Selected Topics in Decision Modeling
Prof. Biswajit Mahanty
Department of Industrial and Systems Engineering
Indian Institute of Technology, Kharagpur

Lecture - 19
Travelling Salesman Problem (Contd.)

So, in our course Selected Topics in Decision Modeling; today we are in our lecture number 19 and we are in the middle of the Travelling Salesman Problem discussions.

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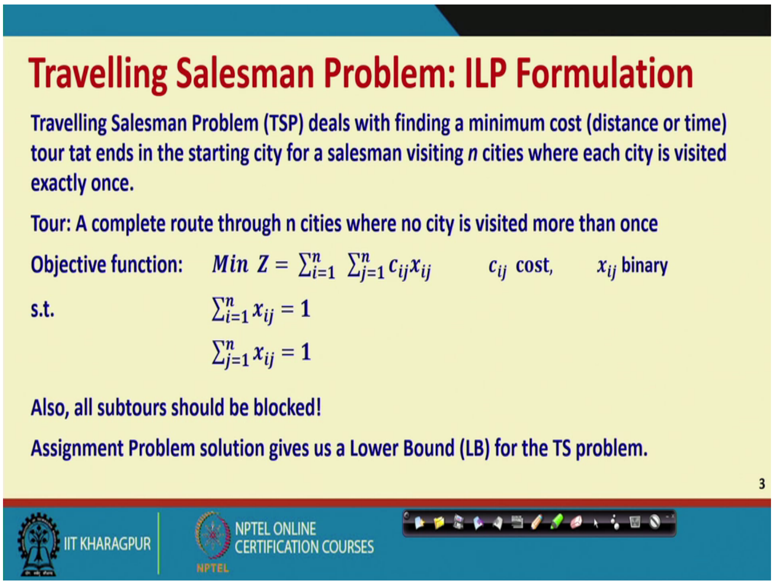

So, travelling salesman problem as you have seen earlier that the formulation of the travelling salesman problem; as an integer linear programming becomes like these. So, usually what happens in a travelling salesman problem; the salesman travels from a given city to a number of cities and goes to all the n cities exactly once and comes back to the city of origin.

So, because of the nature of the problem you know the formulation becomes minimize Z equal to sum over i sum over j C i j x i j where i is the starting city and j is the ending city and x i j will have only 0 or 1 value and subject to the sum over x i j equal to 1, sum over i as well as sum over j; that means, there should be only a single starting city and only one city where it will arrive; the salesman will arrive.

But that much formulation as we already have seen is the same as that of an assignment problem although that two problems are very different. The additional constants that you also find in a travelling salesman problem is that; the sub tours must be blocked, right we should have a complete tour and we should not have any sub tours. So, we have seen before that how to block those sub tours how to right the constants, constraints for the sub tours blocking we are not going to discuss that anymore; only thing to note here the assignment problem solutions gives us a lower bound for the TS problem.

So, we have solve the problem in our last class if you recall where you know we had the assignment problem solution and the same assignment problem solution comes up, comes out as the optimum solution for the travelling salesman problem. So, assignment solution gives a bound, but this bound if it becomes optimal for the travelling salesman also then there is no need to solve any further or in other words the branch and bound is not really required.

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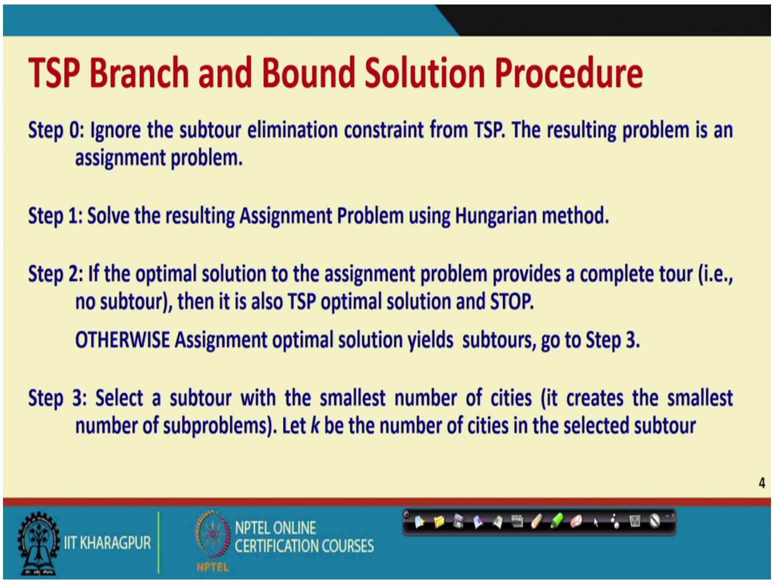

But what happens if it is not? So, as you can see here the steps once again the ignore the sub tour elimination and the resulting problem is an assignment problem, solve the assignment problem by Hungarian method and if the optimal solution provides a complete tour; that is no sub tour then it is also the travelling salesman problem optimal solution so we stop. So, in our previous example we have solved the assignment problem and we stopped.

But what happens if the assignment optimal solution is not giving a complete tour? If we have a sub tour then we should go further. What we do? We take out of the difference I know there would be some sub tours instead of complete tour, take the smallest sub tour; let us say the sub tour has got two legs.

So, we branch the problem the main problem to two branches and if the smallest sub tour has got let us say three branches then we branch into three such branches.

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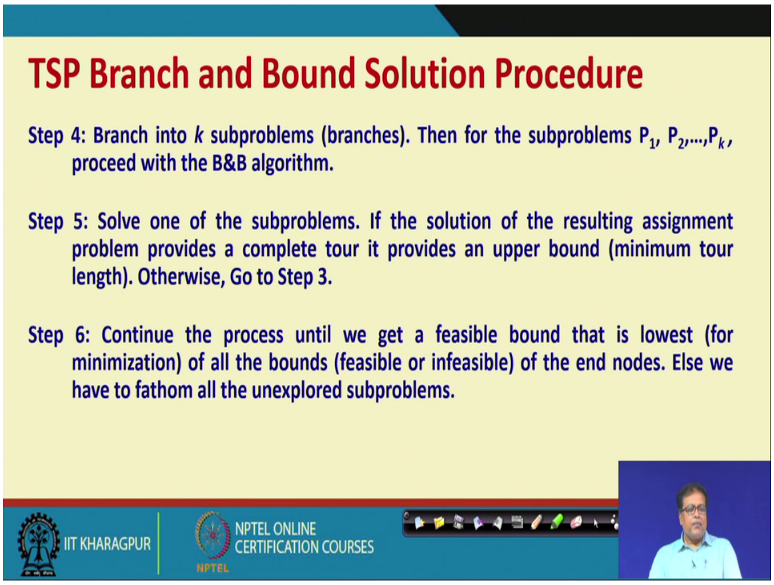

Then assume there are k sub problems or branches; then solve you know the sub problems P 1, P 2, P k and proceed with the B and B algorithm; what is that solve one of the sub problems if the solution of the resulting assignment problem provides a complete tour it provides an upper bound otherwise go to step 3.

Really what it means? It means that if the sub problem really becomes the minimum tour length; how to know that minimum tour? Basically check all the n node that is our step 6, that is it is the lowest of all the bounds of the n nodes, but if it is not then we have to fathom all the unexplored sub problems right.

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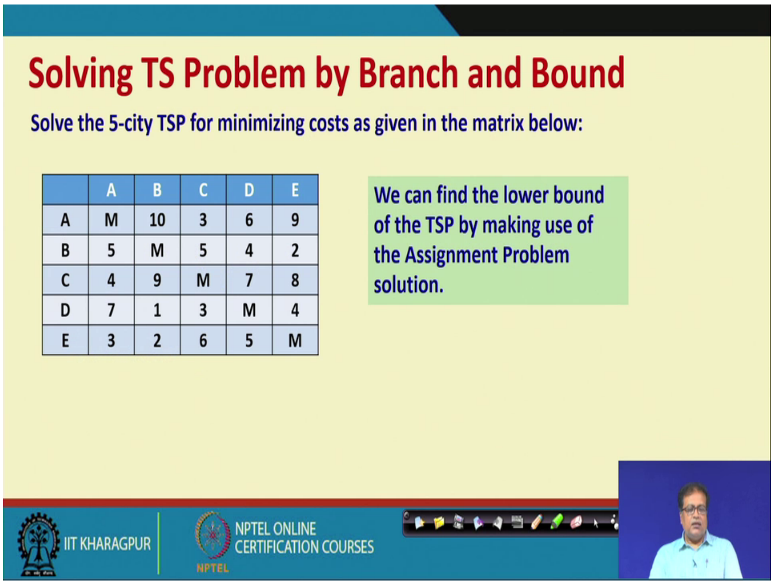

So, those are the steps now let us look at problem and see how these steps are executed. Supposing we have a 5-city travelling salesman problem has given here and we need to solve it. So, first of all we need to find a lower bound by solving the TSP by the assignment problem.

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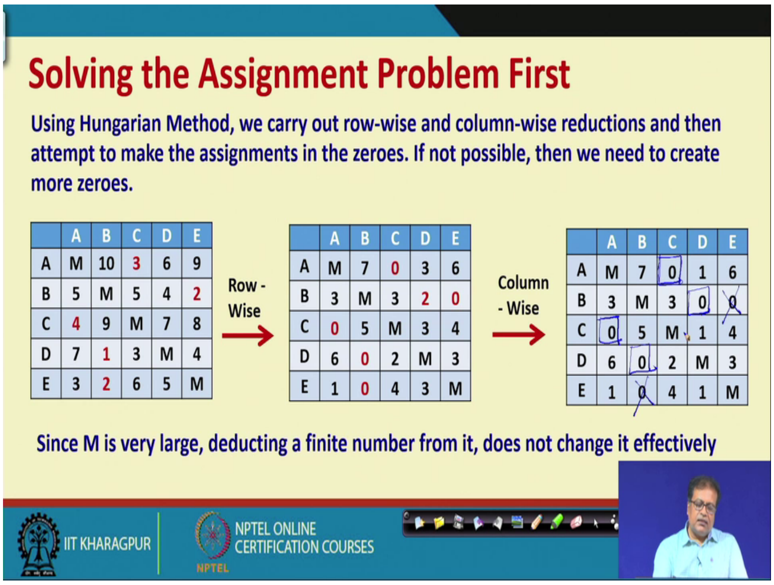

So, let us see how to do that; so solving the assignment problem first so as you can see here the particular problem that we have taken for Hungarian method; what we have to do? We have to do row wise and column wise reductions and after doing the row wise and column wise reductions the 0's that we shall get we should try to assign mean those values. So, see first of all you know you have these matrix here so those which are identified by red those are our column wise minimum, row wise minimum values. So, in the first row the minimum is 3.

So, we deduct 3 from all the elements look here 10 becomes 7, 3 becomes 0, 6 becomes 3 and 9 becomes 6 is it alright. So, we have the row wise deduction of the minimum amount. Similarly, we do for all the rows; so after doing all the rows if you see we have got 1 2 3 4 5 0's is it alright.

Now, we have to do column wise reduction. So, column wise you see 0 is minimum 0 0 2 minimum here and 0. So obviously, if it is 0 medium minimum there is nothing to deduct, but in this case because the minimum is 2 we have deducted 2 and we got 3 becomes 1, 2 becomes 0, 3 becomes 1, M remains and 3 becomes 1. Why M remains because M is such a very large number essentially that A to A B to B C to C we cannot assign. So, they are replaced by very high numbers.

So, see these are our 0's look at the 0's. So, we have all these 0's right. So, all these 0's we have to now make assignment. If you look carefully then you see that some 0's are in pairs right because those 0's are in pairs, you know if you assign in one the other one cannot be used.

So, can we make all the 5 assignments; let us start with some unique 0. So, this one is an unique 0; suppose we make some assignment here right. So, let us do once again supposing we make some assignment here in this unique 0 and then this is also another unique 0 so we make some assignment here. So, we made 2 assignments, but then out of these you know the remaining once suppose moment we have these 0 these 0 is gone, moment we have these 0 then these 0 is gone. So, you see we have been able to make 4 assignments with the available 0's right.

So, we cannot make all the 5 assignments. So, what we have to do now? We have to make more 0's right. So, you see although we are really solving travelling salesman problem, but if the assignment problem itself has given us some difficulty which we did not face in our previous lecture; that time just by row wise and column wise reduction we could make all the assignments.

So, when we have such kind of difficulties; that means, we are unable to make all the assignments simply by row wise and column wise deduction what we need really need to do is you know we have to create more 0's how to create more 0's by you know a clever method by coverings all the 0's and then you know do some matrix manipulation. So, that more 0's are created an existing 0's are more or less not changed. So, what is that method let us see that.

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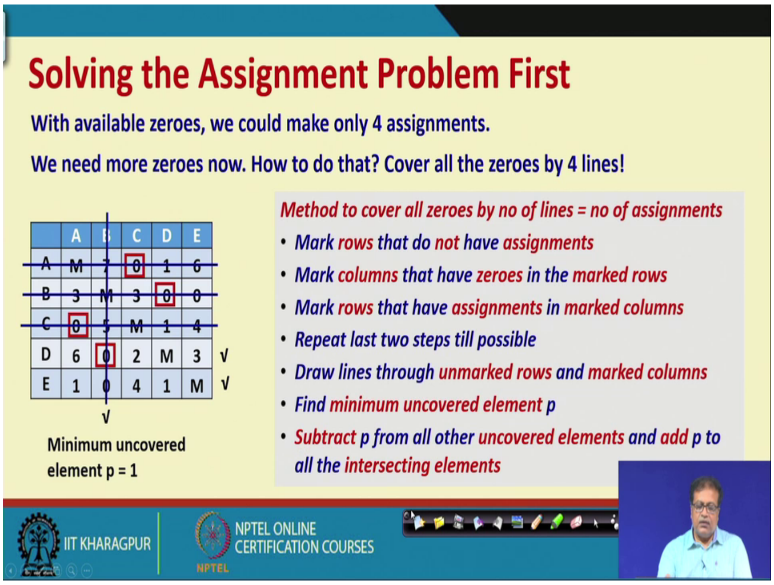

So, you know this method will be like this. So, you see we could make only this four assignments as I have sown you earlier and you know these two 0's are not possible because you know we have some other 0 here we have made assignments then we should do.

So, what exactly we should do now you know here is a method that method is given here and using the see the method really does what; it actually tries to cover suppose we have made been able to make 4 assignments and we have 2 other 0's these method essentially covers all the 0's by same number of lines that is equal to the number of assignments. So, how many assignments we have made 4.

So, we are allowed to have 4 lines to really cover all those 0's right it should be very clear it to you that you know although there is a proof, but I am not going into the proof and all that you know we should not have more number of lines to cover all the 0's because if we have more number of 0’s then the problems optimally will be altered its alright.

So, we should be careful that we should not have more number of lines to cover all the 0’s then the number of assignments we that we have made at a given point of tome. So, what should we do? See the method is that we you are intuitive you can have also draw those lines you know just like that, but this method if you follow it will guarantee you that you can cover all the 0’s by exactly same number of lines those are have actually the number of assignments.

So, the first method make rows that do not have assignments right. So, which is that row that do not have assignments you are clear this is the row right, it does not have assignment, all other rows have some assignment. So, if we say you know choose this particular row that is where the tic is given then; you know you choose mark columns that have 0’s in the marked rows. So, this row is marked so there are some 0’s where you know in the marked row; obviously, this is the column.

So, this column is chosen right. So, this two are chosen the third is marked rows that have assignments in marked columns. So, in these this is a marked columns; so in this marked column there is an assignment that is this one so we marked that row also right. So, we have marked two rows and one column and then the process should be repeated right the repeat last 2 steps till possible; that means now we have marked one more row, is there any other 0? If there would have been 0 we could have marked that column also, but fortunately for us there is no such column so we stop here.

Now, what we do draw lines through unmarked rows and marked column so these are unmarked rows we draw lines and this is the marked columns so we draw lines right. So, this is how you see the four lines we have got and those four lines exactly cover all the 6 0’s that we have got so far right and also the number of lines has become equal to the number of assignments.

Now what, now you find the minimum uncovered element which is see now certain numbers are covered, certain numbers where doubly covered they intersecting and try to see what is that number which is the minimum and uncovered. So, you see these 1 is and these 1 they are the minimum uncovered element. So, call that p and p equal to 1 is it all right. So, next what we have to do we have to subtract p from all other uncovered elements; that means, these ones will be 0's and add p to all the intersecting elements is it all right now why is that; we shall understand this lets go to the next slide.

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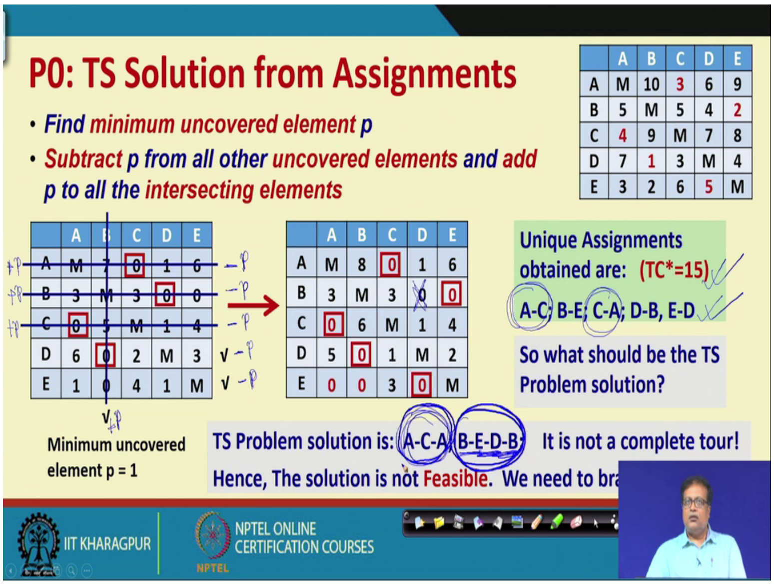

So, you see if you see the next slide then again those things are written; say essentially what you are basically doing you know you are deducting p from all the rows so you see you are doing this minus p, minus p, minus p is it all right.

So, you are deducting p from all the elements is it all right; so you are deducting p from all the elements that is the first job and then you see those covered lines so wherever you have covered lines you are adding p. So, you see this is the matrix manipulation that we are doing the manipulation is that; we are deducting p from all the rows and columns and we are adding p to those rows and column where we have got lines is it ok.

So, essentially what will happen those items which are uncovered they will be deducted by p, those items which are covered their minus and plus will cancels so there will be no change, but those intersecting elements will go up by p. I hope you understand that the uncovered element p will be subtracted, the covered elements the minus and plus will cancel so nothing will happen, but intersecting element p will be added right in this case the value of p equal to 1.

So, what is that that is the resulting matrix, look here 7 has become 8, 5 has become 6 and you know these are the two intersecting element M anyway does not change because

M plus 1 is M only, but those this row other items have not changed, but their 6 has become 5, 2 has become 1 and 3 has become 2 and these are 0 0 3 and 0 and M right. So, with that now we can see that you know we have been able to make all the 5 assignments then note about making assignments; please see that while you make these assignments try to make assignments in unique 0's.

So, this is an unique 0 assign, this is an unique 0 this way so assign right. So, this then you know once you have then this is another unique 0 so assign, so this 0 will be cut off. So, moment this 0 will cut off; obviously, you can make another assignment here right.

So, we have made been able to make all the 5 assignments the assignment problem is solved and the unique assignments obtained are A to C B to E C to A D to B E to D. So, that is the assignment solution and if you look at here the solution are indicated in the original problem and if you are add them 4 plus 3 plus 2 plus 1 plus 5 it is going to be 15 and then 15 is our TC star right and this our solution A to C B to E C to A D to B E to D.

But what it speaks about that travelling salesman problem solution if you combine then A to C you know just look here A to C and C to A together combines to A to C to A just you are clear you know it is you know if you think you will find that this is A sub tour right. A sub tour has been obtained there is another sub tour that is B to E, E to D and D to B.

So, B to E, E to D and D to B can you see that we have got another sub tour here that B to E to D to B is it all right. So, we have you know in this travelling salesman problem basically 2 sub tours A to C to A and B to D to E to B.

So, we do not have A complete tour. So, the solution is not feasible. So, we need to branch further question is how, so how do you branch please remember the steps I have told you earlier choose the one that is having the least number of you know the legs or branches. So, in this case A to C, C to A that is the smallest sub tour this is A bigger one. So, how do you branch A to C block A to C and block C to A right, block A to C and block C to A those will the two branches.

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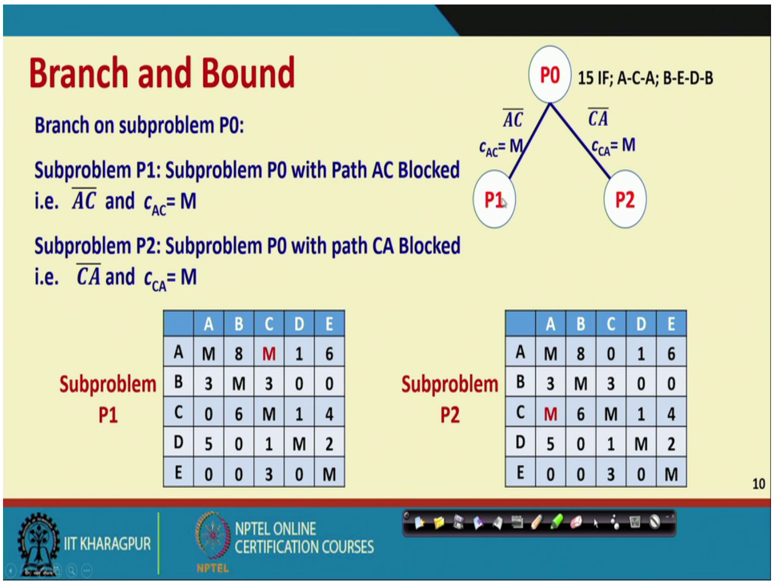

So, let us see those branches now. So, if we do look here this is A C bar; that means, put A C equal to M and C A bar put C A equal to M. So, two sub problem this is sub problems P 1 A C has been put to M, C A C M so that means, the crossed for A to C becomes infinite and here C to A becomes M. So, these are going to be our two sub problems I hope it is clear to you now that how do we go about the two sub problems is it alright. So, once we have those two sub problems now solving P 1.

So, what has happen in P 1 if you look at once again we have put A C bar; that means, A to C we have blocked.

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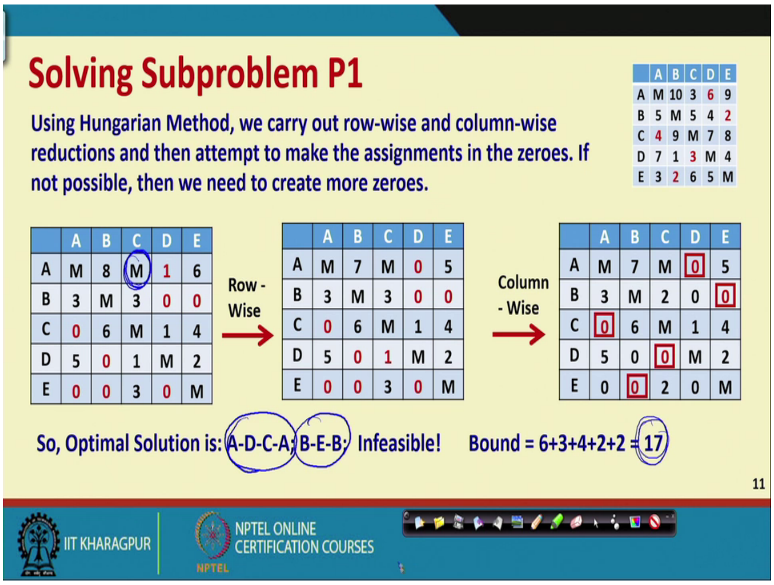

So, A to C has been blocked here just look carefully A to C has become M; now again we solve so I am not going deep into it; that means, row wise again we find out the minimums and we deduct them row wise. So, we do not have to start from beginning we can start from the matrix that we already had. So, if we now recalls you know because A 0 has become M. So, we put like this and we see solution has been obtained once again; what is that solution? A to D, D to C you know D to C to A.

So, again A sub tour is there A to D C to A and B to E to B so B to E and E to B so these are the two sub tours. So, again we have an in feasible solution and the bound value is 17 where from we got look at the original matrix this solution is put there and if you add 6 2 to 8 12 15 and 17.

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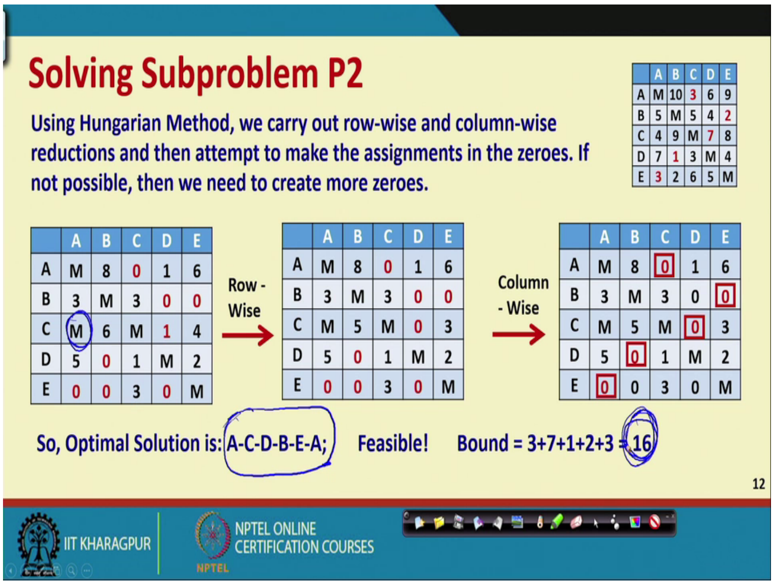

So, that is are the solution of sub problem P 1. What is the solution for the sub problem P 2 right so sub problem P 2 here we have put C to A to M right it is the other one C to A has been put to M and after row wise and column wise deduction we find the solution has A to C C to D D to B to E and E to A.

So, A to C C to D D to B to E and E to A so you are right that what you are thinking I am also saying the same thing that we have now got A feasible solution, feasible solution for the travelling salesman problem right and what is the bound value? The bound value is 3 plus 2 5 plus 7 12 plus 130 plus 3 16. So, the current bound value is 16 is alright so for sub problem P 2.

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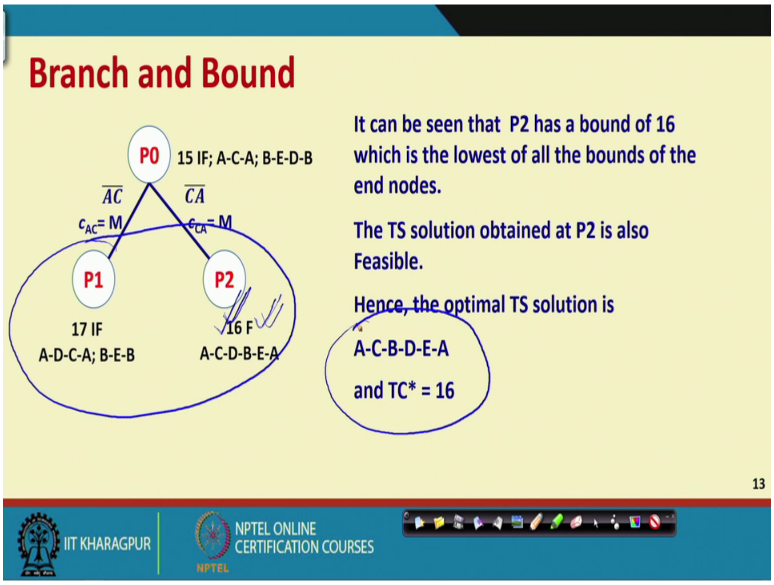

So, if we now put both of these into one place then we see that the P 0 had value of 15 P 1 value of 17, but in feasible and P 2 is feasible and the value is 16. So, what we have to do just you know carefully look at all the branches; what are the n nodes? These 2 are the n nodes.

Now, out of the n nodes which one is the minimum bound. So, look here it is a minimization problem. So, which one is the minimum out of all the bounds which is the most minimum you know 16, that is the minimum. So, out of them; is it feasible? Yes so this is optimal right.

So, find out the lowest of all the bounds and if it is feasible; obviously, that is also optimal right. So, we do not have to branch any further and that is going to be our optimal travelling salesman solution and that solution is A to C to B to D E to A and the total TC star equals to 16 right.

So, that is how the branch and bound algorithm can be used; now this is not the only branch and bound algorithms there are other methods available for branch and bound. In fact, several branch and bound algorithms are available one such branch and bound was the oldest one where we have penalty base method.

There are other methods there are also branch and cut methods and so on and so forth you know there are lot of research that is going on and how to solve the travelling salesman problem in an very efficient and effective manner by using branch and bound algorithm is it alright. But before leaving this topic let us look at another problem and quickly understand how this branch and bound method really works one more time right.

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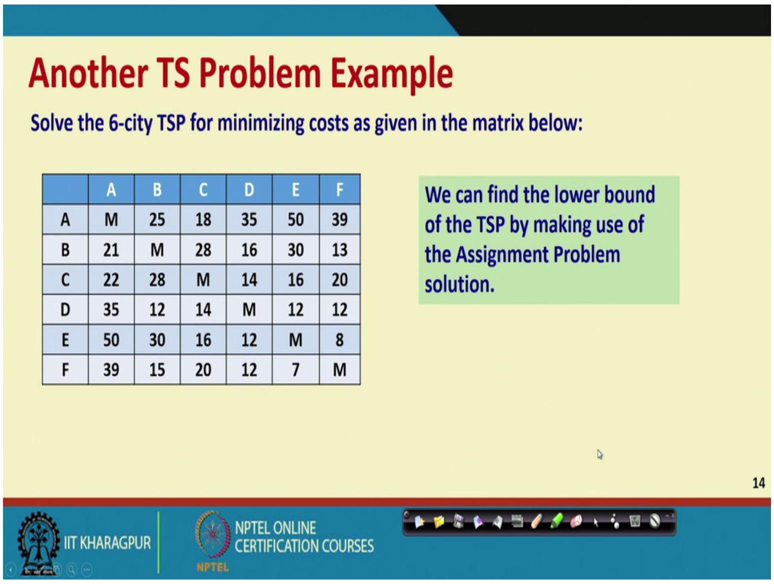

So, let us take another problem suppose we take a 6 city TS problem as given here and again we need to solve this by B and B the branch and bound algorithm. So, we know that you know the lower bound can be found out by the assignment problem solution is it alright.

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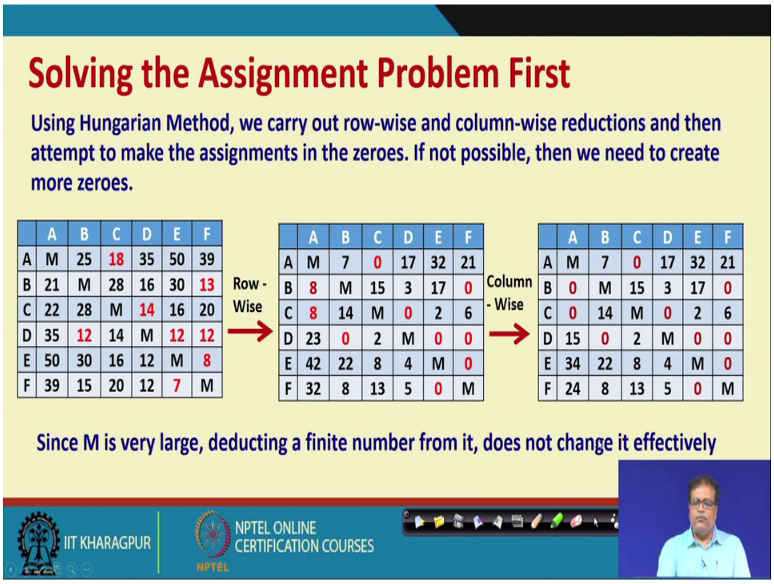

So, how this assignment problem is works out let us see. So, you know is the first if you solve the assignment problem we take the original problem, we find the row wise minimum and then find out certain number of 0's right. So, the C 18 is minimum, so we deduct 18.

So, we get certain number of 0's is it alright and then we see the column wise minimum. So, when we take those column wise minimums you know 8 is minimum here 0 0 0 0. So, this eight become 0 and then we get so many 0's right so; obviously, there are M on the you know the long diagonal. So, we have made row wise and column wise deductions.

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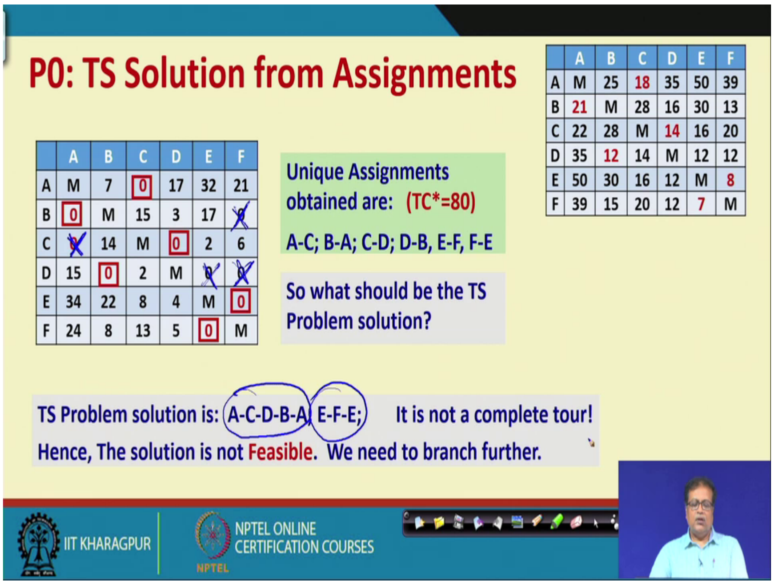

Now, let us see that how do we make assignments right. So, how do we make assignments? First we make you know here A to C because this is an unique 0 right there is this is also an unique 0. So, we make assignments right, this also another one so we make assignments; moment you do that these two assignments are not possible these two 0's are gone because we have made assignment here right.

Now, yeah now what happens that this become unique then right so we make assignment here this is also unique right so we make this. So, this 0 is gone so; obviously, now this has to be assigned; so that is how we make all the assignments you know then that 0 will be not use also right. So, that is how all the assignments are made, but what is the solution that we have got we have got A to C B to A C to D D to B E to F and F to E and if you look at all these figures then we get that TC star equal to 80. So, what is the TS problem solution A to C D to A to C C to D D to B B to A.

So, we have a sub tour here, we have another sub tour here so it is not a complete tour we need to branch further. So, how do you branch now this time you should tell which is the smallest sub tour just look at carefully it should be E to F and F to E that is the smallest sub tour. So, we should therefore, in two branches block E to F and another branch we block F to E right. So, let us see how it is done.

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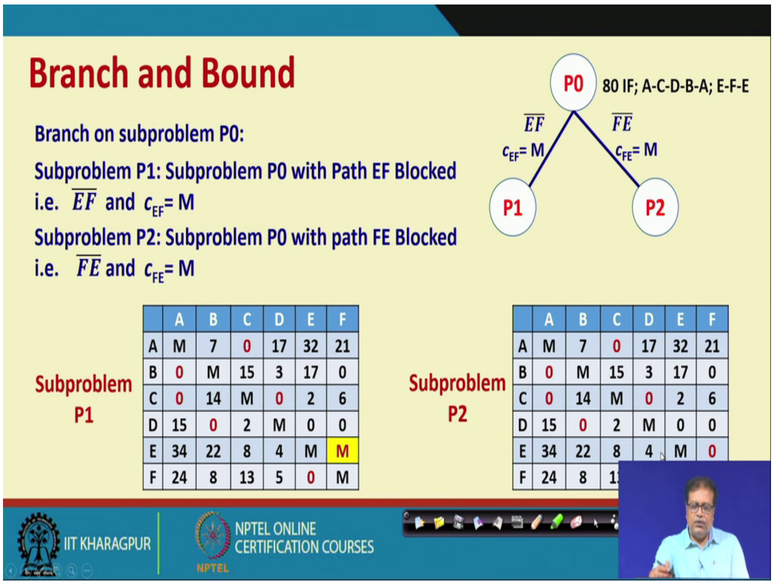

So, we take E F bar and F E bar and so first of all E to F we put M and F to E we put M and then we get to sub problems P 1 and P 2 right.

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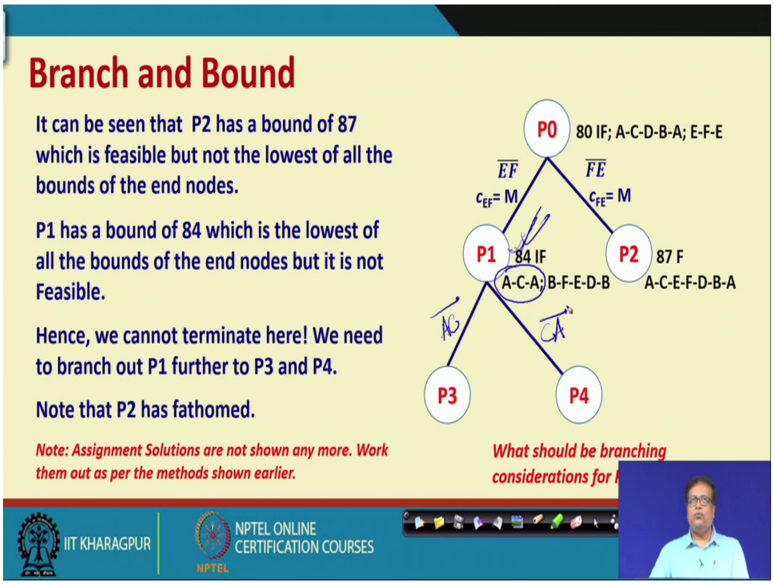

So, this two has to be solved so this is how you know when this two solved. So, assignment solutions we not showing any more you have to work them out please work them out to really find the as the method shown earlier.

So, you see the solution comes as such that for P 1 problem we get 84 as the TC star and P 2 we get 87, but while 87 is feasible the 84 the P 1 is not. So, you see which one is the lowest of the bounds it is 84 not 87. So, out of those two 84 is the lowest so 87 is feasible, but 84 is in feasible.

So, since it is in feasible we have to branch it once again right. So, again since the solution is A to C to A B to f to E to D to B so; obviously, this A to C to A we should then branch A to C bar and C to A bar is it alright. So, these two we should do in our next iteration to get P 3 and P 4.

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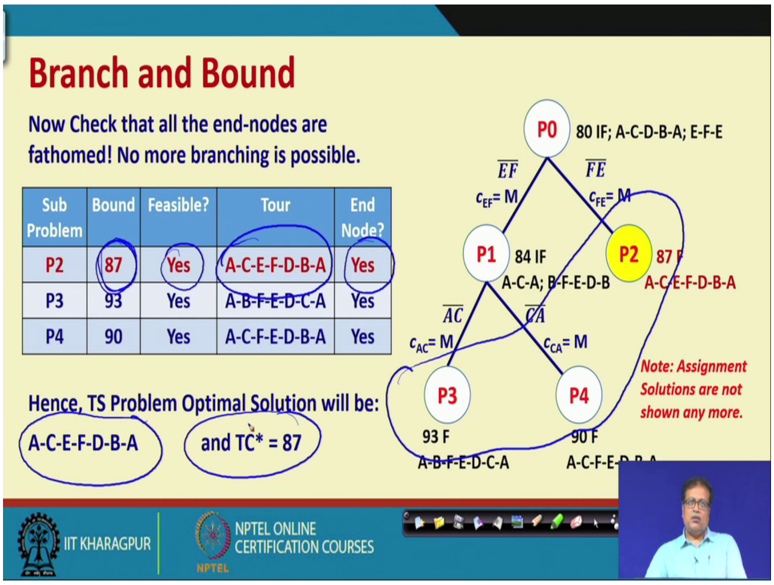

So, that is what has been done here and after solving we find P 3 equal to 93 feasible and P 4 is equal to 90 feasible. So, these solutions are compared look here if I see all the n nodes so these are my n nodes right these are my n nodes and they are all feasible; that means, all have fathomed; since all have fathomed the question here is they are all feasible the bounds are 87 93 and 90 and 87 is the lowest right so that should be our optimal tour and you know A C that is feasible also.

So, that is going to be our optimal travelling salesman solution and the TC star will be 87. So, this is how we solve travelling salesman problem by branch and bound algorithm right with that we stop here.

Thank you very much.