Design and Analysis of Experiments
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Lecture – 56
Analysis of Second Order Response Surfaces

Welcome, we will discuss analysis of second order response surface. In last class I have shown you that how to fit second order response surface given the experimental data in this class we will revisit this for 5 minutes and then I will stay to go to the analysis part and it will take around 35 to 40 minutes of time.

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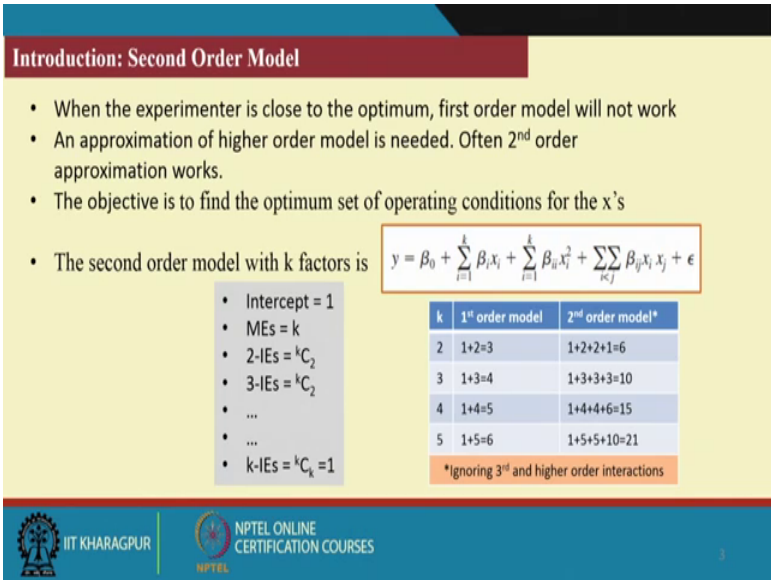

And the primary contents of this lecture is taken from the book written by Montgomery Design Analysis of Experiments. So, you have seen this slide in last class also, that the second order response surface will be like this; y equal to beta 0 plus i equal to 1 to k beta i x i, then the second order term, then the interaction term and error term will also be there and you have seen that when the number of k factors increases, the parameters also increases. Find these other things what we have discussed earlier also

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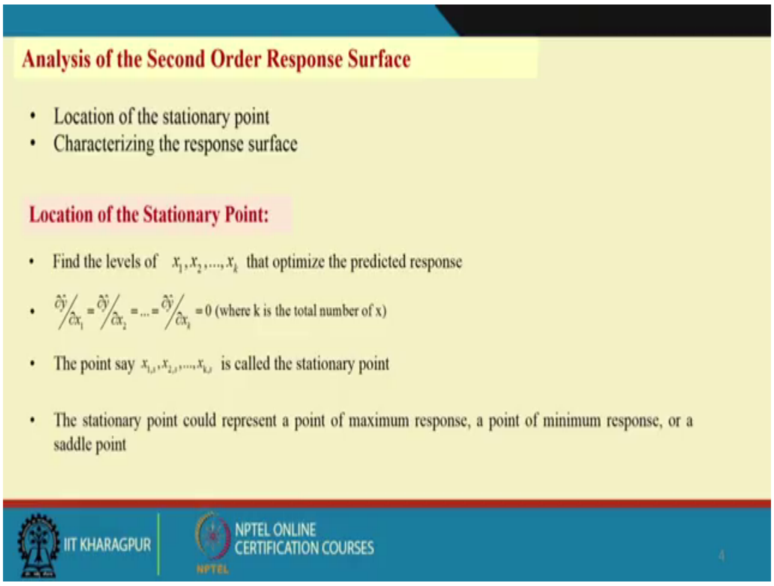

Now, question is that, what are the issue that we should discuss when we talk about analysis of second order response surface. The second order response surface will be used to find out the optimum zone or settings for process operation or the system at which, where should we operate the system or the process. So, that we will know that where is the location of the, where is the location of the maximum yield or maximum response, let such or minimum impurity that also minimum response something like this. So, first rule one is that you have to know location of stationary point.

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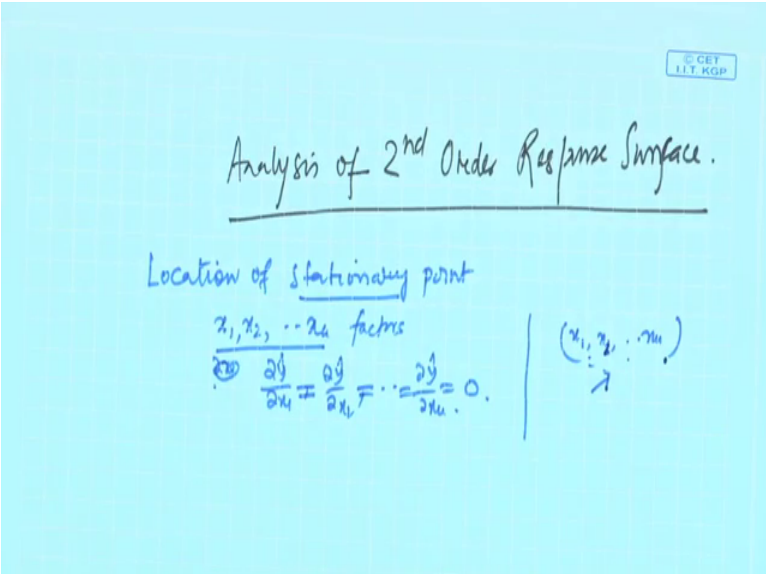

Stationary point means the point where the first derivative is equal to 0 ok. Suppose if we if we consider x 1 x 2 like x k, these are the, these are the factors which basically define the design space, then del x 1, sorry del y cap by del x 1 del y cap by del x 2. So, like this del y cap by del x k; so this all, I think all will become 0 at the stationary point. So, then if x 1 x 2 x k, suppose these are the factors, but at the same time if we say that the point x 1 x 2 then x k, this is the point, because there are a k dimensions and there will be one point where values are x 1 x 2 x k, let it be small x 1 x 2 x k.

This is the point where the derivative, first derivative equal to 0, then this is a stationary point. So, you have to first find out the stationary point and then what is the nature of stationary point, whether it is a point of minimum or point of maximum or point of or a saddle point, where it is not sure point of maximum. It is not a point of maximum or minimum saddle point ok

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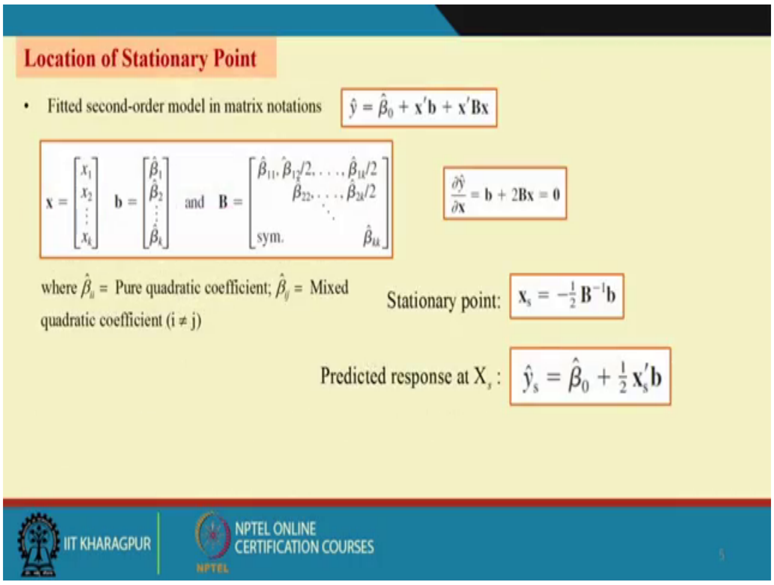

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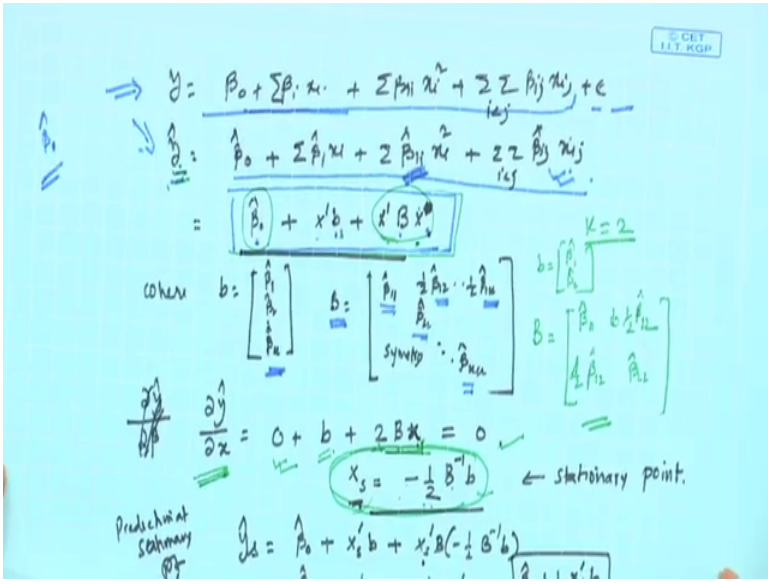

So, now let us see that how do we find out the point that location of stationary point and the related computation. This part is known to you. Now second order equation, when you estimate the beta then we write, these are all beta cap or beta hat. So, then the fitted value is y hat. So, y hat will be this portion except arrow term. So, this is what is y hat ok.

So, now, this one can be represented like this beta 0 cap cap plus x transpose b plus x transpose capital B x, where small b is the estimate of the first order coefficients, the beta

1 cap beta 2 cap beta k cap and capital B is basically it considered talking, it is basically con, it basically contains the second order that coefficient; that is quadratic part beta 1 1 beta 2 2 beta k k and the interaction part like beta 1 2 beta 1 3, beta 1 k and it will be half 1 by 2 1 by 2 and this side will be symmetry. So, ok

So, now if we consider that k equal to 2 that was considered k equal to 2, then your b equal to beta 1 cap and beta 2 cap and your capital B equal to beta 1 1, then beta half half beta 1 2 and then here again half half beta 1 2 and beta 2 2 cap, we all cap ok. So, this is what is capital B now. So, we can say that the point, the response surface is our fitted response surface is y cap equal to this and we say the location of stationary point is 1, where del y cap by del x is 0.

So, del y cap by del x equal to 0. So, if you take derivative with respect to x, then beta 0 being the constant, its derivative will become 0 x transpose b derivative will be small b, and this quadratic term there will be 2 times b into x. So, we put equal to 0 and then you get the value of x, which is minus half B inverse b and this x is nothing, but the stationary point for x ok; so 2 b x equal to minus b so x equal to minus half B inverse b inverse b; that is the point, stationary point ok.

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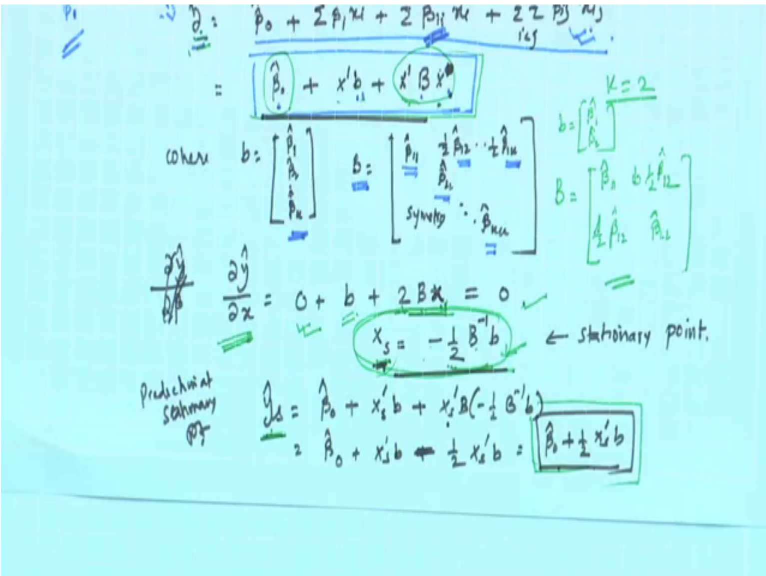

Then using this x s equal to these you will be able to find out that the predicted response at the stationary point will take, will be like this; that is beta 0 half of x transpose b. So, if you want to derive it, derive it from the.

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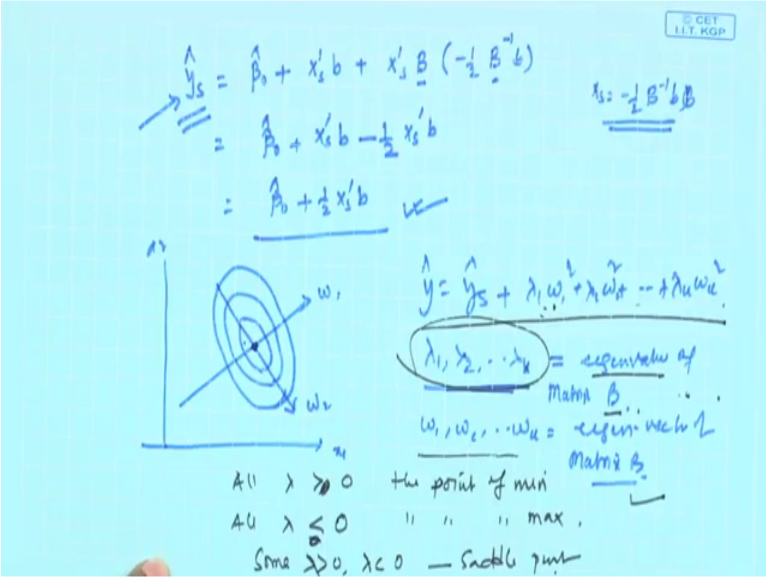

From the basic equation y s cap equal to beta 0 cap plus x transpose s beta plus x transpose s b into x, there I will write minus half B inverse b, because x x equal to minus has b inverse b. This we have seen, we have seen that at this equal to 0, x is equal to this.

Now in this formula we put x x equal to this value and then you are getting like this. So, what is this, then this is beta 0 cap x s transpose b minus half b and b inverse will be i. So, this is nothing, but the x is transfer b again. So, then this one is beta 0 cap plus half of x s transpose b. So, this is the predicted value at stationary point

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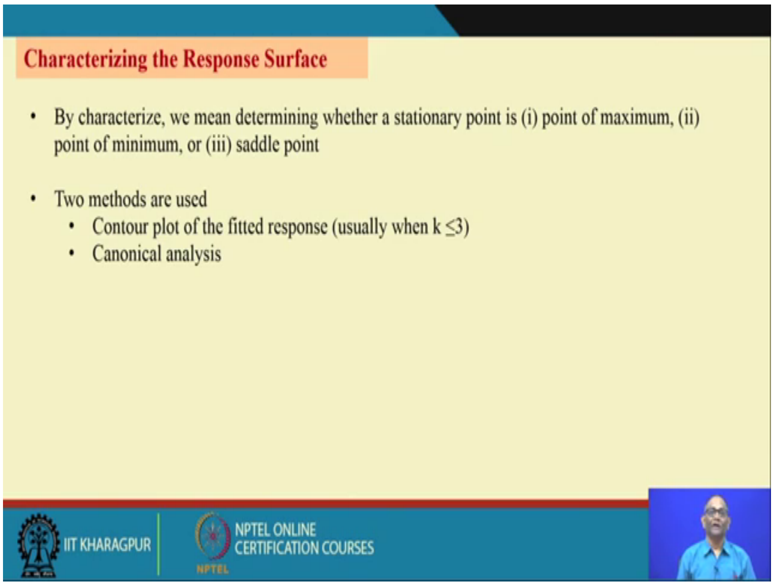

Now, we want to know whether this value is, and is the optimum value or not. So, in order to do; so you do one thing, either you go for contour plot of the fitted surface or canonical analysis, you do, because the stationary point can be a point of maximum, a point of minimum or a saddle point, determining whether the stationary point is a point of maximum point, a minimum or saddle point. So, that is done through two methods; one is contour plot another one is canonical analysis.

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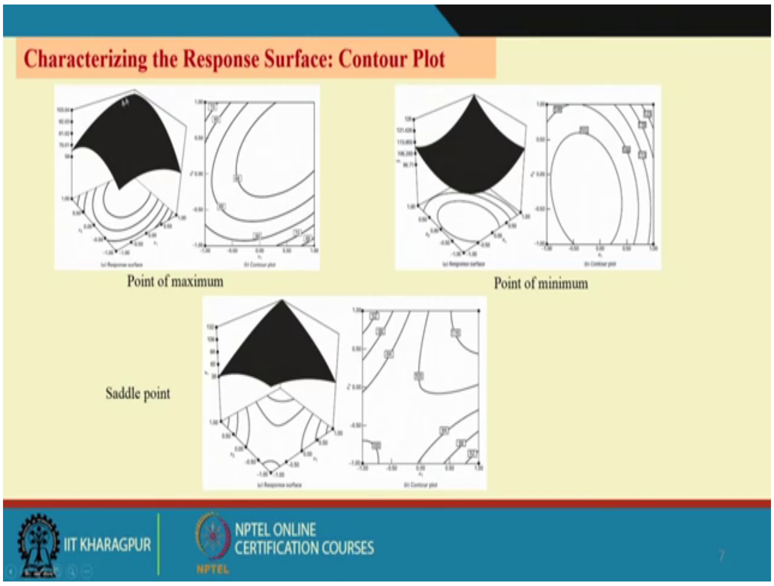

So, let us see the contour plot. I have discussed contour plot several times earlier also when we have we have we have fitted the factorial experiments and then found out the regression equation and then the effect of x on the y through contour plot, we have explained several times. So, here, here you see, this is what is our response surface. So, here there are two variables in coded terms, we have written x 1 and x 2 plus minus 1 2 plus 1 minus 1 2 plus 1 and this is your response variable.

Now, if you see the contour plot, you see the contour plot. Contour plot mean, here it is the contour plot mean all the point on this line will have the same y response value. So, that mean here 96, then you have 88 then 80. So, if you go along this direction or the direction it is basically reducing. So, that mean this is the direction of improvement and somewhere here at the middle it will improve. So, this basically talks about, suppose this point which is somewhere here. So, this is point of maximum, because of this shape.

Now here what happen the shape says, this is valley or in this case point of maxima, saying it is a hill here, it is your valley and you are getting point to minimum also, but the third one if you see the contour plot, you move you go to one direction like this direction, then 100 and 116, here 100 then 84 68 52, here 84 60 52. So, it is, it is basically, it is difficult to tell that whether you are a point of maximum or point of minimum. So, it is a point which cannot be said either minimum or maximum. So, that is the helper.

So, contour plot will help you, but you know that contour plot is possible graphically for two dimensional or at mixed three dimensional case, when k equal to 3 or k equal to 2 or 3, but when k equal to more than 3, it is really difficult for all of us to use the contour plot, we require a better method of doing this work and that better method is known as Canonical Analysis

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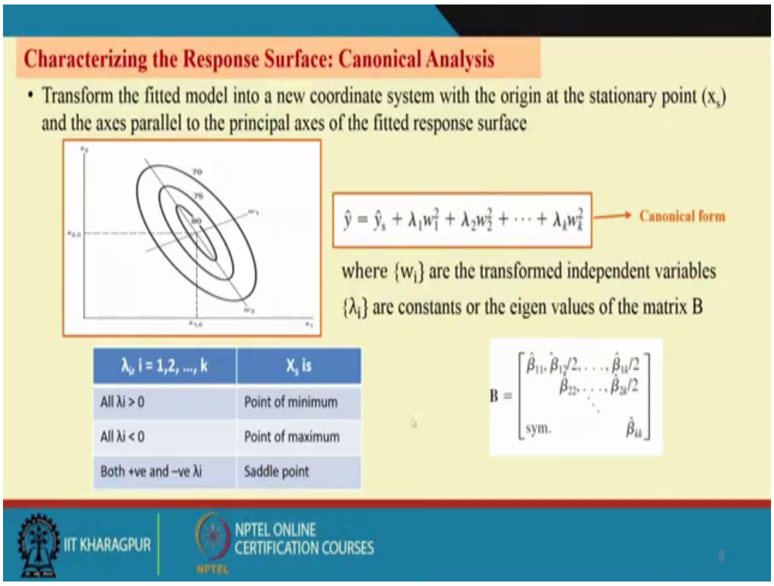

What is the approach here in canonical analysis you see. So, this x 1 and x 2, suppose these are the contour plots. So, at this and the inner ellipse that talks about the response of y equal to 80, then middle 175 and the outer one is 70. So, somewhere here the maximum or ill or maximum response lies. If you see the principal axis of the ellipses and the axis, axis of x 1 and x 2, you see that they are not parallel. So, what in canonical analysis we want to do.

We want to first ship the 0 point to the center of the ellipse and then second we transform the x1 and x2 along, along that principal axis of the ellipse. So, you want to access like this, this one is one and another one is another y w1 and w2, which are parallel to the principal axis of the ellipse, not that parallel to x1 and x2 so you transform it. So, then what happened in the transformed dimension, your respond that response predicted will become that response at the stationary point response, means predicted response at the stationary point plus lambda 1 w1 square lambda 2 doubled square lambda k w k square.

So, this is a special form which is a typical form, which is known as canonical form ok. So, originally we are here x1 and x2, my new dimension is there w1 and w2. Why you are doing like this? Because your ellipse, the response contours they are basically their principal axes are axes are parallel to parallel to that w1 w2 dot w1 w2 the two dimension parallel to the principal axis, we want that.

So, here then you first find out what is the predicted value here and then at the canonical part. So, your y cap will become y s cap plus lambda 1 w1 square lambda 2 w2 square dot dot dot lambda k w k square, where lambda 1 lambda 2 dot dot dot lambda k. These are basically like, this is basically. They are the Eigen value, Eigen value of matrix matrix capital B that you have already seen and w1 w2 w k are the Eigen vector Eigen vector of matrix b

So, just seeing their lambda values you are in a position to find out the. So, seeing the lambda value you are in a position to find out this, whether the stationary point is a point, a minimum point of maximum or point of inflation or that is the saddle point. So, let me repeat here that you, once you do this. So, you are getting this then lambda values also are there. Lambdas are basically Eigen value of b and w or omegas are Eigen vector of b. And then seeing the value of lambda you will note you are in a position to tell whether the, whether the stationary point is point a maximum point of minimum or saddle point.

If all lambda, all lambda greater than equal to 0 then it is point of minimum. If all lambda less than equal; sorry if all lambda greater than 0 and all lambda less than 0, this is point of maximum and if some lambda, some lambda greater than 0. So, other lambda less than 0 then this is saddle point. So, this is much.

So, it is better, it is acceptable better model than the contour, because you can find out the all the Eigen values of b and, but just by seeing the eigen values very objectively you are in a position to tell whether you have gone to the maximum point or minimum point or the, it is not neither of the two ok. Essentially what you require, you require the matrix b and that also we have seen earlier.

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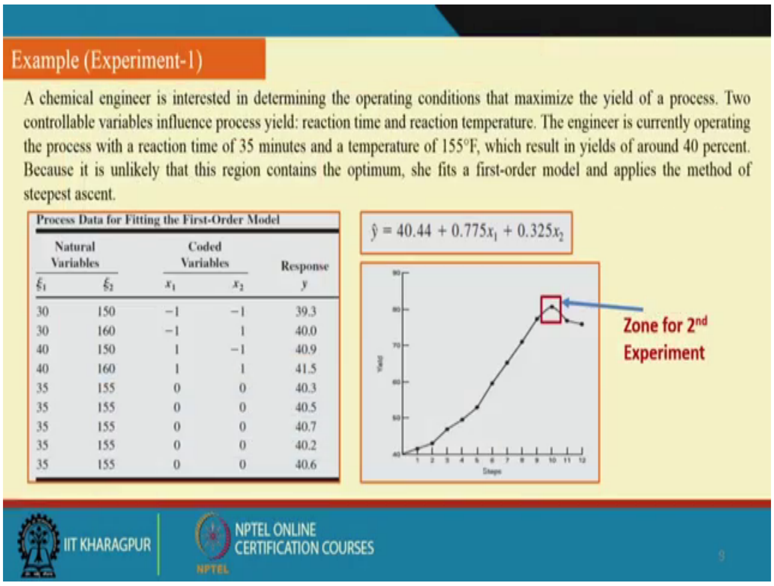

So, with this background let us see the example, this example we have discussed in several classes.

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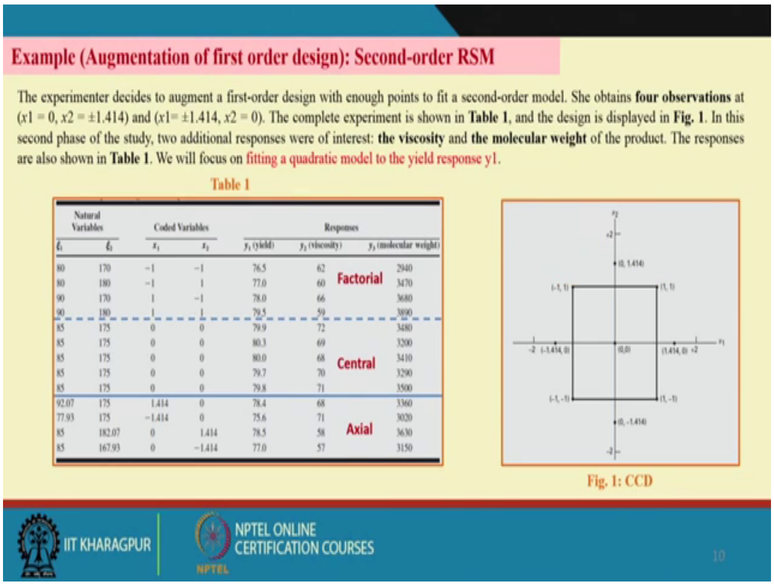

Now, in last class I have given you, shown that the C C D is used here and we have.

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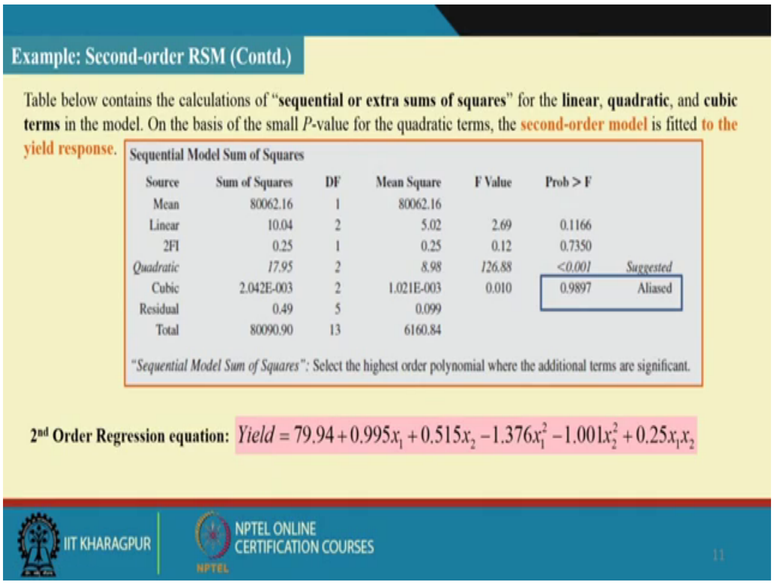

We have also got this, this one, the second order response surface this one. What is your second response surface for the example, example y.

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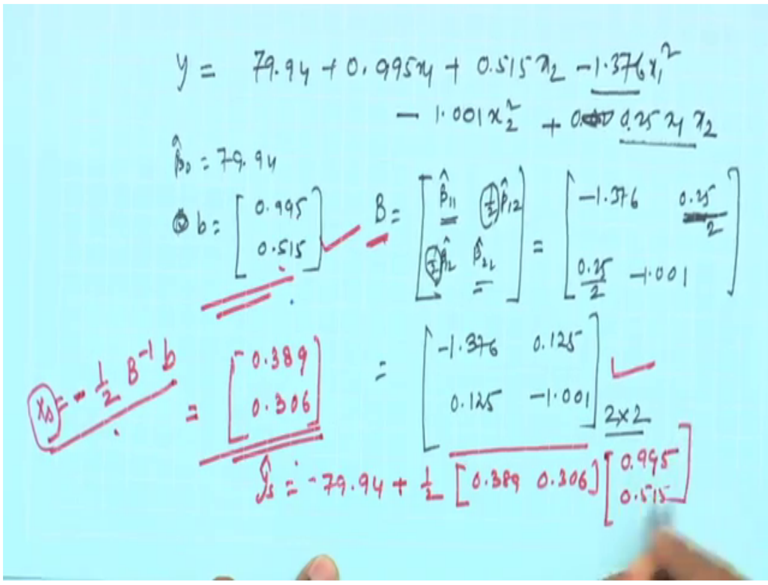

Which is, basically yield is 79.94 plus 0.995 x 1 plus 0.515 x 2 minus 1.376 x1 square minus 1.001 x2 square plus 0.25 x1 x2 ok. So, then what is this beta 0 is 79.94 then b, b is your 0.995 and 0.515 and capital B is what that beta 1 1 cap beta 2 2 cap half of beta 1 2 cap and half of beta 1 2 cap of here. So, this will be, what is beta 1 1 here.

Beta 1 1 means minus 1.376, what is beta 2 2? Beta 2 2 is minus 1.001, what is beta 1 2, beta 1 2 0.25. So, this is 0.25, but please remember there is a half so of diagonal element with half. So, this by 2 and this by 2, another minute B matrix will be minus 1.376 0.125 0.125 minus 1.00 1 2 cross 2. So, let us now analyze, let us now analyze this response surface using the methods that is known as canonical analysis.

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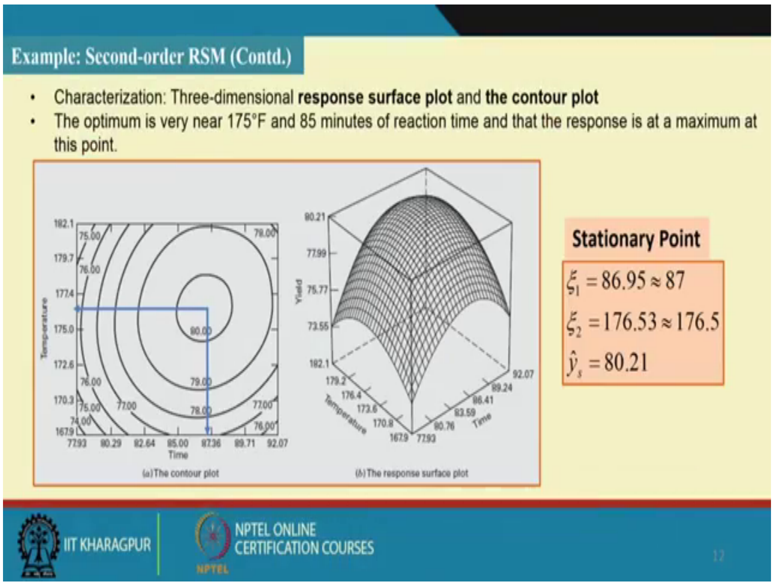

So, first one you see the contour plot then I will show you that you know. So, if you see the contour plot you are getting like this and here is the point; that is the point, stationary point and if you just project across this axis; that is time and this axis temperature. You are finding out the value, value for that time is around 87 and value for the temperature is around 167 76.5, and at this point what is the value of station yield is 80.21 and now ok. From this plot you are finding out this is, this is a point of maximum, because you see that the response surface is a hill like structure.

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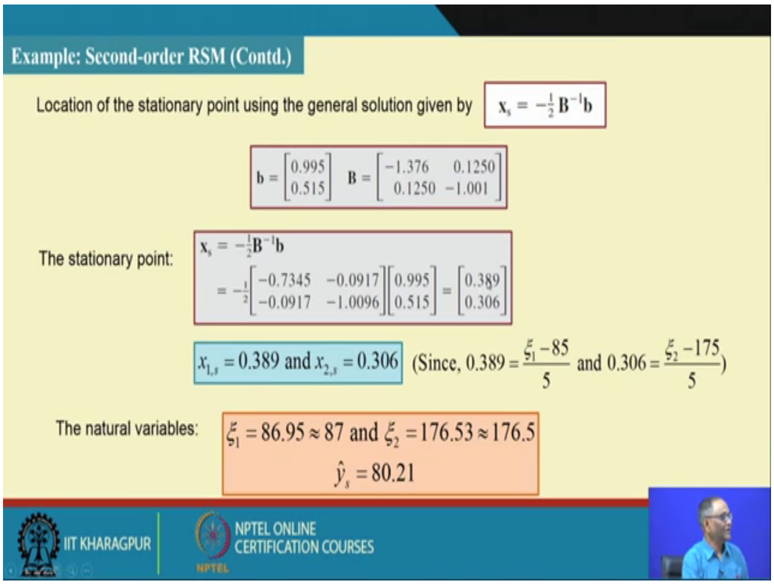

Now, I want to, we want to use the canonical part. So, what is the stationary point x s is minus half B inverse b Now, b part is given to you. So, b part is, b part is small b is this and this one is this. So, what you require to do. You require to find out b inverse ok; so b inverse in to. So, what do you want half B inverse b; that is minus of that is your x s minus of B inverse b is x x.

So, this value, this value when you compute, it will become 0.389 and 0.306. So, B is this, capital B is this, you find out the inverse of it multiplied divided by half minus symbol you use and then you will be getting you know. So, I mean this is the point x, this is the stationery point in terms of design variables, then what you want? You want y s this value. So, what is the y s value we have computed earlier? y s value we have computed that y s that is predicted y at s is beta 0 plus half x inverse b, all those things are known to us

So, if you apply this one then beta 0 is 79.94 plus half your stationary point, its transpose 0.8389 0.306 and into b, what is your b point 0.995 and 0.515.

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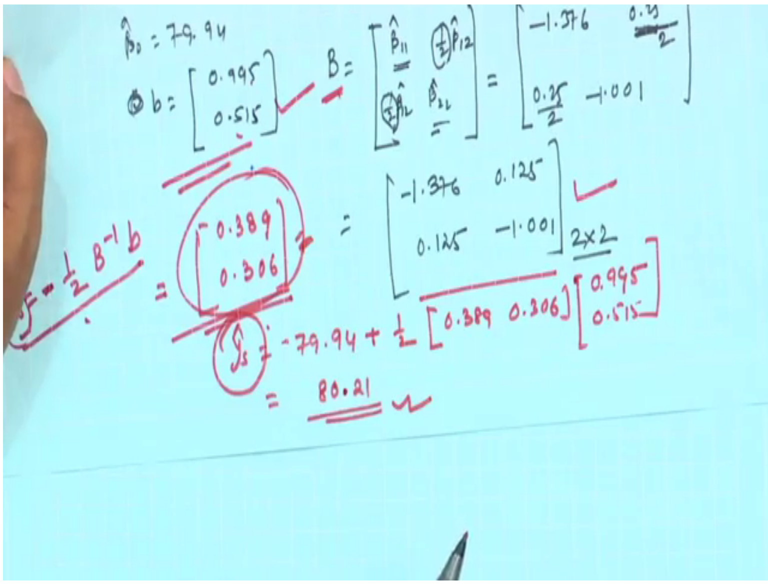

So, this, this value is 80 point how much. It is a 80.21. So, you are getting one end. What is the stationary point, what is the predicted response? A stationary point right. Now what do you want whether this stationary point is a maximum point of maximum, because the contour chart says this is point of maximum.

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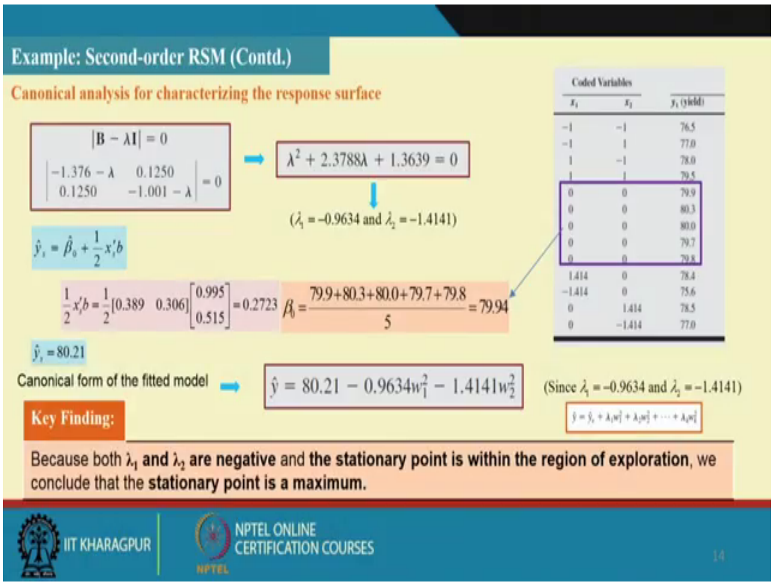

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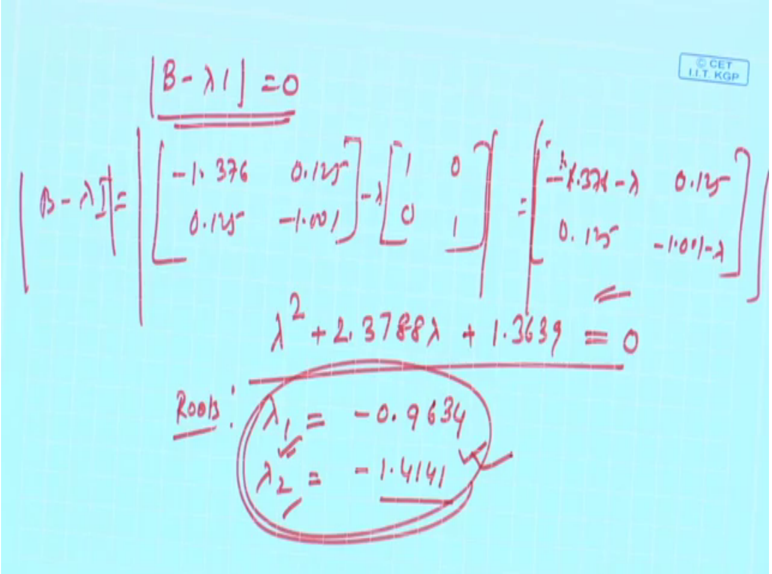

So, in order to do, so what do you require? You require to find out B minus lambda i determinant equal to 0 to get the Eigen value and Eigen vector and your B is minus 1.376 0.125 0.125 minus 1.001, this minus lambda 1001. So, this is what is b minus lambda i equal to minus 1.376 minus lambda 0.125 0.125 minus 1.001 minus lambda. So, this is my matrix ok

So, now you take the determinant and then the determinant of these determinant of this. This will give you the equation called lambda square plus 2.37 8 8 lambda plus 1.3639, this put equal to 0, this you put to 0 and then find out that the roots, find out the roots of this equation lambda 1 and lambda 2. So, lambda 1 minus 0.9634 and this one is 1.4141 ok. So, now, you know you have lambda 1. So, now, all lambda 1, lambda values are negative. So, this is a point of maximum. Once you know the lambda values you know whether point of maximum or minimum is reached or it is a saddle point ok

Now, come back to this. So, both, because both lambda 1 and lambda 2 are negative and the stationary point is within the region of exploration. We conclude that the stationary point is a maximum canonical form of the fitted model. Now here we have shown that how beta 0 is calculated; that is the average value at the center point fine.

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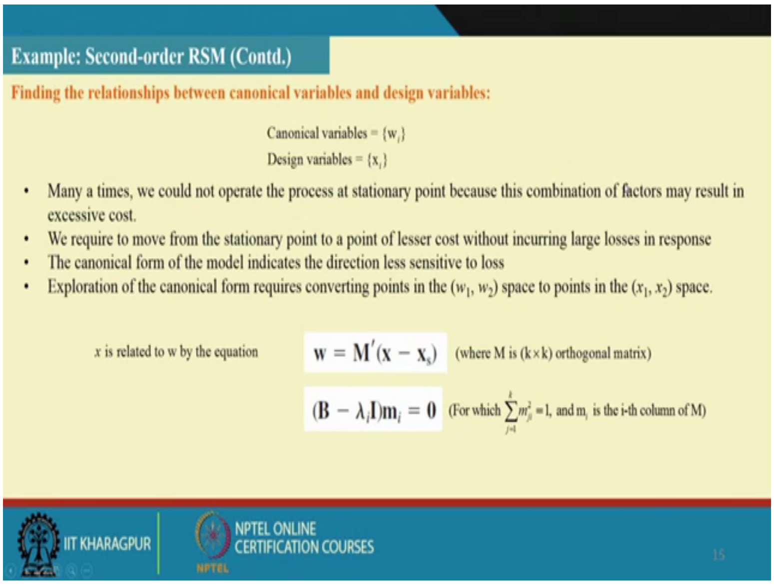

Now, now what happened when we could not operate the process at the stationary point? So, many a times we could not operate the process at stationary point, because this combination of factors may result in excessive cost. We required to move from the stationary point to a point of lesser cost without incurring large losses in response, the canonical form of the model indicates the direction of less sensitive to loss exploration of the canonical forms requires converting points in the w 1 w 2 space to the point in the x 1 and x 2 space, what is happening here. Let us see.

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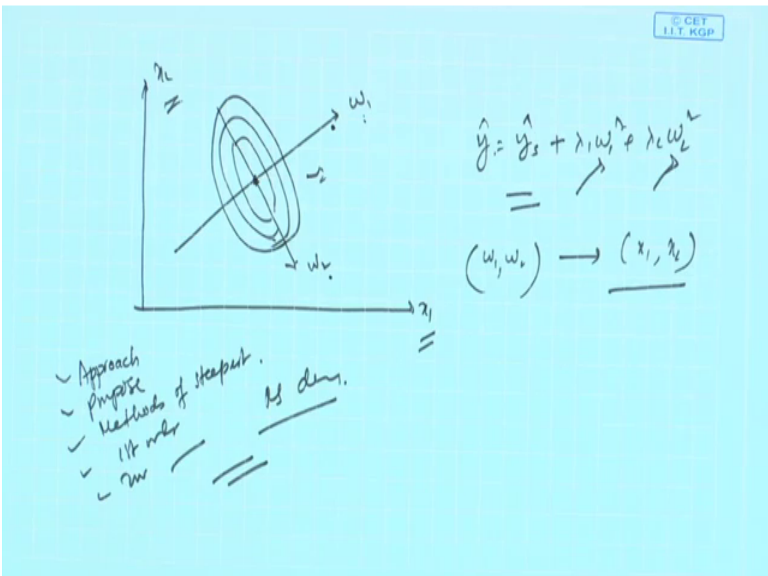

So, you have seen that. So, our stationary point is this and then this one is w1 and this is the w2 axis, and suppose your response surface your contour plots are like this. Suppose you would cannot operate here, because maybe cost is prohibitive or the technicalities is a problem for operators point of view or something, something like this. So, then what happened, you want another point another location another zone, where you will operate the process, you will not incur much loss at the same time, it will become convenient to do it.

So, that is possible, because you know that the directions x y 1 w1 and w2 and you know that which direction if you go. If you go to the direction, what is the loss, minimum loss will be in which then directional direction that also known to you, because you know the function, you know the y cap in with in terms of y s stationary point plus lambda 1 w1 square plus lambda 2 w square we. They go along this diamond selection or the direction or depends on the value of lambda will direction, you will incur less loss

So, this is one advantage of having this. Now second is that ok, suppose you will you will go along w1, because lambda 1 is giving you better situation then question is that, you know those in w1 and w2 dimension, but you have to experiment it. So, you have to convert back to x1 and x2; so w1 and x1. So, w1 and w2, this dimensions it should be, this must be relation with x1 and x2 dimension, this how do you know.

So, there is, there is the concept called, there is a concept called, this w is related with x with this equation w equal to M transpose x minus x s, where M is basically a orthogonal matrix and it is k cross k dimensions, because we have k number of factors, then this ultimately leads to this b minus lambda i i equal to 0

Now, you already know the lambda values put 1 lambda and then find out the m i m value, put another lambda, find out the M value and then what happened. Once you know this m; that means, m the M transpose is known. Now using these equations you will get the relations between x m omega.

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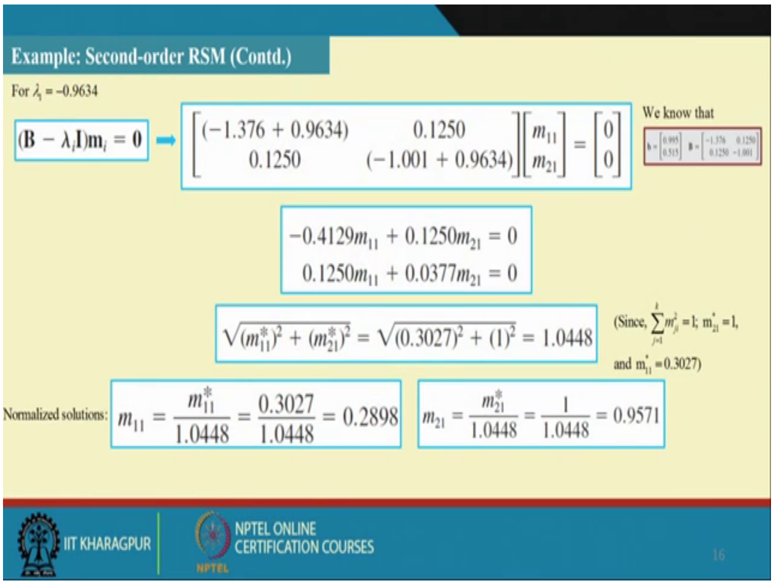

This is what we are doing here. We know b, we know lambda. Now you put the first one what is the; so minus this into this value. So, minus is there minus plus minus is minus minus is plus. So, your resultant matrix is this and it is multiplied with m1 1 and m2 1, because your head is the two dimensional case, then you are getting these two equations. Now these two equations wants you and two unknowns. So, you can solve it, but you will get a situation, where you find out that you will get infinite number of. So, an unique solution you will not get here, and as a result you require to normalize it.

The normalization one is this j equal to 1 to k m i j i square equal to 1. So, that is what we have done m1 1 square plus m2 1 square, this square root is this value and then using this three, this one and the, so this three you are finding out m1 and m2. So, from these two you are getting the relation between m1 and m1 1 m2 1 that is less than you are getting.

Now, using this one, this normalization you are able to uniquely estimate the value of m1 1 and m2 1. This is very simple one is, just you know are know1 thing you write down in terms of relation m1 1 equal to in terms of m2 1 from the first equation and from the second equation you will get the similar thing. So, you need adage in financial number of solution to unique solution, because of this normalization. So, now, m1 1 m2 1 is known. So, with reference to lambda 1 it is known. Now reference to lambda 2, second Eigen value, again find out the Eigen that m1 1 and m2 1.

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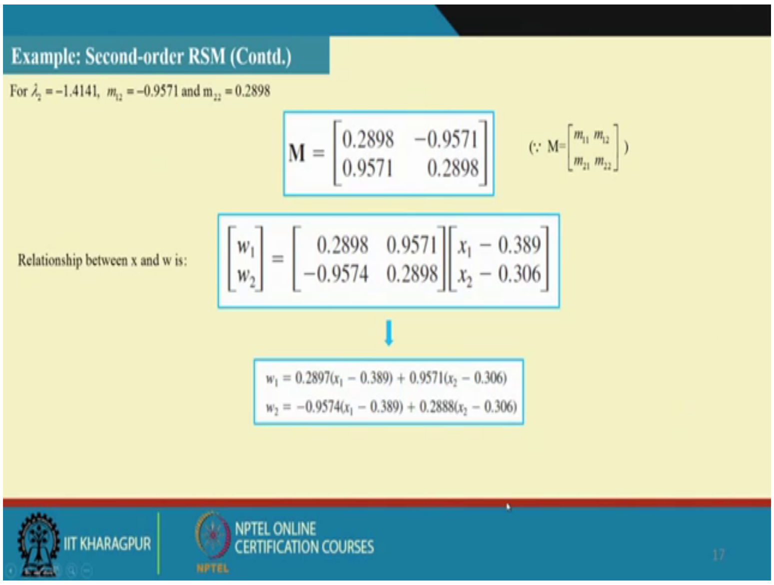

So, then you will ultimately found out the m m is 0 point this, this value you see m 11 28989 0.9571. So, 2898 0.9571 and this is from coming from the lambda 2. Now w1 w2 equal to M transpose x minus x s, this you are getting and putting this you are getting this equation ok

So, put in this you are getting w1 and w2. So, fine if we have w1 value and w2 value, you will get x1 and x2. Once you get x1 and x2 you can convert to the original variable values and then you experiment there ok. So, this is what is the analysis part.

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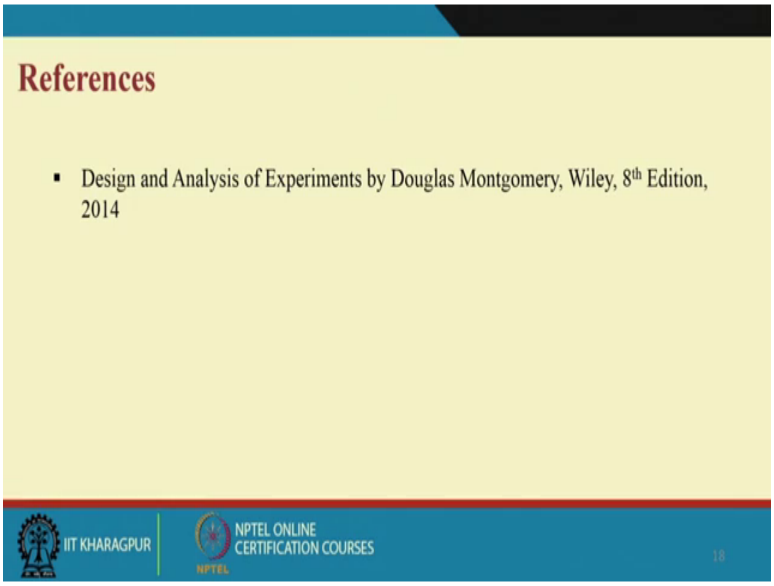

So, response surface methodology, I just conclude that. We have given you the approach, the purpose then we have shown you that to that is method of steepest methods of method of steepest ascent or descent, then first order model regression model, then second order regression model. How to compute all those things we have discussed their response surface designs then we have so now that way.

Then how to analyze the second order response surface and then when you are not in a position to operate the process in the point of maximum minimum or the stationary point, how to go to the next point of operation without incurring much losses. We have given you two kinds of analysis; contour plot base analysis and canonical analysis. So, all those are very very important and I hope that you will be able to answer the questions. So, that will be posted in the exam. And we will show you some of the Minitab based our computation in next few lectures.

Thank you very much.