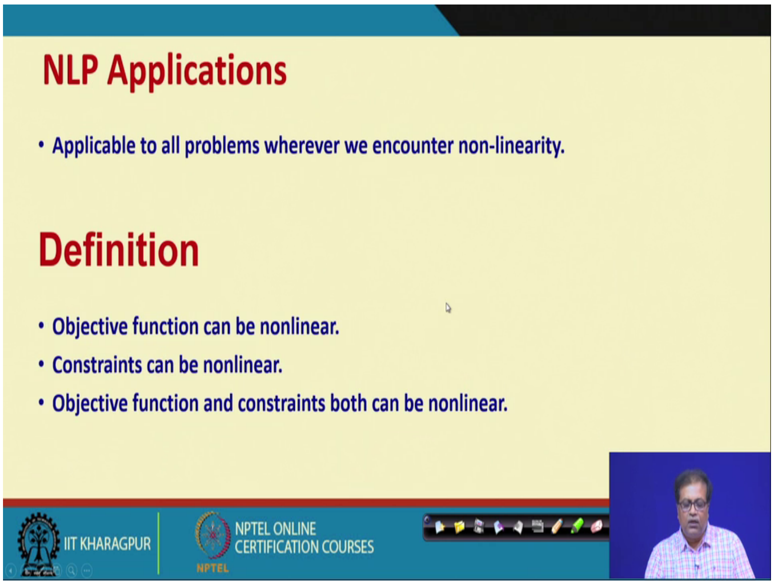

Now, if you if you if you carefully see then these point is you know is not a corner point so that is the basic point to note the even if we have only 1 variable only the objective function is non-linear, look here the constants are still linear here the objective function is non-linear, the optimal point actually does not lie on the corner points right.

So the basic advantage that we have a the linear programming that our such space was finite; we see only the corner points that advantage is gone right. So the basic advantage of linear programming is not available in non-linear programming; so you know you can understand that the solution space the solution methodology has to be different.

We just cannot use the standard linear programming methodology for solving non-linear programming problems. Sometimes even we might have the solution when we might be having a constant lines which are also non-linear maybe the optimal point may be within the solution space; may not be on the boundary also is it alright.

So you know all kinds of things may be possible then the optimal solution could be a corner point, optimal solution could be a point on the solution space boundary or it can be also within the solution space right. So infinite number of points are now candidate solutions; instead of finite set of corner points what was our advantage in linear programming.

So, these are another very important aspect that we should also be aware of. So the applications of non-linear programmings are applicable to all problems wherever we encounter non-linearity. Truly speaking almost all the linear programming problems you know with a little variation there could be non-linearity. Just take a very simple transportation problem we have the cost function which was usually we take c i j cost per unit you know transportation, but supposing the cost is not linear right for 1 unit they cost is something, but for more units cost is not proportional we have a non-linear problem.

So in any situations wherever we have applied linear programming we must remember that those are actually special cases, the general case can be a non-linear programming problem. Now objective function can be non-linear, constraints can be non-linear and or objective function constraints both can be non-linear right those are the possibilities of the NLP problem.

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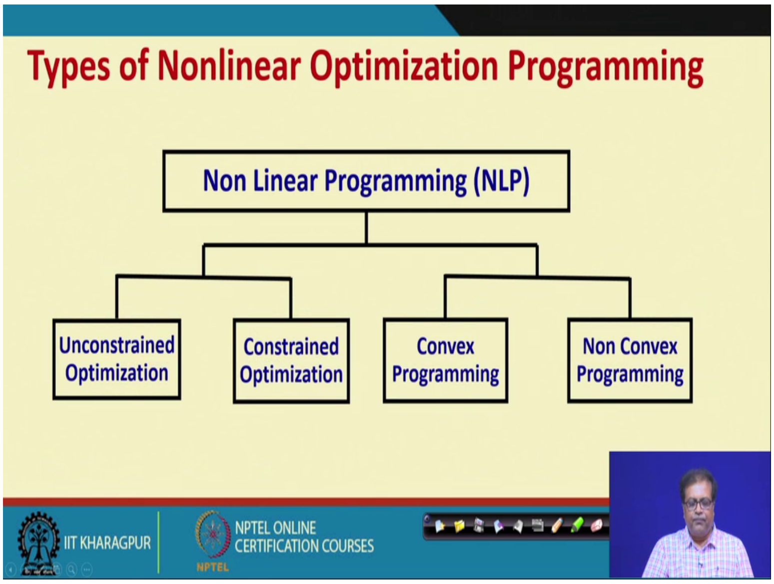

Now, broadly we may classify the non-linear programming problems NLP into this 4 different classification actually not 4 actually 2 2 the problems could be unconstrained or constrained or problems could be convex or non convex is it alright. So there could be unconstrained problem as the name suggest that there are no constrains there is only objective function; the constrained optimization on the other hand there is an objective function and there are constrains also.

It is alright these those problem will be more complex whereas, if we have unconstrained optimization we only have a some function that is the objective function, but on the other hand there could be convex programming and the non-convex programming that will depend on the type of functions that we are you know encountering. So that is how we can have different types of linear programming problem and there could be other variations as well; so all of those variation we shall study as an when we get those kind of problems.

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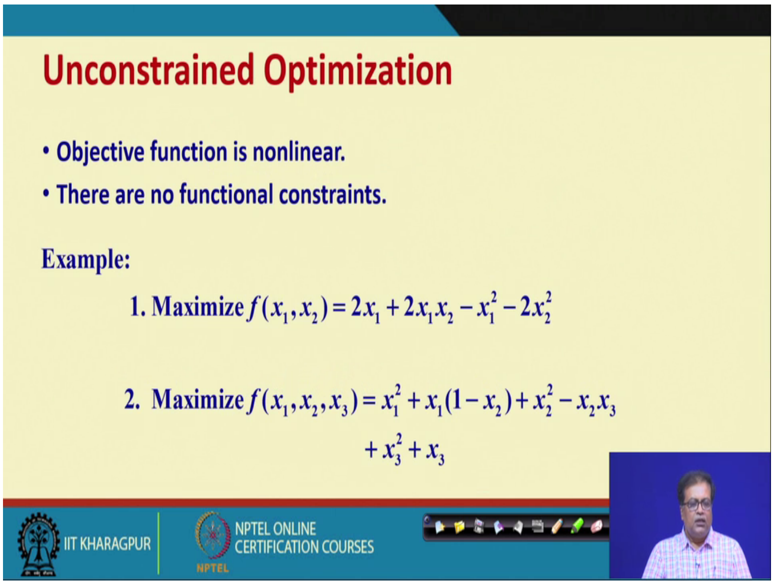

So, let us look at an unconstrained optimization problem, so we have objective function which is non-linear and there are no functional constraints so here are some examples maximize f x 1, x 2 equal to 2 x 1 plus 2 x 1 x 2 minus x 1 square minus 2 x 2 whole square 2 x 2 square. So see these is a non-linear function, but there is no constraints so that is an unconstrained optimization problem. So similarly you can also have a 3 variable problem x 1 square plus x 1 into 1 minus x 2 plus x 2 square minus x 2, x 3 plus x 3 square plus x 3 etcetera so these are all unconstrained optimization problem.

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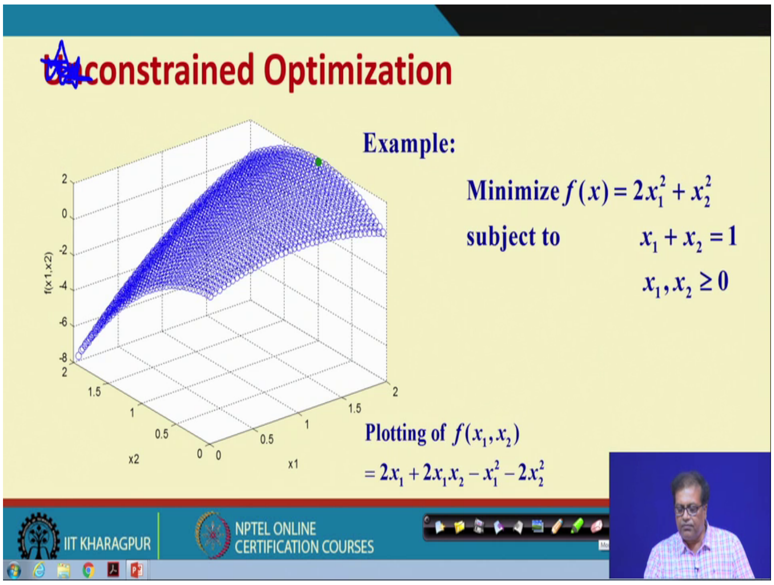

The unconstrained optimization problem so you know the constraint on the other hand, so here if you if you look at this is not an unconstraint so there is a hence has to be you know this is not unconstraint so we cannot so you can cut this part; so this is a constraint optimization problem.

So, the constraint optimization here we have a minimization function and subject to there are some constraint so we have a constraint here and that could be a minimization function. So if we plot you know this curve then you can see that these x versus f x this plot is given and these particular function actually can look like this; so that is a constrained optimization problem.

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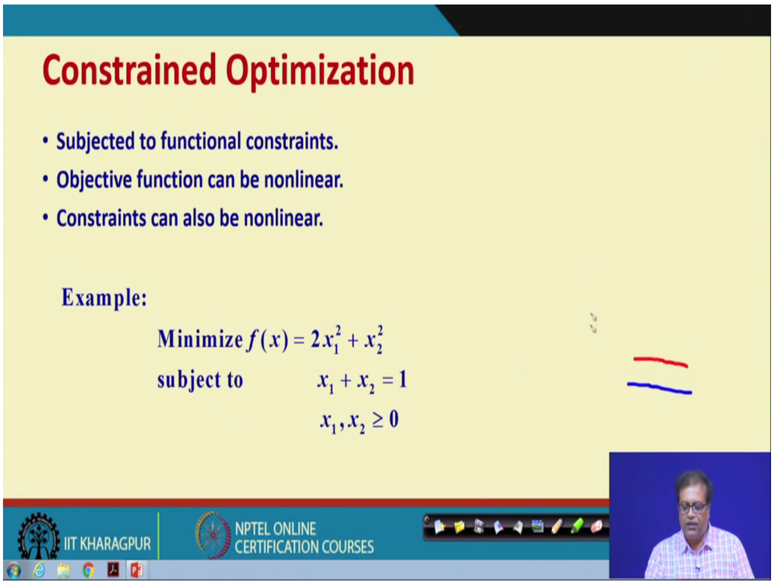

Now let us look at another problem that is you know the constraints this is there is another problem; that is a constraint optimization problem. So what really happens here subjected to functional constraints, so both objectives and the constraints can be non-linear right so when objective functions and the constraints both can be non-linear we have a constrained optimization problem right.

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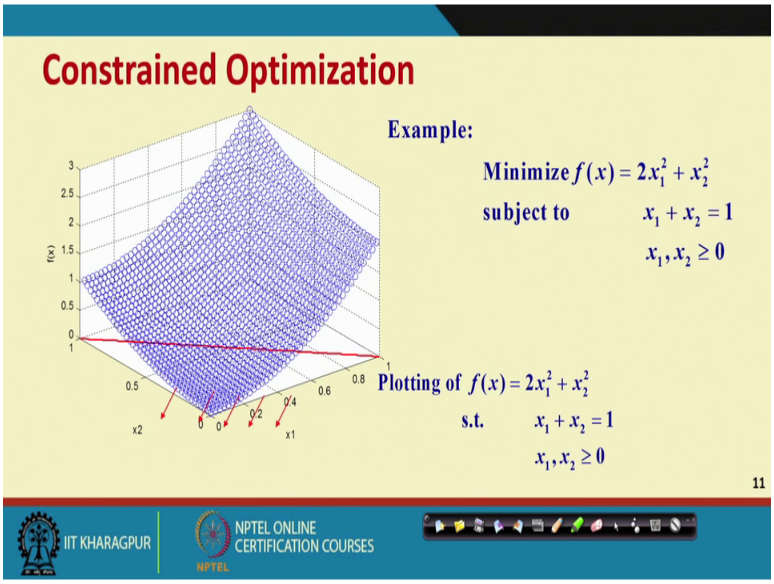

So, there are constraints and unconstrained optimization problem; so let us look at a constraints optimization problem. So see this constrained optimization problem if you see that here the this is the function the function is 2 x 1 plus x 2 subject to this constraints and this is the plot and see this actually this is a x y plane the f x is plotted on this; so you can see since it is a minimization function we have to really see that this is our solution space; so within this solution space the functions which is the minimum value of this function right so that is what we have to see. Incidentally this happens at 0 probably, but anyhow that is that is that is a particular problem so we might have a objective function which could be maximize or minimize.

And there are set up constraints so we can have the solution space the solution space is in the x y plane, but the objective function which is separately shown by the f x function; so where we have the minimum f x value within the solution space. So you can see that this is different from the way we deal the our linear programming problem you know reconsiderations are very different here.

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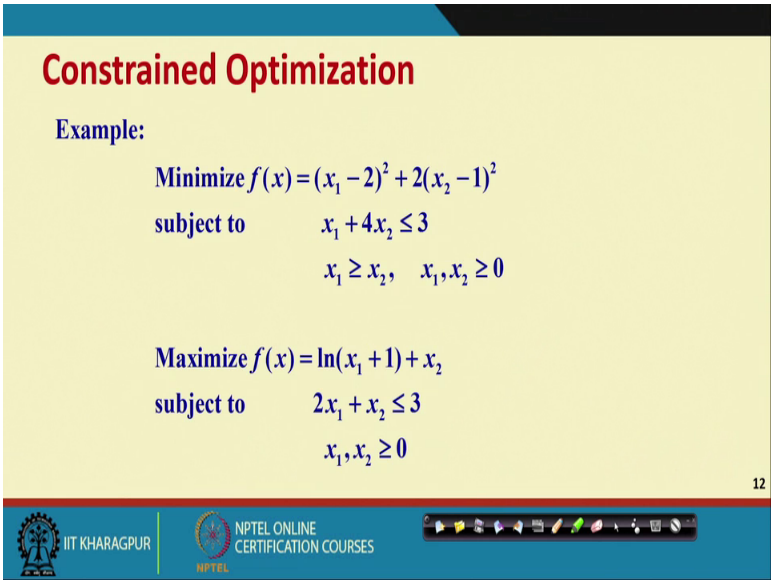

So, these are other examples of constraint optimization problem; so you can have a minimization problem were f x is x 1 minus 2 whole square plus 2 x 2 minus 1 whole square subject to certain constraints or we can have an ln function ln x 1 plus 1 plus x 2 subject to certain constraints; so you can have several types of unconstrained and constrained problems.

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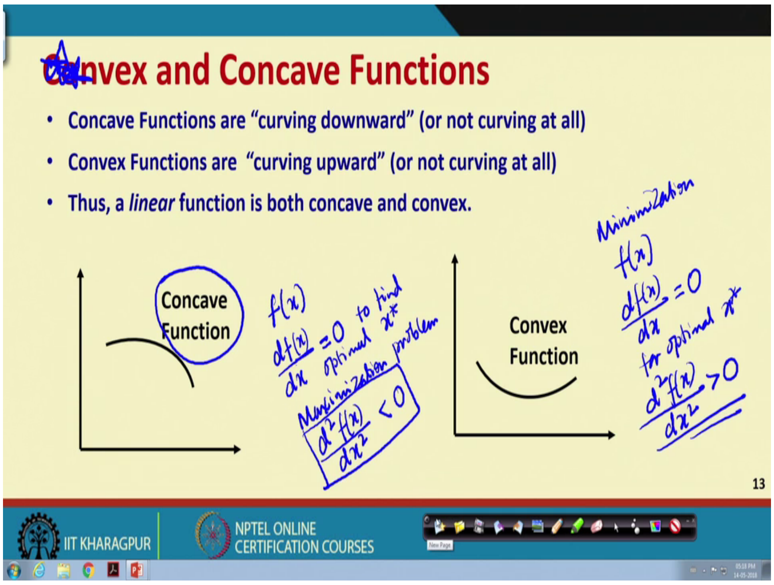

Now a very very important consideration is the convexity and concavity of functions right. So as you can see here the 2 examples concave functions and convex functions the concave functions are curving downwards whereas, convex functions are carving upwards or not curving at all. So in that sense a linear function can be considered both as concave as well as convex right; so because neither they are curving downwards nor they are curving upwards, you see a very important thing to note that if we have a maximization problem the maximization problem will have you know a maximum point.

So you can see this concave function you know these kind of functions were you can find out the maximum of a variable whereas, the convex function you can have the minimum of the function you cannot have a maximum value for the convex function by differential method. So how what is the differential methods let us see that, so supposing if we find the suppose we have the function as f x right and we find what is known as df dx df x dx that is the gradient right.

So what is the method? Method is that we put gradient equal to 0 right so if we find gradient equal to 0 to find optimal that is x star so we find the gradient equal to 0 to find x star, but there is a point the point is that this will be and suppose we have a maximization problem then additionally we have to take what is known as a second hand derivative; the second derivative should be negative right. For a maximization problem the second derivative should be negative that is the point to be noted whereas, the so you can see that really if you are having a maximization problem then your function should be concave objective function; on the other hand for a minimization problem, so minimization we have again df x dx equal to 0 for optimal x star, but note that we must have the second derivative should be positive right.

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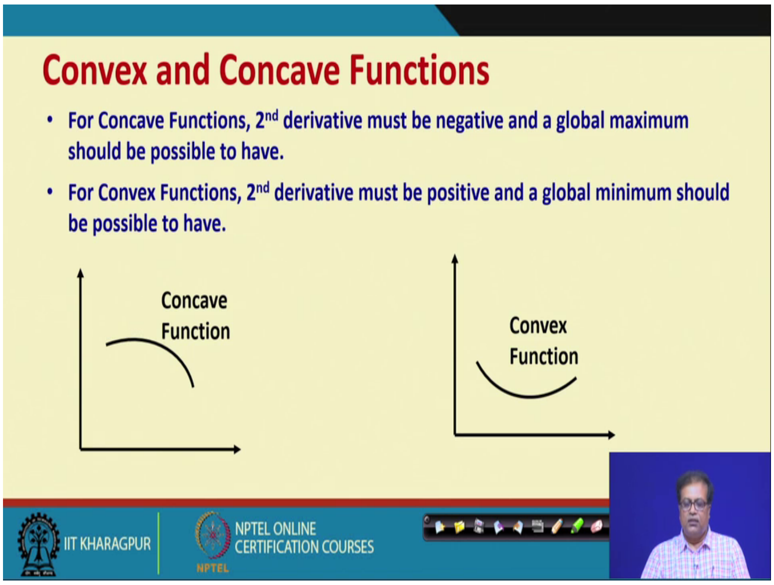

So these are some of the considerations that we must have for the convex or the concave kind of functions right; so therefore, we must have that kind of consideration. So that is what we say that for concave function second derivative must be negative and a global maximum should be possible to have.

So just now I explain that for convex function, second derivative must be positive and a global minimum should be possible to have is it alright. So by taking a suitable second derivative we can find whether it is negative or positive and we may find therefore, the maximum or minimum of a function by search considerations.

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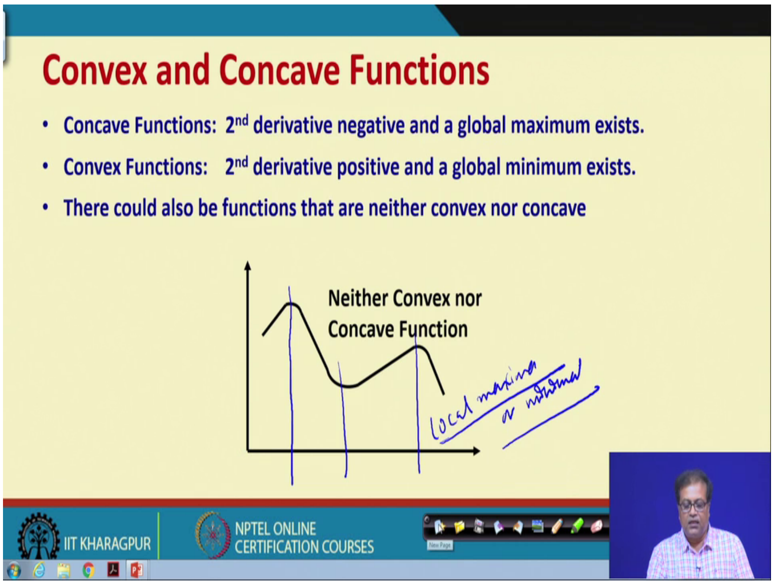

Now, sometimes the curve could be something else, so you see if you look at this curve is it you know going down or going up so you know it is very difficult to tell so it is neither convex nor concave function. So here if you find derivative you know what may happen you may actually find the suppose these points you see their local maxima or minima.

So, for such kind of a problem you might find local maxima or minima and you may not be able to find the global maxima so easily right. So that is why the convexity and concave concavity of the functions are so important; so you know then we might have to use some other methods like evolutionary computing right.

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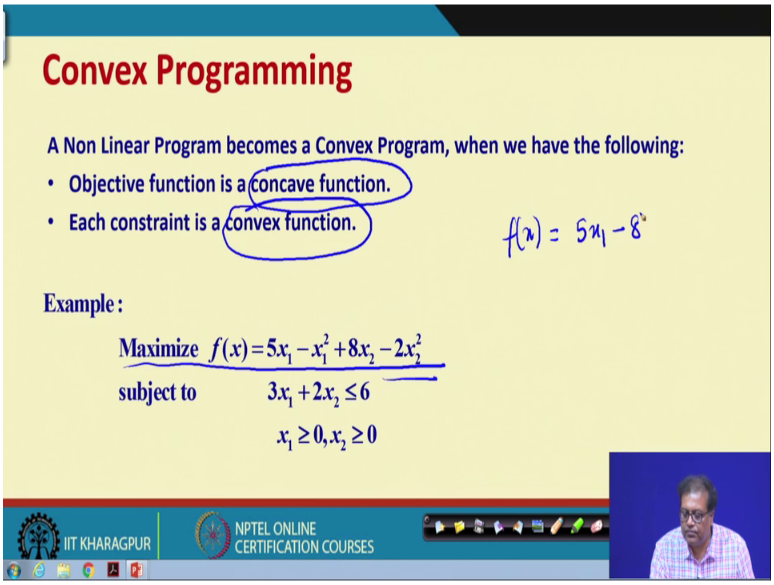

So, therefore, we see that that convexity and concavity are very important considerations particularly for maximization problems concavity is important and for minimization convexity is important. Now that brings us to the convex programming the essentially if you have non-linear program to become a convex program we must have two things number one; the objective function should be a concave function and each constraint should be convex, convex means it could be linear also because linear is both convex and concave so objective function should be concave.

Now, when you see this particular maximize f x equal to say 5 x 1 minus x 1 square plus 8 x 2 minus 2 x 2 square; how do you know it is concave right all that you know if I take its second derivative then the second derivative should be negative right because it is a concave function.

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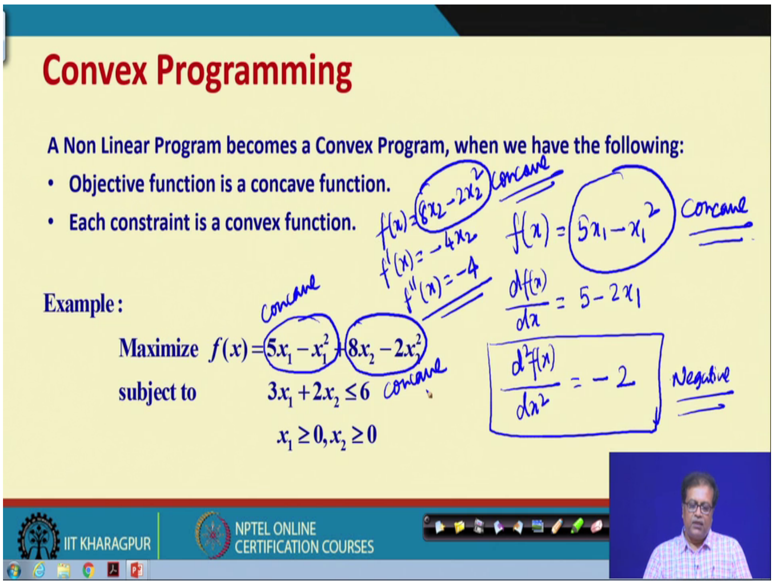

Now let us take a very simple case supposing our f x is only this much f x is 5 x 1 minus x 1 square; so what is the first derivative? First derivative will be 5 minus 2 x 1.

What is the second derivative second derivative will be minus 2 right so this is negative; which means that this portion is concave right. So you can similarly take this second part also like so we have taken the first part we can also take the second part and suppose we take another function f x equal to 8 x 2 minus 2 x 2 whole square then we find f dash x that is the first derivative will be equal to minus 4 x 2 and f double dash x becomes minus 4. So you see so this will also negative so we find that that is also concave.

So, now an interesting thing is that if I can partition a particular function in to separate portions I can show that this is concave and this is also concave then; that means, the total function also concave is alright. So that is how you can also do, but then this is very typical not very simple process so we have to solve such problems as an when we find then right.

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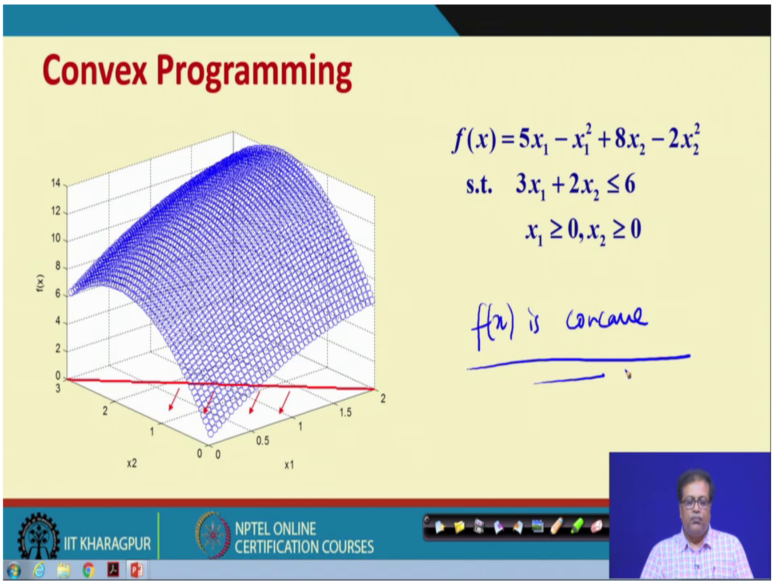

So that is about the convex programming; so you see the same problem again you know I have plotted the function so if you look at the plot then you can see that basically it is a convex kind of a function is not alright, from the nature of the plot you can you can find out that this is actually a convex concave function is not alright. So we find that the f x is actually concave right, since f x is concave it is possible to find what is known as a maxima around any difficulty right.

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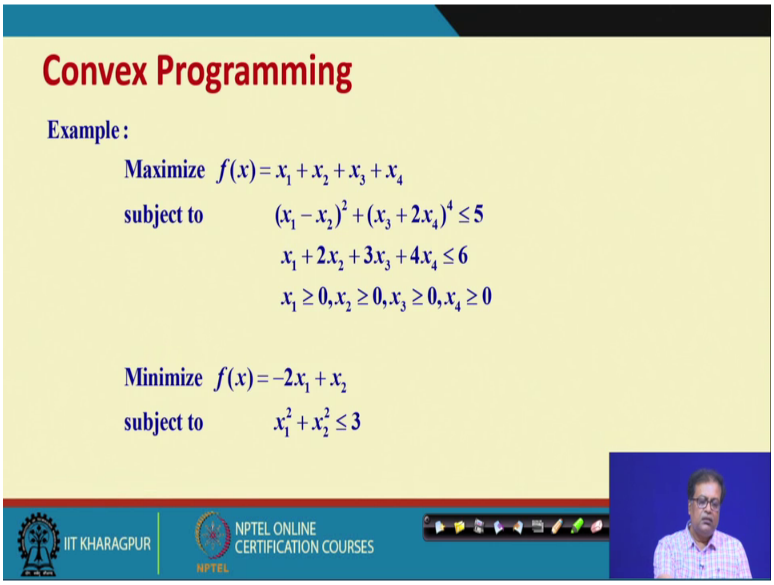

So having said that; let us now see other examples so for example, minimize f x equal to x 1 plus x 2 x 3 plus x 4 and subject to all these constraints or minimize minus 2 x 1 plus x 2 whole square x 1 square plus x 2 whole square greater than less than equal to 3. So there could be several examples of these kind of thing so like this is a maximize pro program things so these are other examples.

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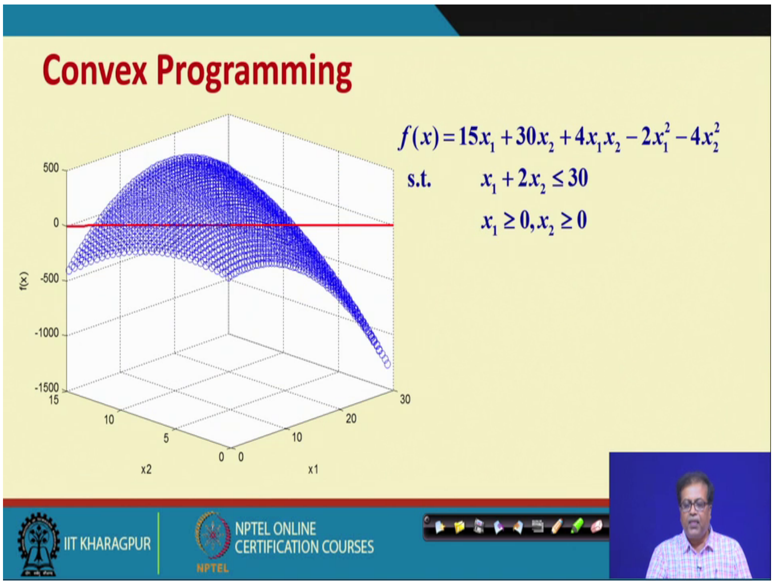

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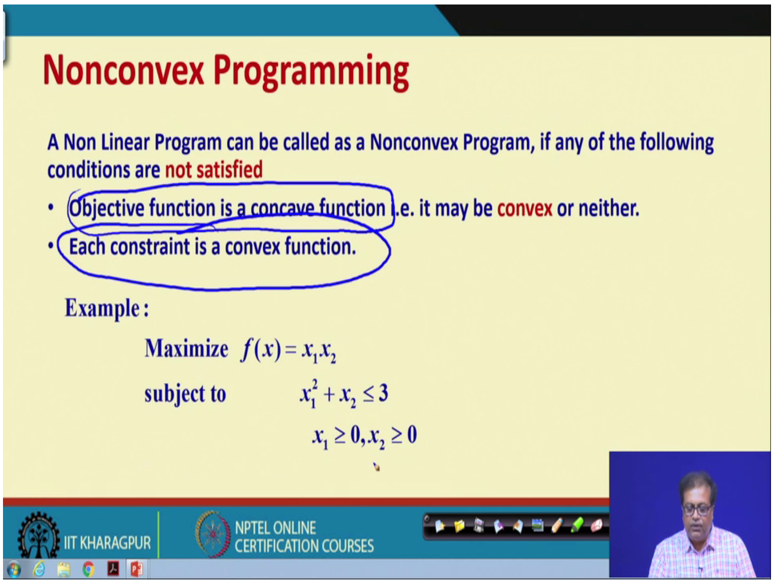

So here is another example of convex programming, so this is the plot that is shown here. The non-convex programming usually it is the convex programming and the non-convex programming so in non-convex programming the those to constraints that we have had that is the objective function is a concave function and each constraint is convex, they may not hold. So when any one of this conditions are not satisfied we have what is known as non-convex programming, for example, suppose f x equal to x 1 x 2 and these are my constraints so what we may have if we if we plot then let us see one such example.

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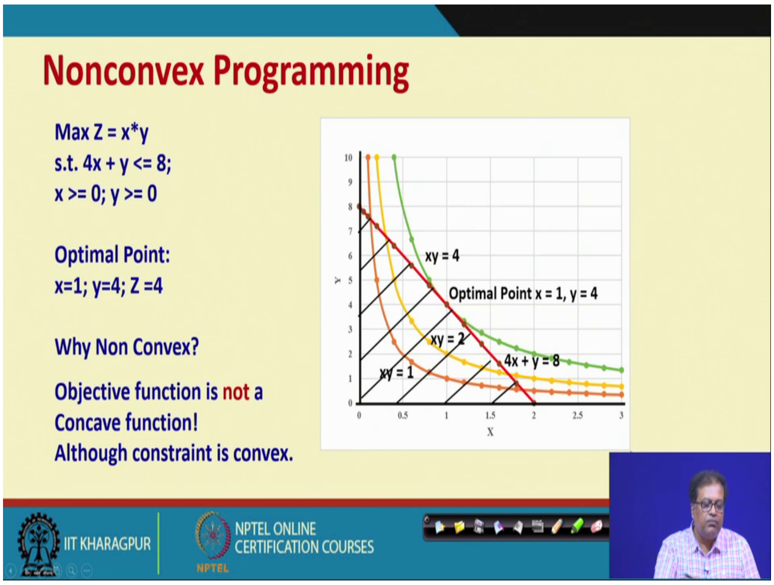

If we see an example these example we have already seen that a maximize z equal to x y and 4 x plus y less than equal to 8; you see the objective function is convex right since the objective function is convex you know we have the objective function concave that is you know not possible. So therefore, it a it is a non-convex programming; so since objective function is not a concave function although the constraint is convex because it is linear, but then we call it a non-convex programming.

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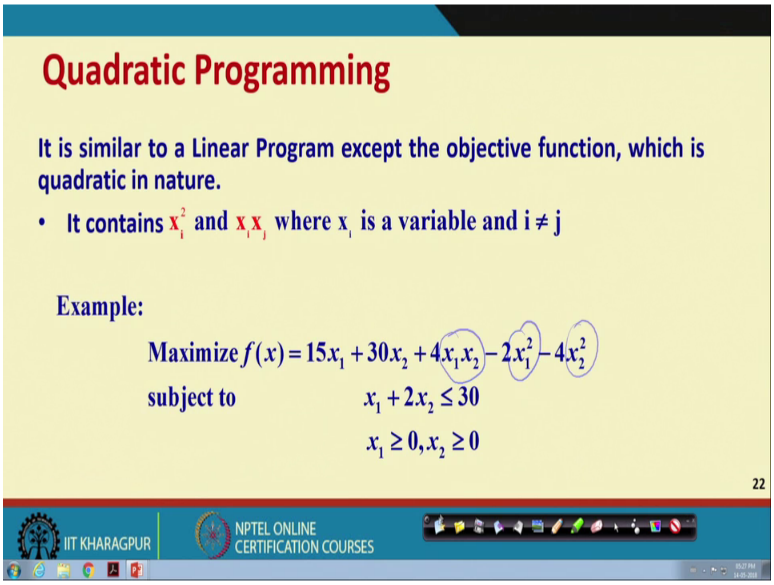

Then we have what is known as the quadratic programming; the quadratic programming is a similar to a linear programming except that the while the constraints are linear the objective function actually contain quadratic terms like see like x 1 square so you can see that we have the square terms and we have the product terms.

So, when we you have such kind of terms we actually call them quadratic programming. So quadratic programming has got their own methods of solving which we shall see in due course of time.

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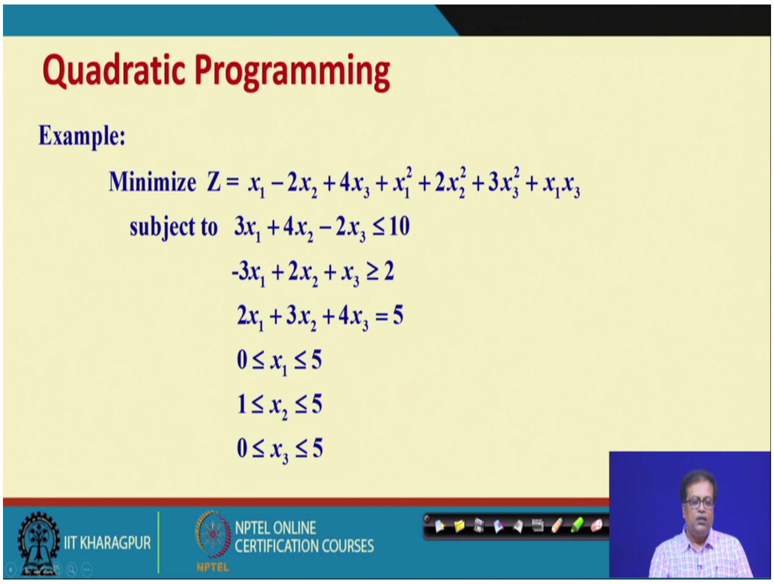

So here is another example such kind of examples again you can see that there are the square terms or the product terms. So when you have such situations we call them quadratic programming alright.

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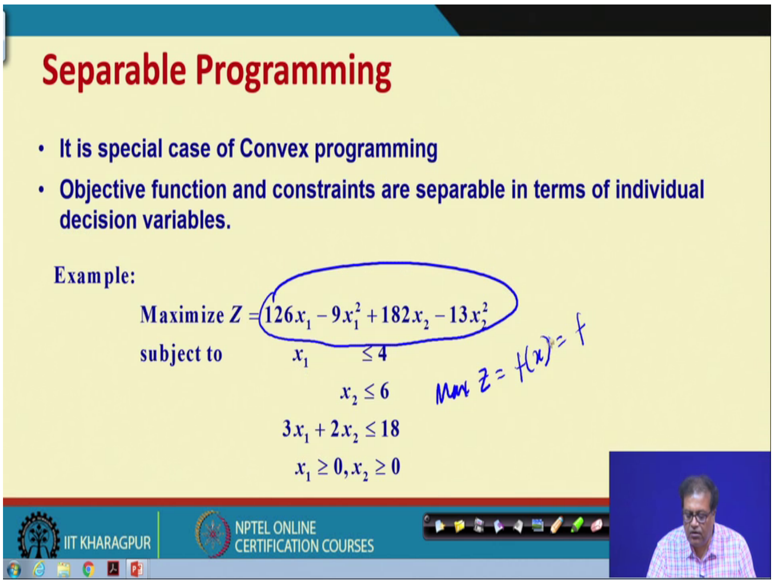

Ah There is another type which is known as separable programming; what happens in special separable programming if the objective functions or the constraints are having multiple variables, but they are actually can be separable in terms of individual decision variables, so here is an example so supposing this is our objective function now, you see maximize z equal to f x now which is basically you know is a function of 2 variables.

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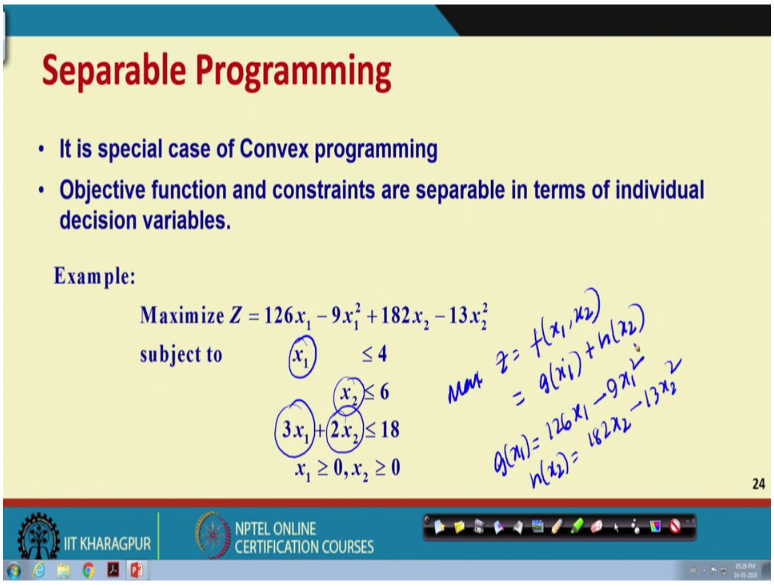

Maximize z equal to f of x 1 and x 2 which is possible to write you know g x 1 plus h x 2 right. So what is g x 1 here; g x 1 is 126 x 1 minus 9 x 1 square and h x 2 equal to 182 x 2 minus 13 x 2 square right. So you can see even this one the constraint this is only x 1, this is only x 2 this is 3 x 1 and 2 x 2 they can be separated.

So, when you can do such separation into functions of individual variables then those kind of programming are called separable programming. So we shall see later on how such kind of problems could be solved right so leave it here in the in this particular lecture we introduced various types of non non-linear programming problems.

Thank you very much.