Selected Topics in Decision Modeling
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Lecture - 17
Branch and Bound Technique

So, in our subjects Selected Topics In Decision Modeling, today we are going to have lecture number 17 that is on Branch and Bound Technique. So, in our previous lecture, we have just started the topic and today we shall see the particular technique in more detail and solve some specific problems.

May be somewhat simpler problems in the beginning, which will be our precursor to really solve a classical problem that is a travelling salesman problem. Is it all right? And you know travelling salesman problem, branch and bound is a predominantly important method, for solving the travelling salesman problem really to get an exact solution.

Having said that, let us revisit the Branch and Bound Technique once again, and then see some examples, then how exactly they are really taken up.

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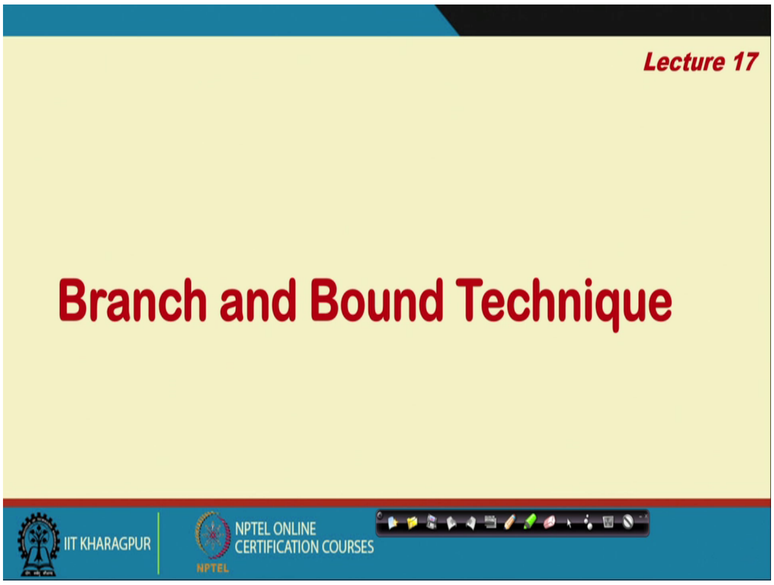

So; Branch and Bound Technique that is our lecture 17 today.

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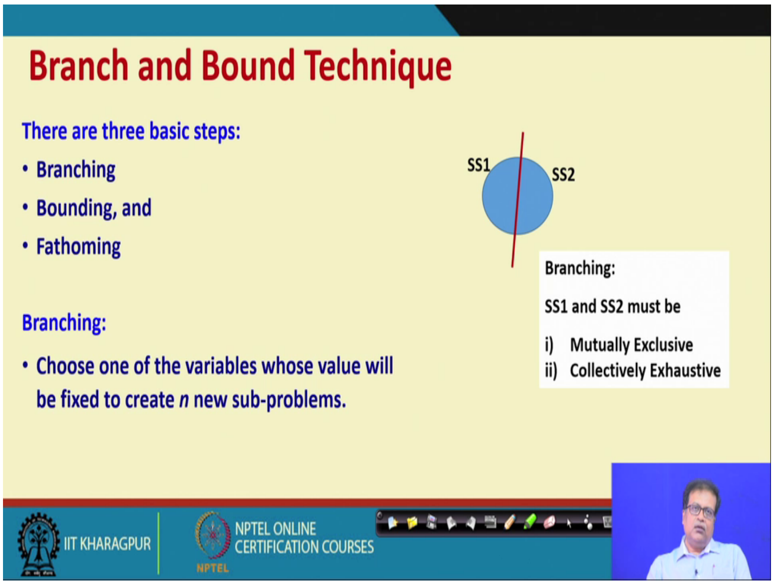

So, as we have said the essentially what happens in a branch and bound technique, we have a solution space that is shown here in this blue colour. Now, a solution space has a solution, now the question is that we really want to solve the final optimal solution for the total problem. Is it all right? But then we need about a quick value, we need there should be a method available to us, by which we are able to get a bound to this problem.

So, if it is a maximization problem, we need a bound which could be the maximum possible value, is it not? A kind of upper bound and minimization problem we need something which is lower bound; that means, a lowest possible value and it should be possible to get it quickly. Obviously, such a solution will not be feasible, if it is also a feasible solution like it happens in the assignment problem solution which we shall see later becomes a quick solution to the ts problem, if it is a feasible, we shall see that later.

But most often, you find the quick solution which you get by some method intuitive method, which gives us the bound is not optimal, in a sense that it is not feasible. So, what should we do then? Then we should be branch this particular solution space an example is here, we have branched it into two you know two parts. It need not be two it could be three or more also.

So, supposing we divide these into two, these two sub problems that is SS1 and SS2, they must be mutually exclusive and collectively exhaustive; that means, they should not be any overlap and together, they should have the complete problem. Is all right?

What we do then, then we try to see again by the same intuitive method a quick bound for SS1 and SS2. Now, if that particular value also be feasible; obviously, you know we then compare out of the different bounds which we have got for different such substance, which one is the best possible value. Incidentally the feasible value that we got if it is also the best possible value; obviously, we have got the answer is it all right.

So, let us see, how we go about it. So, as I said that there are three basic steps. The Branching, Bounding and Fathoming, the branching choose one of the variables whose value will be fixed, to create n new sub problem. So usually, that is what is done we take variable value to get the sub problems.

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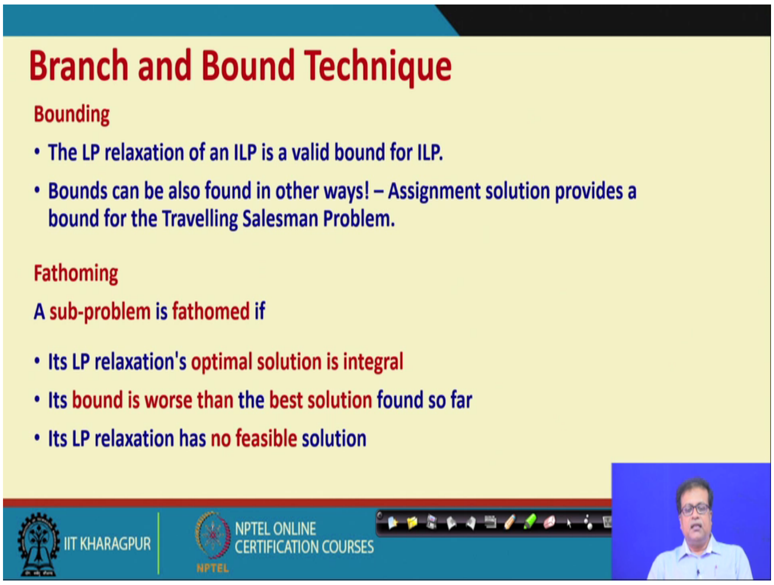

Then the bounding, as I said there should be quick way to find the bound; that means, the best possible value whether feasible or not, like example is the LP relaxation of an integer linear programming is a valid bound for a linear programming problem. It can also be found in other ways like I said, the assignment solution provides a bound for the Travelling Salesman Problem.

The fathoming is another thing that you see where do I end, suppose I divide a particular problem into sub problems, then to which level do I keep dividing; is not is there something, that we just cannot divide the sub problem anymore. Is it all right? That is where we say that this sub problem is fathomed means, the end has been found; it there is no point really dividing the sub problem into further sub problems right so, that is fathoming.

And in, in the case of the LP relaxation the optimal solution, suppose we find an integral solution as, I said that feasibility is found, then that sub problems is fathomed or if we get to a bound which is worse than the base solution found so, for is it not. See the idea is if you divide the sub problem, further and further; obviously, we are not going to get any better bounds, we shall only get worse bounds, then that we have at present.

So, if the current bound is already worse, than what we already have at some other sub problems there is no point, you know really going deeper into it, there is no point branching it further.

So; that means, that particular sub problem is fathomed, and the other part that we due do LP relaxation, but even that LP is infeasible. See we are not talking I LP in feasible just now, I said that feasibility, but that feasibility is in the context of I L P, LP for an I L P problem becomes a quick method to find the bound. But even L P is in feasible; that means, you just cannot proceed further; that means, that sub problem is also fathomed fine.

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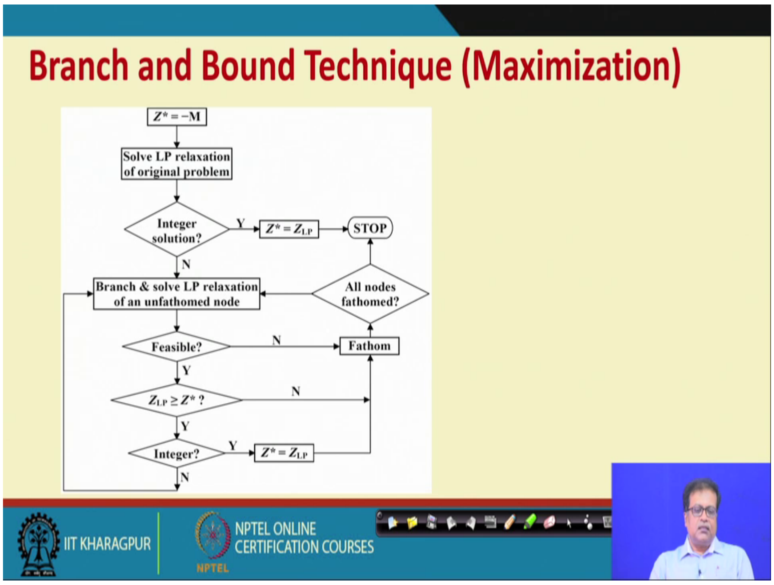

So, now, that we understood here is the same thing whatever I said in the form of a chart right. If we have integer solutions in the first place itself then, obviously, we will stop and whatever is the current bound that is itself is the final solution.

Otherwise branch and keep doing till all the sub problems are fathomed. In fact, we shall see the later, that is not really necessary to fathom all the sub problems is it ok. There are situations where even without fathoming all these sub problems we can stop a branch and bound algorithm, but we shall see them as and when we get them.

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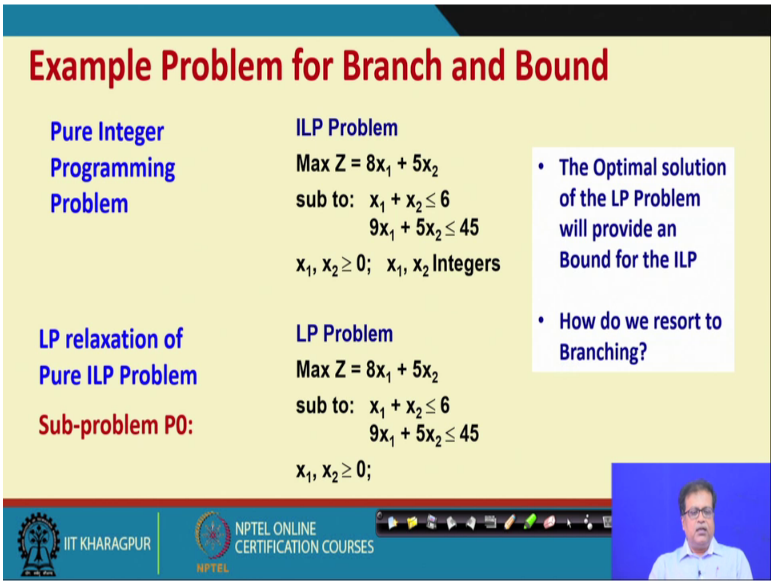

So, now, let us we take a. particular problem. So, supposing I have an ILP problem like these that is maximize Z equal to 8 x 1 plus 5 x 2 subject to x 1 plus x to less than equal to 6, 9 x 1 plus 5 x 2 less than equal to 45 and x 1 and x 2 they are all greater than 0 and both are integers.

So, you know we quickly get about; how by LP relaxation right. So, if the solution of the LP is found, what does it mean the LP solution; obviously, if it also the ILP solution then you know you cannot get any better than that because LP really gives an optimal solution. But if the LP solution is not a feasible ILP solution, then we have to branch further.

Please remember as we branch we are not going to get any solution which is better than the LP solution. Is it all right? If it is so, then obviously, the LP there is must be some problem is it or not we may not have solve the LP properly right. So, optimal solution of the LP problem we will provide a bound of the ILP. And how do we do branching then? So, let us see that.

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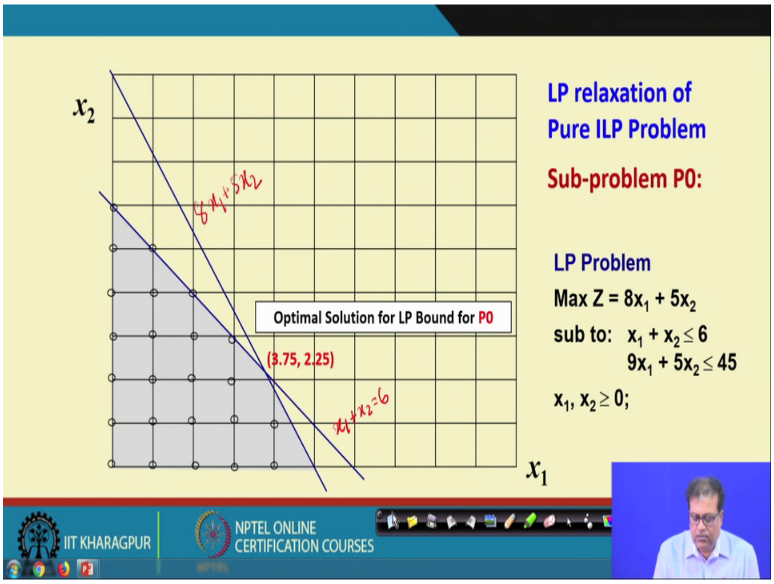

So, the first stage is to solve the LP part, so, I am not really going into the simplest part of it; you can also solve it by simplest. I am just giving you the graphical illustration write a quick graphically, how it looks. But then as you know that if the LP problem is more complex, you cannot solve graphically, only 2 or 3 variable problem you can think of solving graphically.

For bigger problem, you have to go for simplex is it all right and as you go for simplex to solve the LP, the solution of the LP then we can examine, and we all though we cannot draw the graph, but we must have an idea about the solution spaces that we are going to have.

Anyway this a quick examples. So, you know we are have trying to see in the graph itself. So, this is that LP part so, you know we have these two constant lines one is this first one that is your x 1 plus x 2. So, let us see you know this is the constant line the first constant line which one is x 1 plus x 2 equal to 6, and this is the second constant line, which is 8 x 1 plus 5 x 2 equal to and that is a objective function.

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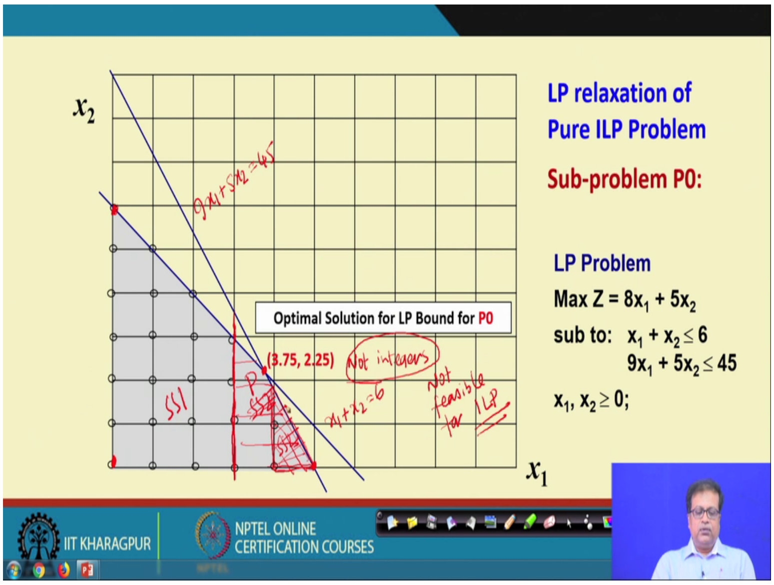

So, not this one so, you know this is the 9 x 1 plus 5 x 2 equal to 45 right and the other one is x 1 plus x 2 equal to 6.

So, these are the two constant lines. This two constant line cuts at these point. So, 3.75 2.25, and the if you looked at the solution space, then the 4 corner points are my candidate solutions, basically they are called basic solutions and they are also feasible. So, they are basic feasible solutions. So, optimal solution is one of them incidentally this is 3.75 and 2.25.

So, I am not going deep into it that you know we can evaluate those 4 points you know, and then see that this point is giving us the best possible value that is the maximum possible value anyhow. So, these particular point let us call it P so, P is our optimal solution for LP bound for P 0. Is it all right?

So, you see what is P 0? P 0 is the LP relaxation of the original ILP problem. So, in the context of ILP, is this point is the P 0 solution feasible. Look at the solution 3.75 and 2.25, it is not integer right not integers.

So, since they are not integers, not feasible solution for ILP from an integer linear programming point of view the current point is not feasible. So, what should we do now? We should then you know resort to branching.

So, how do I divide this particular problem into two different sub problems which you know really do not lose any of the integer solutions. So, you see all the integer solutions are marked here. So, we cannot lose any of those integer solutions. But then you know; we can definitely do something which will cut off the current optimal solution because if we have this current optimal solution, you know again the same solution will come; can you understand that supposing we divide these in the some way say for example, these particular line.

So, supposing I cut the solution space in such a manner that these portion is one right these portion is one, and the remaining portion is another. Supposing I make it SS1 and I make it SS2 does it solve our purpose; you see what you get.

You get that SS2, again the same point will become optimal, But then this is a non-integer solution. So, there is really no need to have SS2 in these manner, SS1 is fine, because that portion is unexplored in that sense whether any of those integers and all those integers are covered by SS1. So, if we take just imagine, if we take these two integers on the other part right, these portion.

So, these small triangular portion if I take and call it SS2 not this one right and these hatched area if I call as SS2, and these entire portion if I call as SS1; you know it cuts off these current optimal solution, which is non-feasible from ILP point of view. And these SS1 and these SS2 together covers all the integer solutions. So, that mutually exclusive SS1 and SS2 and collectively exhaustive; both the things are taken care of.

I hope you understood, how we had done the branching in such a situation. So, having seen that now let us see the how it looks.

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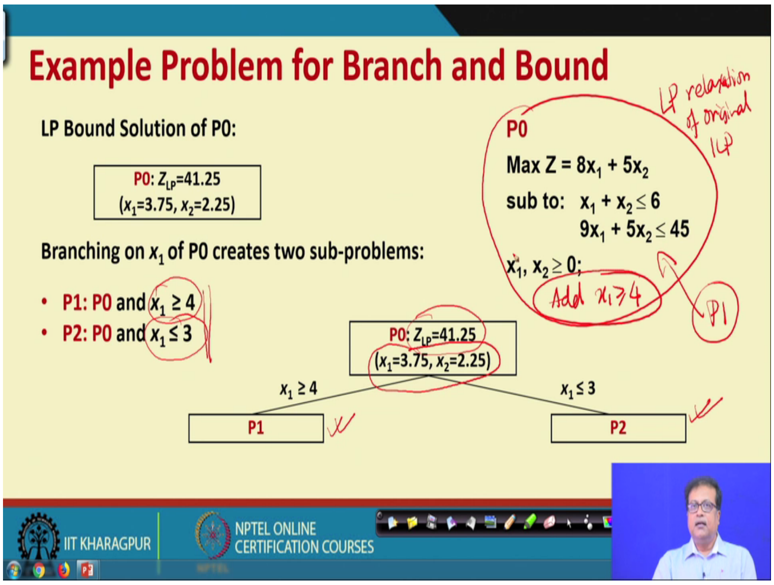

So, exactly that is what we have done that the original LP solution is here, the LP solution shows 3.75 and 2.25.

So, if you calculate 8 times these and 5 times these, then these is 41.25 is our current bound that is a maximum possible value, that we can achieve about the solution. But then we cannot achieve this it is very clear because you know obviously, this is not feasible from ILP point of view.

So, as I said these are the two branches P 1 and P 2. One is x 1 greater than equal to 4 the other one is x 1 less than equal to 3. Is it all right? So, these we call the two sub systems P 1 and P2. So, what is P 1 then? See all that is P 0.

So, now this is P 0. What is this one? This is the LP relaxation of the original ILP. So, this is the LP relaxation of the original ILP and you know if you add these particular constant x 1 greater than equal to 4 then you get P 1 right, and instead if you add the x 1 less than equal to 3 then you get P 2. So, P 1 is an objective function and 3 constraints apart from the you know the non negativity constraint.

So, now I hope you are clear what is P 1 and what is P 2 right. So, once we have understood what is P 1 and what is P 2.

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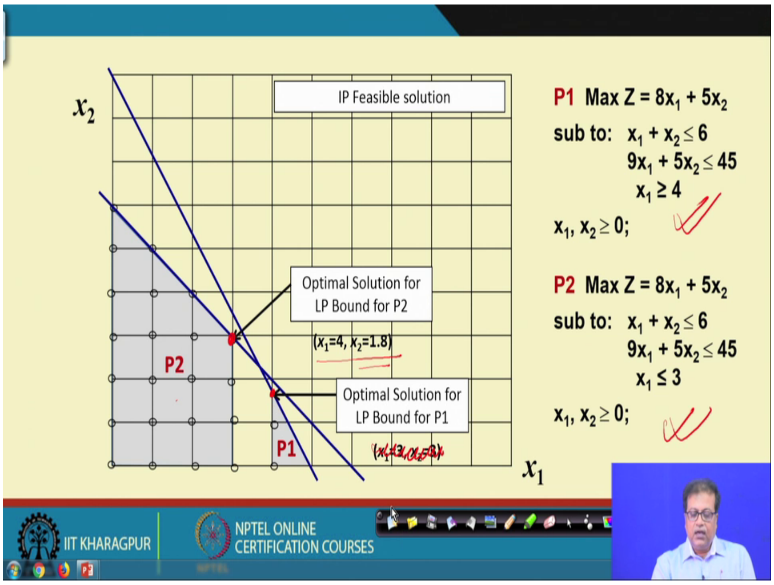

Now, please look exactly, what we said that this is P 1 and this is P 2 right, so, yeah. So, this is P 1 and this is P 2 and you know P 1 is here x 1 greater than equal to P 1 and 4 and x 1 less than equal to 3 that side is P 2.

So, these are the two sub systems right the P 1 sub system and P 2 sub system, then what we have to do. Again we need to find out the integer solutions for them, but you know we have to really find bind bounds of them by finding the LP solutions.

Now, look here you know the optimal solution for LP bound, then we find that you know the solutions for these one; the optimal solution becomes 4 and 1.8, the not this one. So, you know the optimal solution for LP bound for P 1 is these point and optimal solution of these is this point.

But these point is not the integer point, these point is 4 and 1.8 that is a optimal solution for P 2. How did you get it? Obviously, by you know really seeing all the corner points evaluating and all those things which I am not repeating again. And this is the optimal solution for the problem P 1 right so, these two solutions then we write and we see then in our you know the branch diagram the tree diagram.

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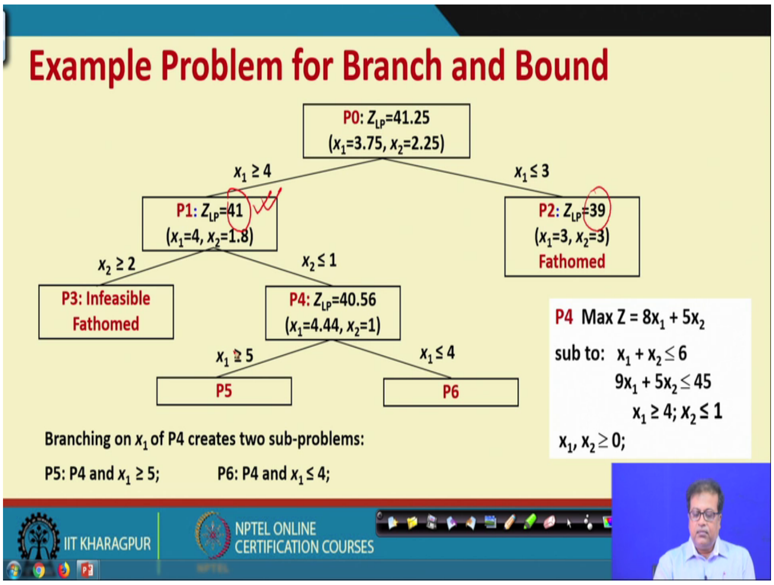

So in the tree diagram, we see that for the P 1 the optimal solution is you know x 1 equal to 4 and x 2 equal to 1.8 and P 1 is 41. Whereas, for P 2 here it is actually x 1 equal to 3 and x 2 equal to 3, that becomes our optimal solution. So, you know once we have it then we have the you know, the out of the two if I see which one its a maximization problem. So, which bound is higher.

The bound for P 1 is higher so; obviously, one thing that we should see that is our look here the solutions space that we have here that is.

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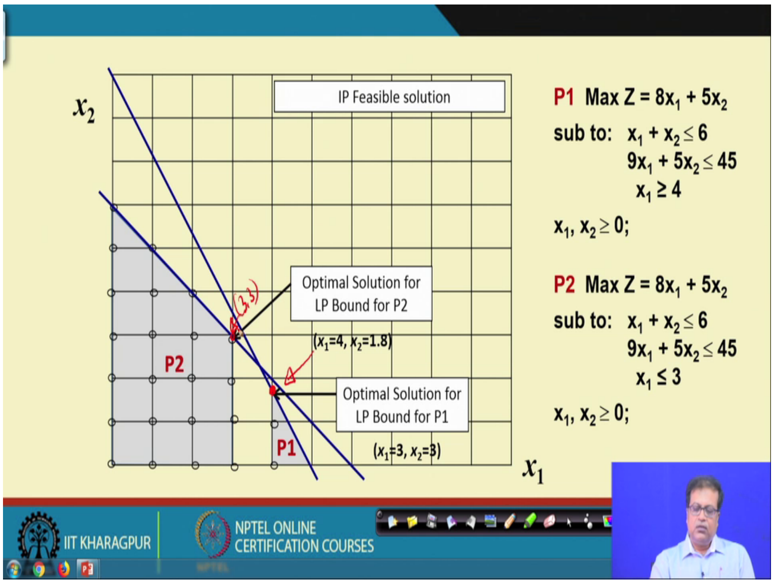

So, that is what we have and let us see the solutions space once again. So, once we have this two problems then we find actually that you know this is that let us do a clarification. So, this point is not 4 and 1.8 really the this point is 4 and 1.8 and this point is this 3,3.

So, now, hopefully it is clear that the optimal solution for P 1 is this point that is 4 and 1.8 and optimal solution for P 2 is this point, which is 3,3 right. So, once we see that hopefully now we are cleared that what are the two solution space for P 1 and P 2 right.

So, having said that let us, now once again see that what we have got. So, what we have got is that for P 1, 4 and 1.8 and for P 2 3 and 3 and these are our bounds. For P 1 the bound is 41 and for P 2 the bound is 39. Now, since 41 is higher so, you know out of P 1 and P 2 which one should be branch further; Obviously, P 1 because this is an infeasible solution.

Supposing P 2 would have been higher, then we could have stopped you know why because you clear we have two bounds, we have found an integer solution here in P 2 which is incidentally 3,3. And that solution is also the best possible bound that we can have, but that has not happened we find 41 has higher. So, we have to again divide them further.

So, how do we divide them? Let us see that in our next slide. So, if you look at the diagram. So, look here now what we do that these portion is P 4 so, what is P4 so this was our you know the SS1. So, please look at this carefully, so this is our P1 so, these portion was our P1. So, which one was our P 1 this is our P1.

Now, this P 1 has got two integer solution one is this another is this. So; obviously, you can take P 4 and the other part that is your the previous part that is your x 2 greater than equal to 2 and x 2 less than equal to 1.

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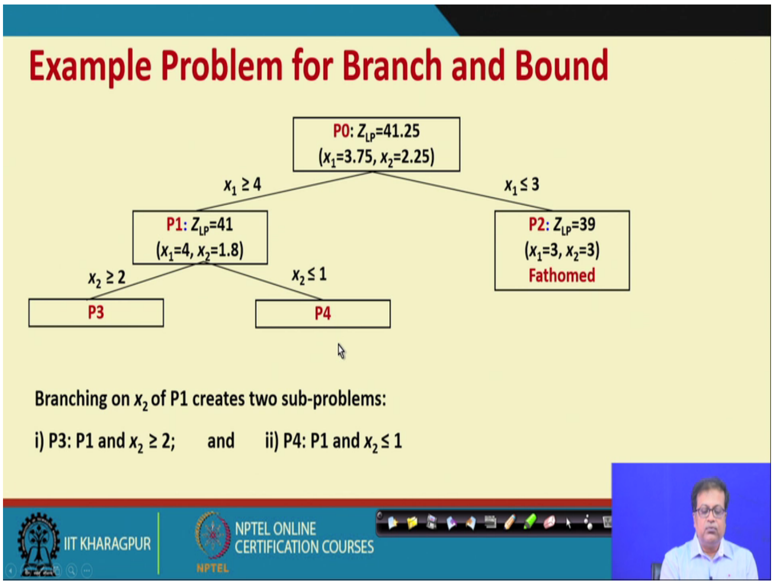

But that x 2 less than equal to 1 that point really cannot be found. Why it cannot be found? Because when you try to do that you know, you really find that these P 3 part cannot be you know this x 2 greater than equal to 2 part and x 1 greater than equal to 4; no optimal solution could be found, there is no feasible solution space.

Whereas, there is a feasible solution space for P 4 and therefore, we again find the optimal solution for P 4 and we find that point is 4.44 and x 2 equal to 1 and P 3 is not feasible. So, when you put these again then you find that these portion is 4.44 and 1.

So, again we make further sub systems, that is x 1 greater than equal to 5 and x 1 less than equal to 4. How to do that? When you do it then the branching on x 1 on P 4 creates two sub problems that is P 5 and P 6. So, I am not elaborating further let us say the solutions that we have got out of all these different branching and bounding procedure.

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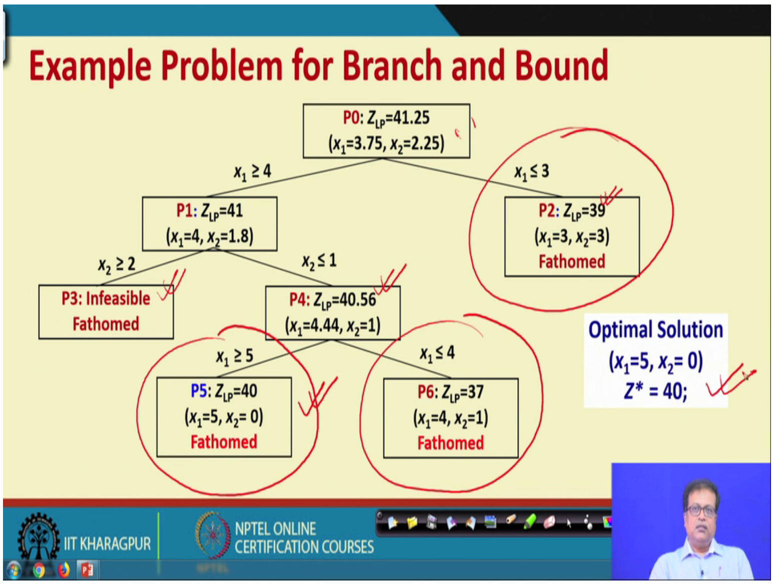

So, this is our final tree in that tree you find that was our original value that was P 0 and we find in P 0 you know the Z LP was 41.25 and this was 3.75 and 2.25.

When we have branch this way by keeping all the integer solutions intact, I find 41 as a bound and here 39; 41 came out to be higher. When you further break them then you find that we have an infeasible solutions in P3, but there is a feasible solution in P 4 where again we found 40.56.

So, again when you compare just compare these value 39, 40.56 and infeasible. So, this need not be see further it has bottomed out or fathomed so, 40.56 is higher. So, we have to branch it so, again when we branch we find an integer solution here, an integer solution here.

So, finally, compare this three and tell me which one is the best. All of them are integer solutions, so the solutions are feasible and out of them it is the Z LP you know that is 40 that was x 1 equal to 5 and x 2 equal to 0 that gives the best possible solution. So, this is going to be our optimal solution x 1 equal to 5 and x 2 equal 0. Is it all right?

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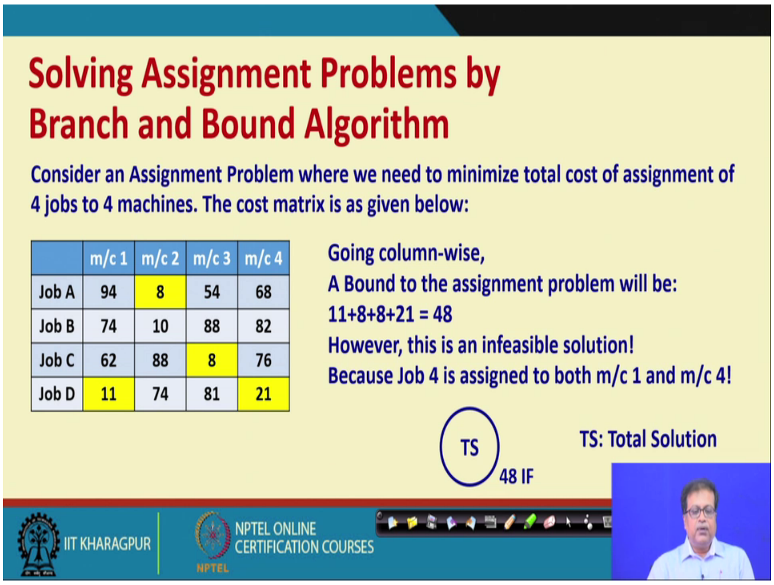

So, this is how we solve branch and bound problem. Is it all right? So, now, let us talk about another problem you know, let us see how we solve let us say an assignment problem through the branch and bound algorithm.

So, here is our assignment problem where we need to minimize total cost of assignment of 4 jobs to 4 machines, and this is our cost matrix. So, what we need to do we have 4 jobs and we have 4 machines. So, what are these? These are the costs of assignment, how can we use the branch and bound algorithm to solve this particular problem.

Usually assignment problem has got a very you know well known method called the Hungarian method and we need not really solve, assignment problem by the branch and bound algorithm, it is not really required.

But this is here an example of how you can really do that. So, we must have a quick method of finding the bounds. See this is a minimization problem, now if I really see column wise now, look here the column wise the lowest here is 11, lowest here is 8, lowest here is 8, lowest here is 21.

So, if you add them up you know we get 48. So, can you really have a solution which is lower than 48; answer is no.

So, therefore, you know this is definitely a bound that means, you cannot have a solution better than these and however from a point of view of assignment problem you know this is infeasible. So, how do I branch it further is it all right. So, we discuss that in our next lecture.