**Selected Topics in Decision Modeling**
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**Lecture - 15**
**Cutting Plane Method**

So, today in our course Selected Topics in Decision Modeling, we are going to discuss Cutting Plane Method, that will be our 15th lecture right.

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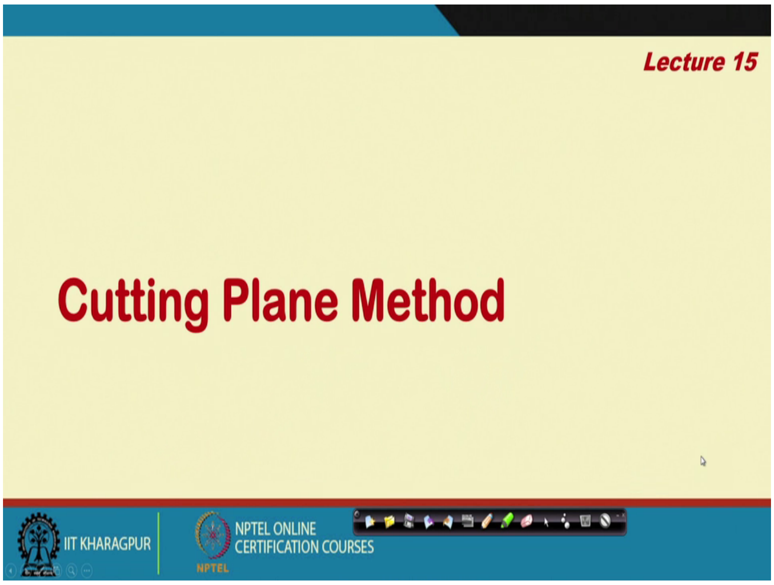


So, as you can see that in the cutting plane method.

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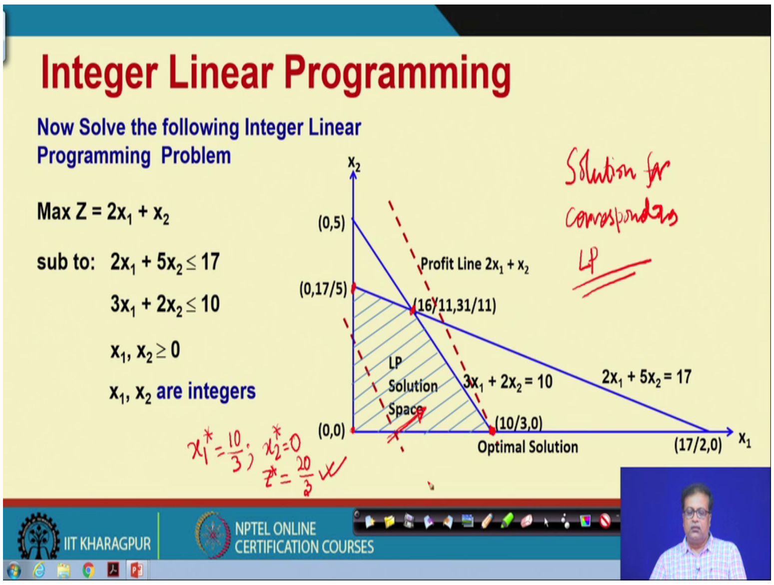


You know in the last class we talked about solving linear programming problem integer linear programming problem which is like this, that we have an objective maximize Z equal to 2 X 1 plus X 2 subject to 2 constants are given both are less than equal to constraints and X 1 and X 2 there are 2 variables, both of them are greater than equal to 0 and they are also integers.

So, how do we solve such a integer linear programming problems with the help of linear programming. So, we shall discuss this problem at 2 levels, at in the beginning we will try to find a graphical solution of the corresponding linear programming problem and we see how we can modify these graph by making use of cutting plane method and you know successively reducing the solution space. So, as to have keep all the integer solutions in the intact and cutting down the current optimal solution is all right.

So, from the linear programming we shall find the optimal solution let us say initially by the graphical method. Then, we cut out the what you call your the optimal solution, but keeping all the integer solutions intact right. And keep on doing this you know a number of times till we get an integer solution in the linear programming itself moment, we have that then our optimal solution that is optimal integer solution is found. But you know you can do it in a 2 by 2 variable problem very easily, but moment you have a bigger problem these method will not be applicable because you cannot use graphical methods to solve such problems. So, what should you do under that situation we have to follow the simplex procedure?

So, and next you know after these after these graphical method explanation we shall then see how we can do this by making use of simplex right. So, let us see how we do this through the graphical method?. Now you see this is the you know the graphical solution for these linear programming problem, you know this one the solution that you see here this is not the integer programming solution right. So, the what you see here this is a solution for solution for corresponding linear programming problem right. So, this is the solution for corresponding linear programming problem.

So, what we do we take the X axis and the Y axis and then you know these 2 X 1 plus 5 X 2 equals to 17 we draw 1 line, 3 X 1 plus 2 X 2 we draw another line. So, as you draw these 2 lines and along with this you know X 1 greater than equal to 0 and you know X 1 this side is X 1 greater than equal to 0 and this side is X 2 greater than equal to 0 and then we find this solution space right.

So, that is our LP solution space. Now, in this LP solution space you know we know that the basic visible solutions are you know this corner points, you know this one this one this one and this one. And you know either we can evaluate one of them will be optimal, because optimal solution will lie at one of the corner points. So, either we evaluate the value at this 4 corner points or simpler method is also there in these case that we know we find the profit line. So, this is our profit line this is also the same profit line. So, you know move the profit line right parallel to itself.

So, as you move the profit line parallel to itself, you know the at some point the profit line will touch one of the corner points and after that it will leave the solution space. So; obviously, that is the point where you will get the maximum value of the function that is the objective function. So, that will be our optimal solution.

So, you will know all these are very well known to all of you whoever have studied linear programming. So, our optimal solution will be then 10 by 3 that is X 1 value and 0 that is X 2 value. So, what is the optimal solution X 1 star equal to 10 by 3 X 2 star equal to 0 and what will be the value of Z star, you know 20 by 3 because 10 by 3 into 2 and 0 into 1. So, that is our optimal solution.

But, we do not want the optimal solution we want the integer optimal solution is it so; obviously, 10 by 3 is slightly bigger than 3 not an integer. So, this solution we shall not accept. So, what should we do see this is our LP solution space, we should see how many integer solutions are covered within this LP solution space all right let us see what are they. So, you know cut it out and then go to next page and here you see the same solution space is shown here and all the integers' solutions are indicated you know by a dot.

So, these red dots they are all our integer solutions within the solution space. So, you can see like you know this is 0 0. So, we have this is 0 0 then this is 1 0 2 0 and then this point is 3 0, here 0 1, 0 2, 0 3, and etcetera etcetera for example, this is 1 1 this is know 1 2 this is 2 2. So, sorry this is 2 2 this 1 is 1 2 (Refer Time: 07:55) this is yeah this is 2 1 all right

So, all these points you can identify. So, these are all the integer solutions which are within the LP solution space. So, you can see that you know your optimal solution will be in 1 of those integer solutions it includes 0 0, but you know 0 0 cannot be optimal. So, these corner point cannot be optimal because this is not integer, these corner points cannot be you know optimal, because that is also not integer and these corner point which is the current optimal solution this is also not integer all right.

So, we want to keep all the integer solutions intact and we want to cut out solution space minimally that will cut these current optimal solution and will keep all the other integer solutions intact.

So, how do we cut this solution space?. Any guess how do we do that? You know think over this is the nearest integer solution which is available. So, you know how do we cut can we cut by taking an like this one suppose, I have this line and then I cut these portions of the solution space just into just look at this; this will cut the current optimal solution. And this will be our new solution space this portion will be cut off you know this is the cut off portion, this is a minimal portion that we can cut off and, but then it keeps all the integer solutions intact is that all right.

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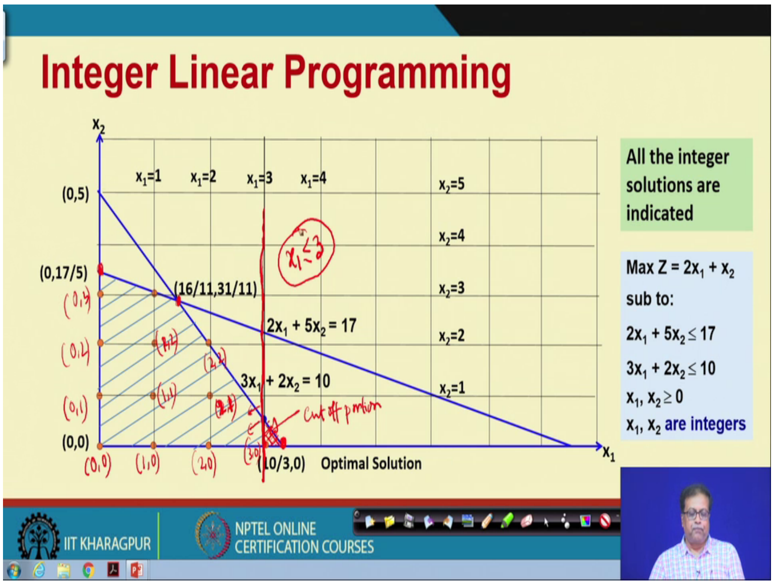


Now, interesting thing is that it can be done easily here, because it is a 2 dimensional problem and we can examine the graph very easily, but then if we have to solve it by simplex there should be a simpler way of finding these particular line is it all right. So, if you see what is this line this line is X 1 equal to 3, but you know we have to take this side. So, basically the constraint you are adding is X 1 is less than equal to 3 right.

So, that is the constant that must be added in these; so, as to accomplish what we are having here right. So, let us see what it is.

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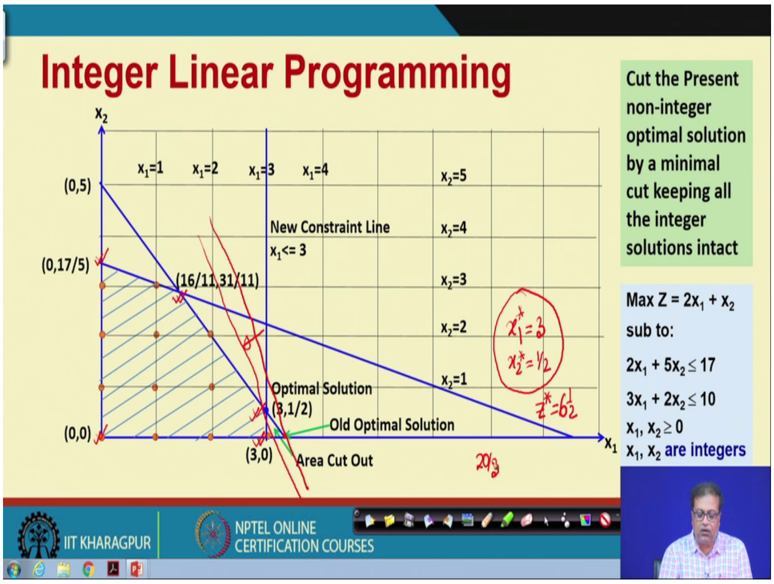

So, you can see here this is precisely done here and then after you do that then what are your corner points now? Now, your corner points are 0 0 this one the other corner point is you know your 0 0, this is one corner point, this is another corner point, this is another corner point, this is another corner point, and this is another corner point.

So, we have 5 corner points; now the optimal solution at this point of time is your you know this point where the optimal solution is 3 x 1 equal to 3 and x 2 star equal to half is it all right. So, this is our new solution space what will be the value of Z star now because 2 X 1 that is 6 and X 2. So, this is 6 and half

Look here this 0.3 0 here the Z value is 6 right. So, this point is having higher optimal value and as we have seen in our previous plot the you know the profit line goes like this; obviously, they are outside. So, this is the where you know. So, you see your earlier profit line suppose I just draw those profit lines. So, you see roughly this is that was our profit line.

So, now, the profit line has slightly gone inside right. So, like earlier our optimal value was 20 by 3 I think is it not. So, that was our optimal value let us see what was our optimal value in our previous slide. So, in our previous slide you know our optimal solution was 10 by 3 and 0. So, 20 by 3 that was our optimal solution and now our optimal solution is 3 and half; that means, from sorry this is from 20 by 3.

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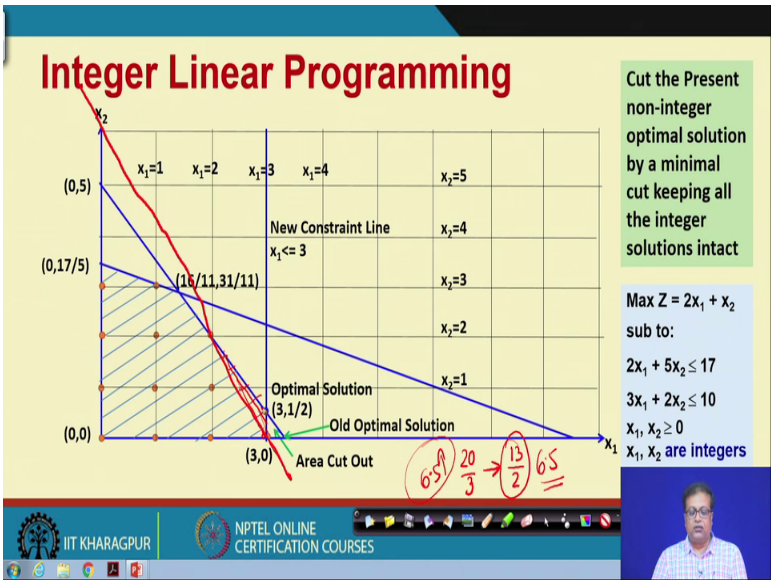

It has come down to 3 and half that is 6 and half so, 13 by 2 all right. So, 20 by 3 to 20 by 3 is you know 6.5 is 19.5. So, it is above.

So, these value is exactly 6.5 and this value is more than 6.5 right. So, from value which is above 6.5 our new objective function value has fallen to 6.5. So, what have we done we have in a way reduced the you know the objective function value; that means, our LP solution was the best.

But then in the search of an integer solution we have come to a point we definitely found a solution, but then again it is not integer right, but then we should not loose heart again we employ the similar procedure. So, what should we do? How do I find out another line that cuts off the current optimal solution and at the same time you know retain all the integer solutions how do I draw the next line?

So, again if you think then the next line actually may join these 2 integer solutions and go further is it not. So, is it a line like this? So, you know this line if you really think of these lines joins the current integer solutions around these points and the area which is here probably if you cut off this portion. Then you know you still retain all the integer solutions and you cut off the present optimal solution. So, that should be our next step. So, let us see how do we do that? So, again we do that.

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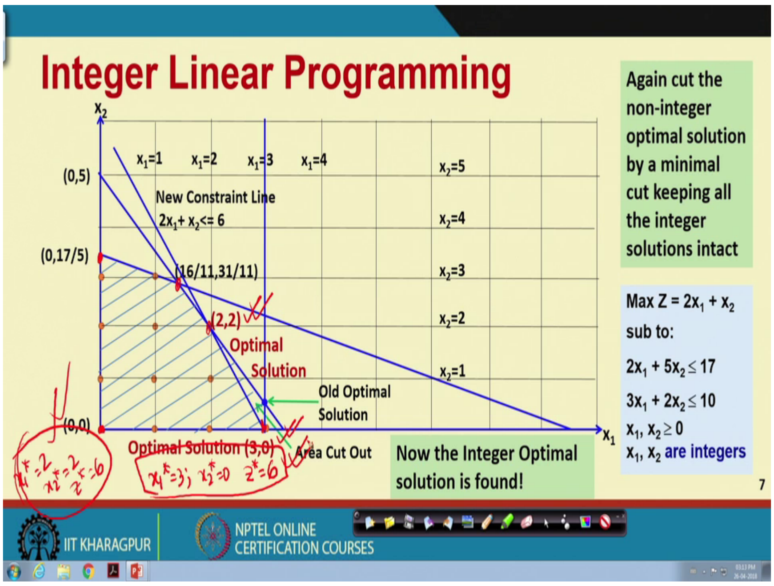

So, look here then we have drawn these line which is your the next constraint line and this constraint line can be measured as 2 X 1 plus X 2 less than equal to 6, you know this line that joins these point the current sorry the optimal solution this will be cut off, this was the earlier optimal solution old 1 and these portion of the area which is cut out. And, now the new corner points that are 1 2 3 4 and 5 right so, the new optimal solutions

Now, what is a optimal solution interestingly these new line which is 2 X 1 plus X 2 less than equal to 6 is parallel to the objective function right since it is parallel to objective function. It is very clear that you know these new optimal solution will be both this 2 by 2 2 and 3 0 right. So, the new integer optimal solution will be found. So, when you cut out then now the new optimal solution are 2 X 1 star equal to 3, X 2 star equal to 0, and Z star equal to 6. So, this is one optimal and the other optimal is x 1 star equal to 2 x 2 star equal to 2 and Z star is also 6. So, these are our 2 new optimal solutions right

So, and both are integer. So, since an integer solution is found we have found out the optimal value. So, this is the method really speaking that we solve the LP, we find the optimal solution then progressively cut the solution space in such a manner. So, that all the integer solutions are intact, but current non integer optimal solution is cut off right. And when the LP returns integer solutions; obviously, that is the optimal solution for the integer linear programming problems.

So, this is how the cutting plane method is used, but then I only showed the graphical part of it right how should it look in a simplex? So, that is our next task. So, let us without wasting time let us see how it is done?

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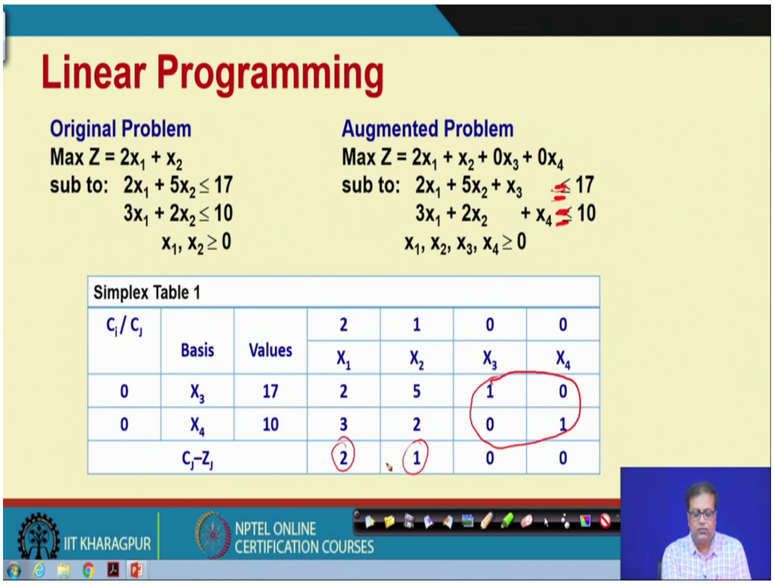


So, see look here you want to solve it by the simplex methodology.

So, again what we do this is our original problem, then we make the augmented problem. So, the augmented problem is going to be like this you know what we do this is our original problem, then we make the augmented problem by taking 2 more variables those are slat variables X 3 and X 4. So, we have 2 X 1 plus 5 X 2 plus X 3 equal to right, 17 3 X 1 plus 2 X 2 plus X 4 equal to 10 X 1 X 2 X 3 X 4 greater than equal to 0 and this is our augmented problem.

Which, we write in simplex table 1. So, this is our basis X 3 and X 4 the current values are 17 and 10, this is our matrix X 1 and X 2 are non-basic variable, X 3 X 4 are basic variable, they form the unity matrix that is how simplex is done? And the matrix is written 2 5 3 2 and you know these are the profit coefficients and these are also profit coefficients to make 1 unit of X 1 you need 2 X units of X 3 and 3 units of X 4 at a cost of 2 into 0 plus 3 into 0, that is 0 and 2 minus that 0 C J minus Z J. So, that part is Z J is 2 minus 0 equal to 2 and here it is 1.

So, this is how we put the first simplex table? Now, usual method is followed to solve this problem.

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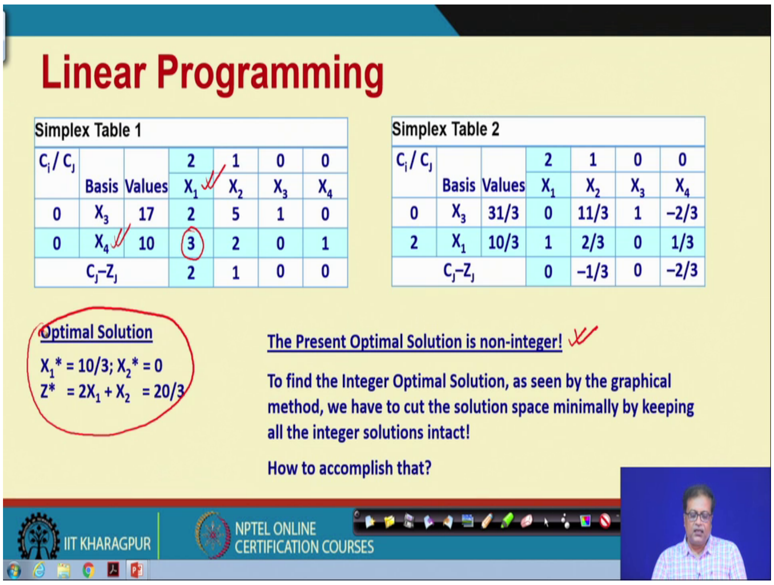


And, then we see that the you know the 2 is more profitable. So, X 1 is going to be our entering variable and 17 by 2 is 8.5 10 by 3 is 3.3 3. So, 3.3 3 is lower. So, X 4 is the bottle neck resource. So, X 4 will be our leaving variable. So, we found the entering variable X 1 and leaving variable X 4 right. So, these two are found this is our common element. So, using them we can find the final solution, that is this simplex table 2. Now, X 1 and X 3 they are having canonical form 0 1 1 0 X 2, they become now basis X 3 and X 1 X 1 and X 4 X 2 and X 4 they will be now non basic and the values are computed.

For example, you know 10 by 3 10 by 3 this 3 becomes 1 2 becomes 2 by 3 0 1 by 3, then this is the canonical form this value, which was 17 17 minus 10 into 2 by 3. So, becomes 31 by 3 is it all right.

So, like this all these computations are done and we found that optimal solution right. So, optimal solution is found and this is a non-integer. So, present optimal solution is non-integer. What is that solution? The solution is X 1 star equal to 10 by 3 X 2 star equal to 0 and Z star is 20 by 3. Please recall in graphical solution first we found the LP optimal solution, which is the same value.

All right, the same LP optimal solution we have found out, but now graphically we kept all the integer solutions intact and we cut off the solution space manually I mean minimally so, as to keep all the integer solutions intact, but also to cut the present optimal solution. Now, how do we do that right? How do we do that? That is our most important task now, how do we do that?

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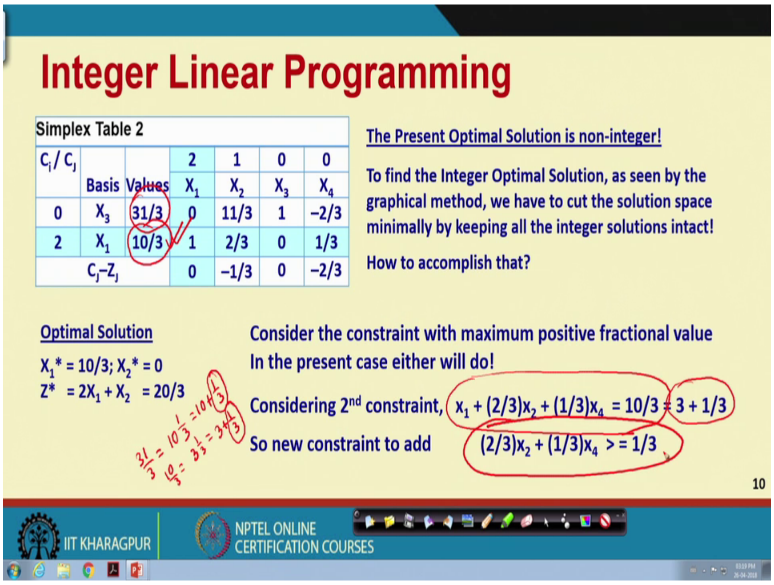


So, look here now again look at it very carefully. What is to be done? That you know look at these 2 lines and look at the values. So, these values are you know there are 2 values the 31 by 3 and 10 by 3. Now, what you do you know the out of these value find out, which one is having listen very carefully, which one is having maximum positive fractional value. You see what is 31 by 3, 31 by 3 equal to 10 and 1 by 3, which is 10 plus 1 by 3.

So, this is the fractional part and 10 by 3 equal to 3 and 1 by 3 equal to 3 plus 1 by 3. So, this is the fractional value. The question is which one is having maximum positive fractional value. In this case both are equal. So, you could have taken either first one or second one any one anyhow you have taken second one. So, the second thing is equation I mean second one is chosen. The second one can be written in this form, you see X 1 plus 2 by 3 X 2 plus 1 by 2 X 4 equal to 10 by 3 is it not. So, that you can write this is our second constraint.

Now, this second constraint can be written in this way 10 by 3 can be written as 3 plus 1 by 3 is that all right. So, the new constraint to add you know this will be interesting, you take only the fractional part, see X 1 is not fractional because it has an coefficient of 1.

So, it is the fractional coefficient part should be taken only. So, fractional coefficient is 2 by 3 here it is 1 by 3 and here the fractional value is 1 by 3. So, this is our new constraint to add all right. For the time being quickly let us see what could have happened if we would have chosen the first one.

So, again let us see that supposing, we chose the first value, because there was a tie you see this can be written as 11 by 3 x 2 plus x 3 minus 2 by 3 x 4 equal to 31 by 3 that is the equation.

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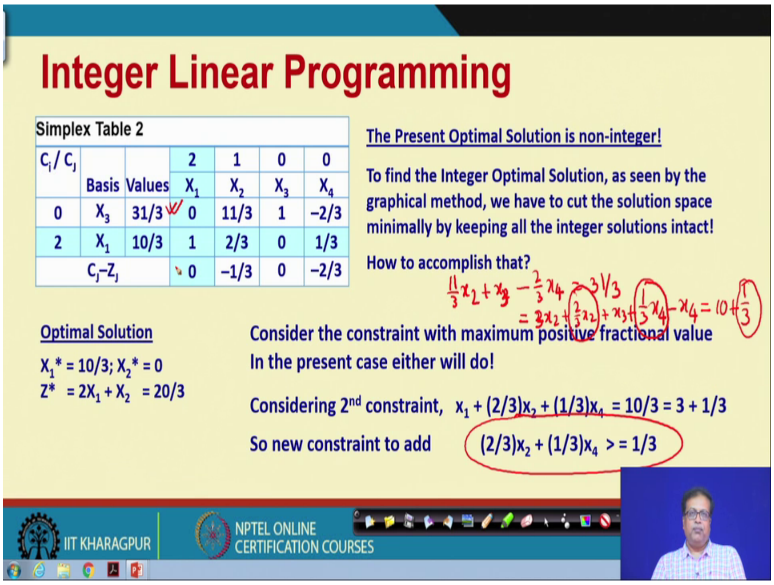


You know 11 by 3 x 2 plus x 3 x 3 plus minus 2 by 3 x 4 equal to 3 and 31 by 3. So, these can be written as so, 11 by 3 it becomes 2 sorry 3 x 2 plus 2 by 3 x 2 x 3 can be written as x 3 and minus 2 by 3 x 4 has to be converted to the positive fractional part.

So, what you can do you can write 1 by 3 x 4 minus x 4 equal to so, 31 by 3 so, 10 plus 1 by 3. Now, take the only the positive fractional part in the coefficient and in value. So, this is a positive fractional part this is not to be taken this is a integer portion this is also integer portion. So, you know this is also integer portion.

So, this is the positive fractional part in the coefficient and this is the positive fractional value. So, taking these three into account again you would have got 2 by 3 X 2 plus 1 by 3 X 4 greater than equal to 1 by 3. So, we would have got the same constant if we would have chosen the first constraint line also, but look here this point is coming because in both these constraints the fractional value part is equal, positive fractional value is equal that is why it is the tag.

Point to remember that you know whenever we have tie like this in the positive fractional value, it does not mean the both consideration of both will give the same constraint line to add it may not be it will depend on case to case. It is incidental, that we have got the same constraint line from both the same constraints in another problem another thing may happen is it all right.

And depending on that procedure may be more lengthy less lengthy, but whatever it may be the solution should be possible to find fine. So, we know the constraint to add.

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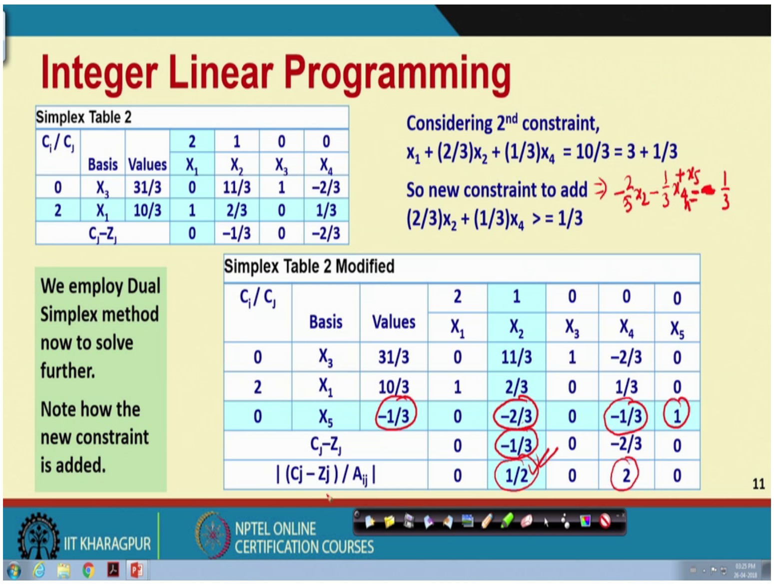


So, if we know then let us move over to our next slide and then we see, how do I add the new constraint? You see what we do is very interesting? We write it the negative way right we, write in this manor minus 2 by 3 x 3 minus 1 by 3 x 4 equal to 1 by 3. So, this less than equal to will be then greater than equal to and then here we add you know a variable say x 5, which is our slag variable. So, same thing is done here. So, minus 2 by 3 X 2 you see minus 2 by 3 X 2 minus 1 by 3 X 4 plus X 5 equal to minus 1 by 3.

So; obviously, this 1 should be minus also because 1 by 3 then will be because we written in the negative way. So, that should be minus also it is ok. So, that is how we have put that constraint here all right. Now, moment we do that we have to employ what is known as the dual simplex method. We do not have scope to explain what is dual simplex method exactly, but just remember I am telling you the procedure the procedure is that we write this C J minus Z J, values and then this C J minus Z J values, we divide by the corresponding you know part. Suppose this is the minus look at these the living variable will be from the values, which is most negative here only 1 value is negative. So, this X 5 will be leaving variable.

Now, the corresponding coefficient for the X 2 is minus 2 by 3. So, you might divide the C j minus Z j by A i j; that means, corresponding value in the in the row then we get the absolute value of C j minus Z j by A i j, and this is half here, and this is 1 here, 2 here we take the minimum the half is the minimum, that is the procedure.

So, the minimum therefore, X 2 should be leaving right. So, understood what we did we have got the new constraint to add. The new constraint is put in a negative way then put in the here and then use dual simplex, because we have a negative value, then choose that particular constraint as leaving variable, which is having the negative value in this case only 1. So, this will be the leaving variable and the entering variable will be the minimum value of C j minus Z j by A i j. So, this is how we do?

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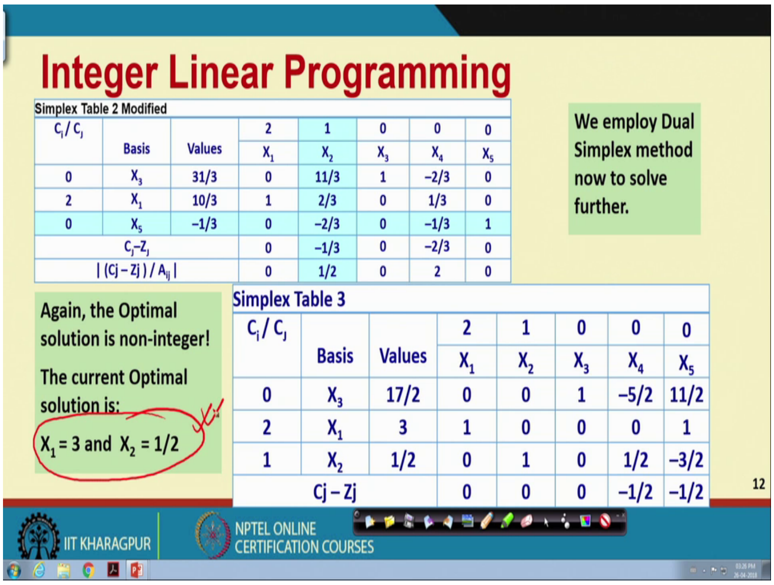


So, once we do that then we find in the next slide, that this is going to be our new solution space. So, what is our optimal solution we have found at this point? We found X 1 equal to 3 you know X 1 equal to 3. So, this was the table by solving we have found this 1 by usual simplex, but then we found we have found X 1 equal to 3 and you know X 2 equal to half recall that this was our next point right.

So, this is the next value that we have found.

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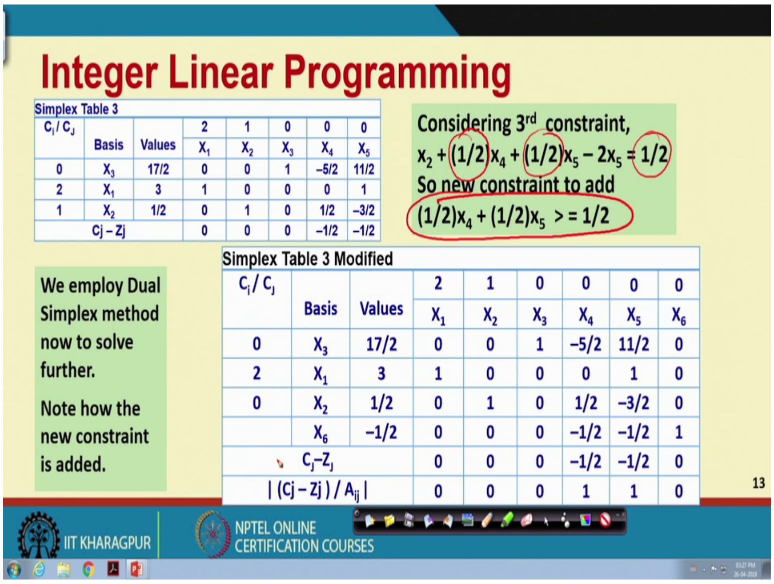


Now using that, again you know we have to see that the values which is the minimum possible maximum possible fractional value. Again, we have a tie. So, anyway we have taken the part constraint, which is half. So, these half will remain and this is minus 3 by 2 can be written as you know half X 5 minus 2 X 5.

So, this is the fractional part right. So, this half this is the positive fractional part and this is the positive fractional value. So, new constraint to add will be half X 4 plus half X 5 greater than equal to half. Now, it is again added in the negative way. So, X 6 minus half. So, we add it you see the size of the problem has gone up minus half minus half minus half and X 6 1 and that will have the canonical form again we compute C j minus Z j by A i j right and we have the modified simplex 3 table.

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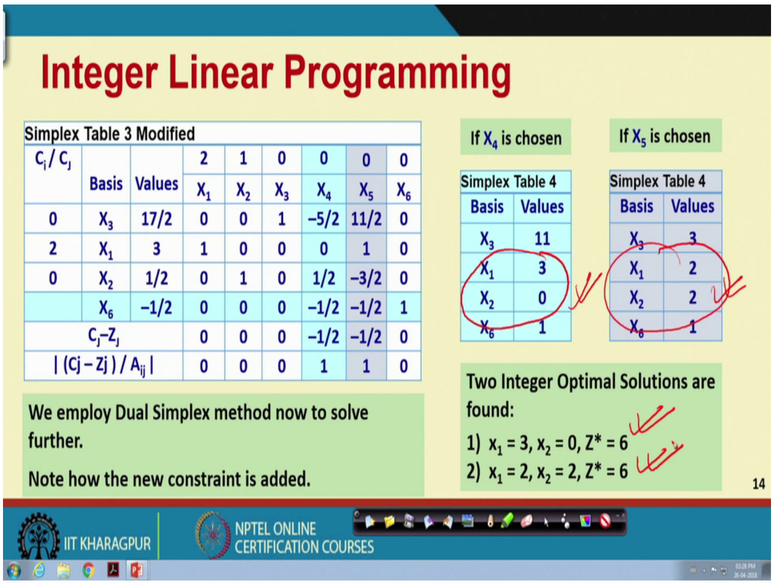


So, then once we have these then this is our modified simplex table, when you solve by dual simplex the good thing is at this point we find the integer solutions right.

We find the integer solutions and there is a tie in the entering variable right. Depending on which one you choose following the dual simplex we find these are the 2 solutions right. This is one and this is one and look here in this solution and these solution we have got the 2 integer optimal solutions.

So, the good thing about dual simplex that once the solution is found at that level there is no need to you know really compute the whole table, why because in dual simplex the problems are always optimal you are moving from op you know optimal you are you know you are keeping the moment you have found something fusible, you know the problem is always optimal, but you are looking for fusibility.

So, moment you have found a fusible solution then you stop. So, we found a fusible solution which is integer. So, we stop is all right. So, a I have also you know shown both graphically as well as through simplex, how do you use cutting plane method to find integer, linear, programming, problem solution right.

Thank you very much.