Design and Analysis of Experiments
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Lecture – 47
Fractional factorial design: One quarter fraction of the 2k design

Hello, welcome. We will continue fractional factorial design. Today, I will discuss in detail with a tutorial on 2 to the power k minus 2 fractional factorial design, which is one quarter fractional factorial.

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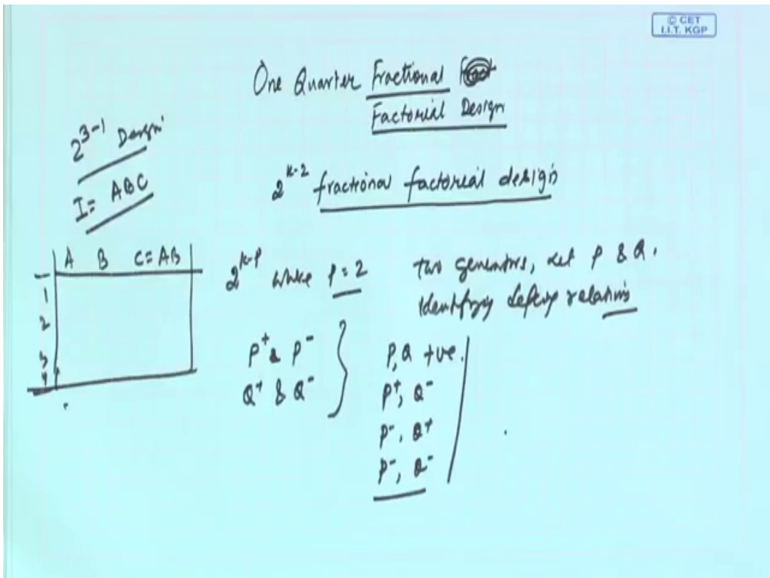

The content we will introduce or other way I can say that we will revisit the fractional factorial design.

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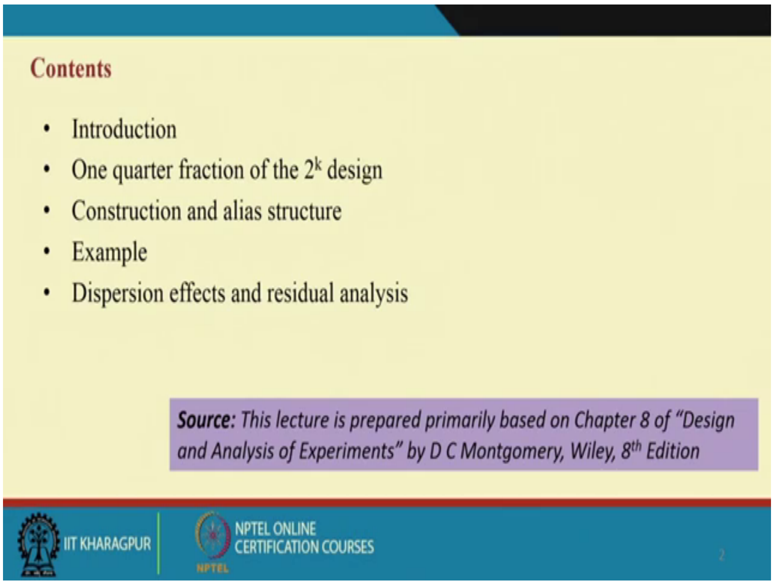

And, then we will start one quarter fractional factorial design. Well I will show you the construction of that design and the finding out the alias structure. We will see one example 2 to the power 6 minus 2 resolution 4 design and then we will discuss the dispersion effects and also we will do some kind of residual analysis for further insights of the on the experimental results.

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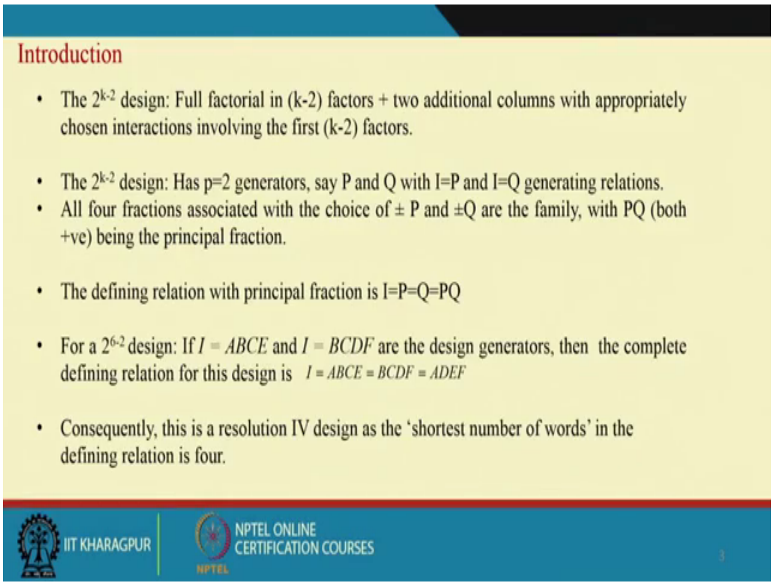

So, one quarter fractional factorial design means 2 to the power k minus 2 fractional factorial design, where it is basically from the general fractional factorial design where P equal to 2. So, as P equal to 2 there are 2 generators 2 generators let P and Q which will help us in identifying the defining relations; So, other way identifying defining relations defining relations. So, how we will do this the generators? First is P can be positive P negative Q positive and Q negative.

So, we have 4 combination both P, Q positive both P and Q positive or P positive Q negative, P negative Q positive P negative Q negative. So, this way a family of a family of generators will be created and we have seen earlier also what we have seen in 2 to the power 3 minus 1 design we have seen that I is the defining relation is equal to ABC where actually the way you have done you are done A, B and C equal to A B that was the that was the design and that is why we are having 2 to the power 4 1 2 3 4 different observations experimental settings.

So, in this example our we have we have basically a one fourth fraction. So, P equal to 2. So, we have we have 2 generators, ok. Now, let us see the slide. So, 2 to the power k minus 2 design, it is full factorial in k minus 2 factors and two additional columns with appropriately chosen interaction involving the k minus 2 factors, I will explain what is this later on here 2 generators say P and Q with I equal to P and with I equal to Q is the generating relations. All four fractions associated with the choice of plus minus Q are the family with PQ both positive being the principal fraction the defining relation with principal fraction is I equal to P equal to Q equal to PQ, ok.

So, what is the principal fraction? Principal fraction is 1, where both P and Q are positive.

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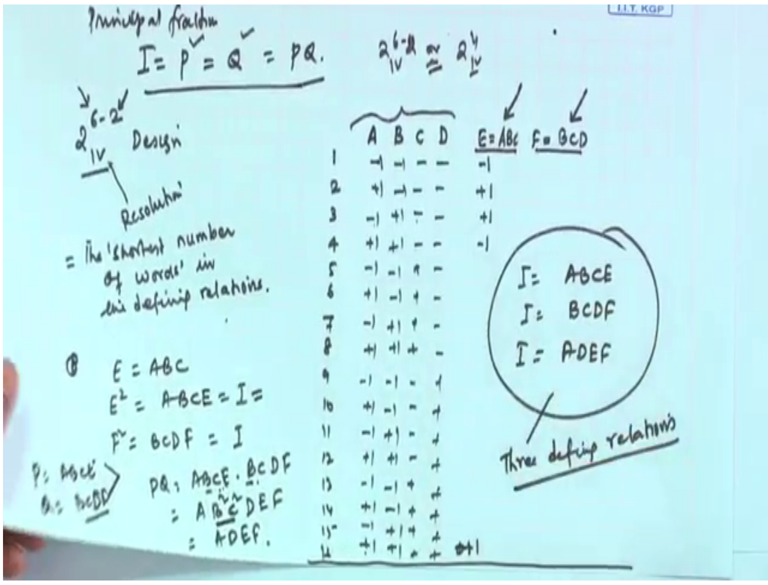

And, the defining relationship with principal fraction principle fraction I will be equal to Q equal to PQ. What is this with reference to one example we will see. Suppose, we are interested in 2 to the power 6 minus 2, 4 design I hope all of you know this 4 this is resolution and this is the number of factors and this is what is the fraction and by resolution we say the shortest resolution means the shortest number of shortest number of words in the defining relations, that is what we discussed earlier.

Now, so, with reference to this design; So, what do you want to define went to define PQ and as well as PQ. So, that can be seen with the, with this matrix. So, what we will do here it is 2 to the power 6 minus 2 design. So, that means, 2 to the power 4 full factorial design plus some additional manipulation.

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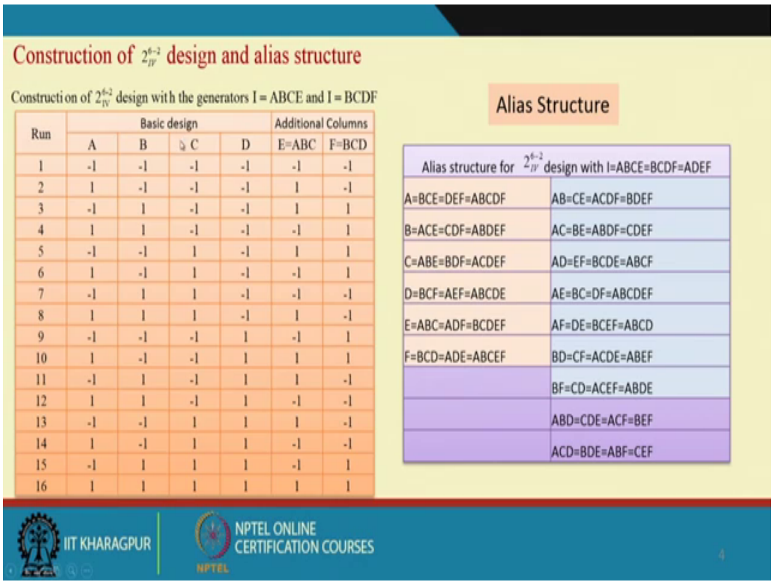

So, what do we you create the design you create first find out the basic design basic design means 2 to the power 4, 4 factorial design and then for another two factors there is some additional columns like E and F. So, what you are doing here then, how you are constructing this 2 to the power 6 minus 2, 4 design.

So, you write down 4 factors A B C D. The reason is it is 2 to the power 6 minus 2, 4 means it is 2 to the power 4 in the sense that if I it is not exactly same in the sense that that 4 factor levels it is complete. So; that means, 2 to the power 16 runs 1 2, 2 to the power 4 means 2 to the power 4 means 16 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 and 16. So, you write down that minus 1 plus 1 minus 1 plus 1 minus 1 plus 1 minus 1 plus 1 minus 1 plus 1 minus 1 plus 1 minus 1 plus 1 minus 1 plus 1.

So, B minus 1 minus 1 plus 1 plus 1 minus 1 minus 1 plus 1 plus 1 minus 1 minus 1 plus 1 plus 1 minus 1 minus 1 plus 1 plus 1. So, this structure you know ok. So, in the same way minus minus minus minus then plus plus plus then here minus minus minus minus minus minus minus minus then up to 8 then plus plus plus plus plus plus plus plus like this. Here plus plus plus plus minus minus minus minus plus plus plus plus. So, put 1 in between. So, this is full factorial with reference to 4 factors. Now, we have another two factors one E and another one is F.

So, what do you require to do you require to require confound E and F with some higher order interaction, ok. Suppose, you do it is with A B C and F with BCD then you will be able to get able to fill up this and you will be getting the getting the structure. How do you do? ABC. ABC means minus minus minus, obviously, this will be minus 1 plus minus minus this will be plus 1. So, A into B into C into minus plus minus this will be plus 1 plus plus minus this will be minus 1 in the same manner minus plus plus plus plus then this will be sorry this will be plus 1.

So, in this manner, F also you will clear and then this table will be generated you see around 1 to 16 then basic design involving ABCD and you created this plus minus and then additional column E equal to A B C and F equal to BCD then you are getting the plus minus plus 1 minus 1 accordingly, by multiplying the corresponding columns ; like E as confounded with ABC. So, A E E is the A column into B column into C column. So, that will give you minus 1 here plus 1 here like this. So, this is what is the design for 2 to the power 6 minus 2, 4.

Now, you may be saying that why should I confound E with ABC why not E with ACD or B with A ABD something like this that is also possibly. So, you can do, but what we require here we require the that E and F this two factors where for which are basically additional factors; So, considering a 2 to the power 4 full factorial design. So, you have to alia alias with other way you have to confound with higher order bit preferable higher order interactions.

So, by saying this what I mean to say, so, there can be other possibilities of confounding E and F, but this ABC and BCD this, this works well also, and if this is the case then what is the what is what is your P suppose our this C the generator is coming from this. So, E equal to ABC; So, if you multiply this with E then E into square equal to ABCE which is I and another one is F square will be BCD into F this is also I. So, if we say this is basically ABP PH ABCE QH BCDF then multiplying this 2 what you will get PQ equal to ABCE into BCDF now A is fine B and B B square C and C, C square D is fine E and F this will become I into I, I so, ADEF ADEF.

So, then what are the defining relation for this for your case here; one I equal to ABCE another one is BCDF another one is I equal to ADEF you have three defining relations, defining relations. Using the defining relations you will be able to find out the alias structure ok. So, as there are three defining relation it is possible and you will get three alias structure for every factor will be aliased with three other effects, every effect will be aliased with three others.

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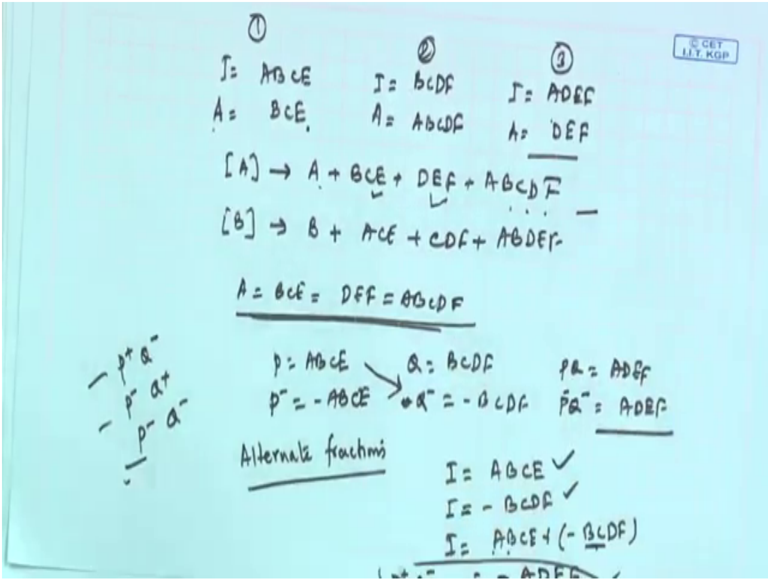

For example, if I say I equal to A B C D you multiply with A then it is BC what is this is the structure alias structure A is aliased with BCE if I take the second one B C D F multiplied by A you will get it is A B C D F if you consider I equal to ADEF you multiplied with A you will get a square, so, D E F. So, then in this design when you estimate A you basically estimate A plus B C E plus D E F plus ABCDF,. So, A equal to BCE DEF ABCDF. Similarly, you will get B B will be your B plus ACE plus CDF plus A B D E F DEF.

So, in this manner then what will happen that this defining relation 1, 2 and 3. So, what do you do you multiply every effect with the defining relation you will be getting the alias structure, ok. So, you can write in this way or other way we can write A equal to BCE equal to DEF equal to ABCDF, this is also the way we write alias structure,. So, the alias structure for this particular design with principal fraction is given in this table. You see the table first we have given the alias structure for all the main effects A B C D E F and then we are given the second order interaction effects AB AC AD AE AF BD and BF and then the third order interaction ABD ACD like this.

So, interestingly you see that here is a color combination, this color for the main effects, this color for the second order interaction effect; this is for the third order interaction effect. If you see that a all the main effects are not aliased with any second order interaction effects, but they are aliased with third or higher order interaction effects. Here you see the second order interaction effects are aliased with or confounded with the second order and here the fourth or the fifth order interaction effects and the third order interaction effects are aliased with other third order interaction effects with this design.

So, this design; that means, if we assume that that the three third or higher order interactions are insignificant then these B this third or these values BCA all those things they will become almost 0 then this is an estimate of the main effect. So, you will be able to estimate the main effect, but if you assume that the third order interactions are significance some of the thing or you in reality or in practice you find that they are releasing weekend then it will not be true.

But, from the sparsity of effect principals you know that higher order interactions are usually negligible. So, as a result I can you can say this design gives you fairly fair estimate of the main effects considering third or higher order interaction effects are negligible. But, that is not true for the second order interaction effect because second order interaction effects are also aliased with another second order interaction effects. So, you cannot distinguish this two effects, but there are Occam's razor principle, principle there are heredity principal. So, using all those things and also through the expert knowledge you will be able to identify which of the effects are significant which of the effects are not significant ok.

So, far we have discussed the principal fraction, but I said you that there are three more one is that P plus Q minus P minus Q plus and P minus Q minus so, if I say that P equal to A B C E then P minus will be minus ABCE if Q s BCDF then Q minus will B minus B C D F and if then obviously, if PQ is our ADEF then, obviously, P minus Q minus will be also A D E F ok.

So, if you choose any one of the three then these are these are basically alternate fraction alternate fractions. So, when you use alternate factions. So, your things the sign conversion things will be different for example, if you use the first one like P plus and Q minus you see what is happening defining relation is like this ABCE I equal to minus B C D F and another I is ABCE into minus B C D F. So, A is not there B, B cancel. So, this one is minus A D E F. So, this is basically for P plus Q minus. So, that mean P plus Q minus the defining relation is this.

So, ultimately defining relation is this and this and accordingly, the alias structure will change.

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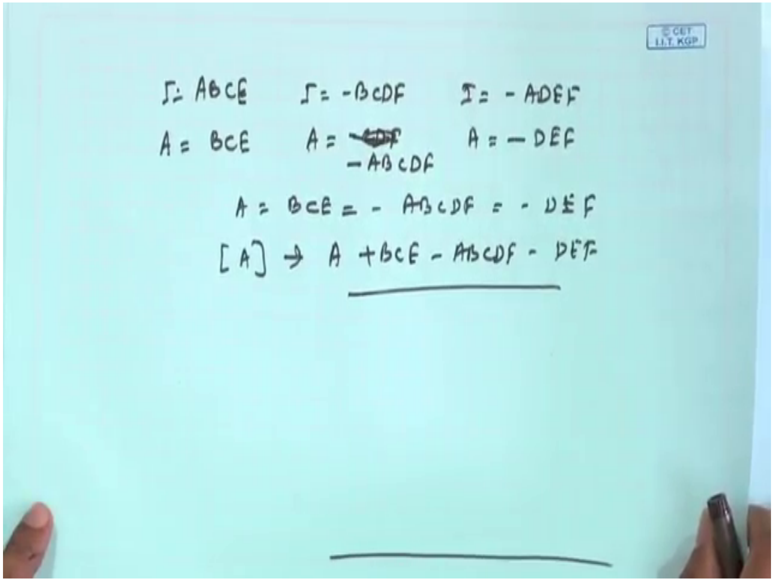

How alias structure will change if I go for this design; So, first one is ABCE that is I, second one is I equal to minus BCDF third I is generalized one ADEF if you multiply with A you are getting BCE here, but what you will get here you will get here minus CDF. What you will get here you will get here no here not minus CBF if you multiply by ABC this is ABCDEF here if you will get minus D E F. So, that mean it is A equal to BCE equal to minus ABCDF equal to minus DEF otherwise if we denote in this manner we will write A plus B C E minus ABCDF minus DEF.

So, your alias structure the sign conversion, there will be some changes in the sign, ok. So, that is why you have to be careful about what kind of that when you use the fractions what fractions you are principle or alternate fraction and then accordingly defining relations also you have to be very very careful,.

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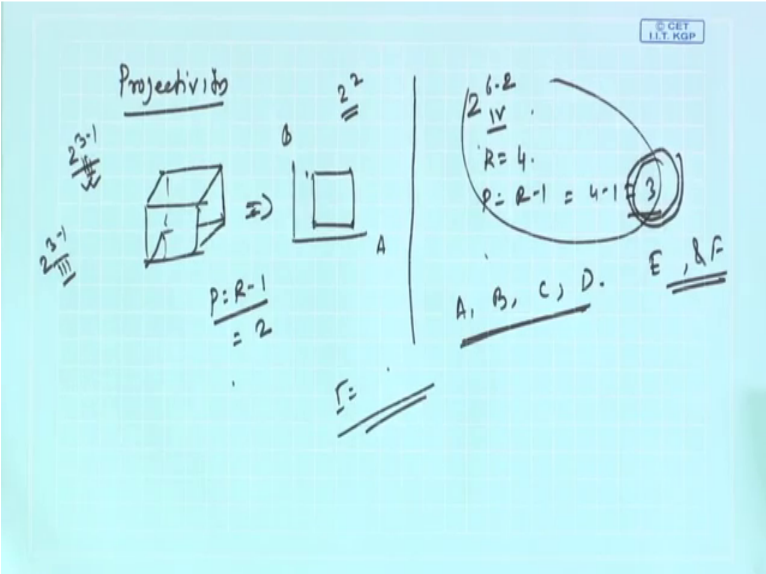

Another important thing is the projectivity. What is projectivity? in last class I I I said that if you do a 2 to the power 3 minus 1 design fractional factorial design then what will happen, you may find out in a B there is there is full factorial suppose this is A and this is B you will find out this.

So, that means, it is 2 to the power 3 design it is full 2 to the k factorial 2 to the 2 factorial. So, that means, high order fractional will become lower order full and that is the projectivity depending on the resolution. So, it will be the projectivity level will be understood if it P equal to R minus 1 in this case this is 3 design 2 to the power 3 minus 1 three resolution that will be equal to 2. So, that mean every 2 dimensions a 2 factor cases it will be full factorial, ok; obviously, with two levels.

So, then what will what will happen to projectivity when we are talking about 6 minus 2, 4 design. So, obviously, you see that what is the resolution? Resolution is 4, R equal to 4. So, if P equal to R minus 1. So, that means, 4 minus so, that means, 3 so, obviously, in some factors cases some cases three factors will be getting full factorial and some cases you will be getting it fractional factorial. So, that is what is the projectivity.

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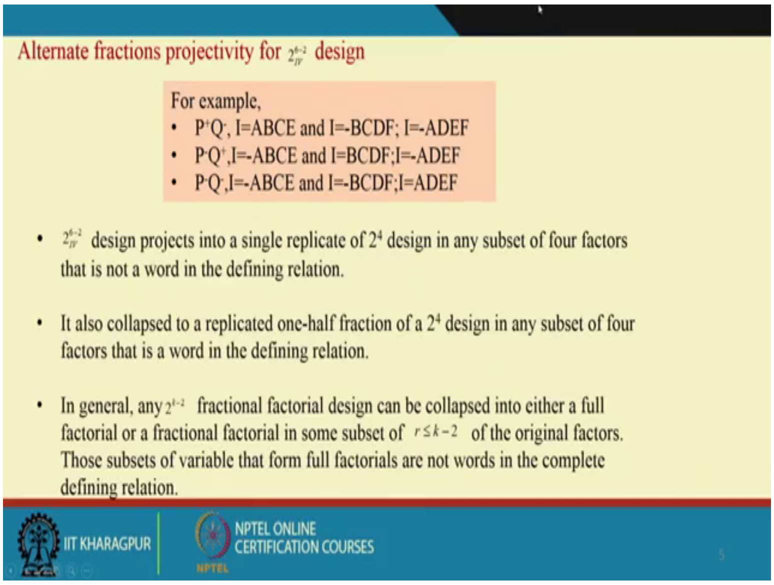

In case of fractional factorial design you see a 2 to the power 6 minus 2, 4 design project is a single replicate of 2 to the power 4 design in any subset of 4 factors that is not a word in the defining relations, ok. So, what I said here that; obviously, actually what I said this is this is from the resolution, but actual thing is that you have seen from the design point of view ABCD they are always in the full factorial then E and F we have gone for additional columns.

So, what I mean to say here, that means, the where the where; so, in the defining relations you have some words some factors if those factors. So, you take a subset of ABCDEF which comprises the factors that are not in the in the defining relation then you will be get you will be getting 4 factor full factorial design for that 4 factors.

But, if you go if some of the factors are in the defining relation then you will be getting subset of 4 factors. So, that mean fraction of fraction of 2 to the power 4 design that is what is one of fraction of 2 to the power 4 3 design that is what is what we meant to say that. So, let me repeat here that a 2 to the power 6 minus 2 design project into a single replicate of 2 to the power 4 design of any subset of factors that is not in a word of the defining relations.

It also collapsed into a replicated one of fraction of a of a 2 to the power 4 design in any subset of 4 factors that is word in the defining relations and in general, what happened if 2 to the power n minus 2 fractional factorial design can collapse into either a full factorial or fractional factorial in some subsets of r less than equal to k minus 2 of the original factors those subsets of variable that form full factorial are not words in the complete defining relations.

So, that means, say when your fractional factorial design you will get two kinds of projectivity one with the full factorial with respect to the basic design kind of thing that the 4 4 resolution 4 then the 4 factors and if their those factors are those factors are not included in the defining relations and if some of the factors are included in defining relations and for those it will be the that one half of the 2 to the power 4 design in this particular case. In case of k minus 2 to the power k minus 2, this is also is similarly true.

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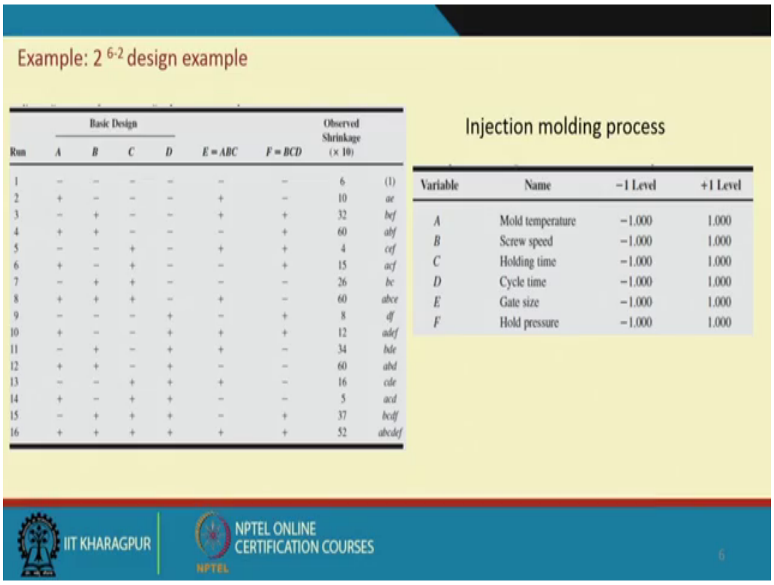

So, now very quickly I will I will explain that it is a basically injection molding process, where parts are manufactured using this process. So, there are there are there are six factors mold temperature, screw speed, holding time, cycle time, gate size and pressures and they are coded values are like this and we have taken first ABCD in the basic design because it is 2 the power 4 full factorial case.

So, what is that 16 design this I have already shown you and E equal to ABC and F equal to BCD, these are additional columns created and the experiment was conducted and the observes in case from part to part that is shown and that is measured and it is multiplied by 10 and the result and quantity is given here, ok.

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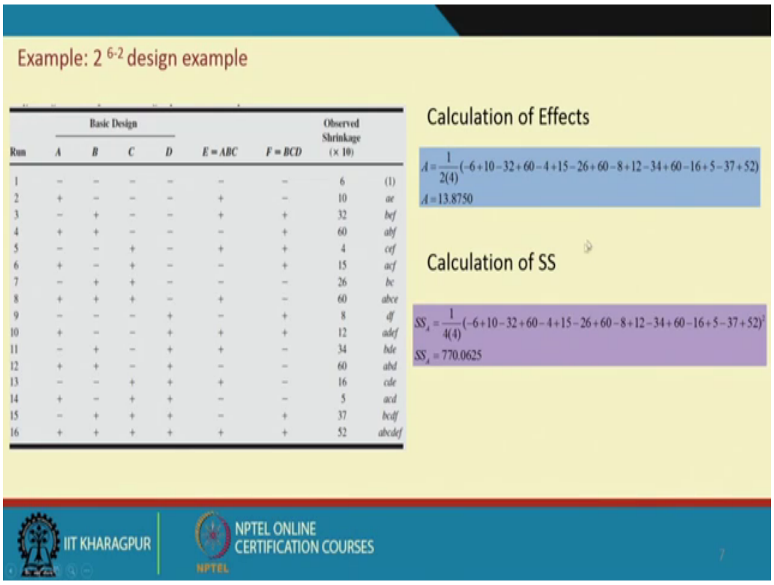

So, how do you calculate the effect? The effect you will calculate using the contract so that and accordingly you have computed; so, and 1 by a 2 to the power k into n. So, accordingly and into contrast that n equal to one and you are getting basically like a single replicate in all cases then calculated a square and that is a 2 to the power k into n n square and in case of effect k minus 1 in case of say SS into the power k into n that will be divided.

So, this is the procedure we have adopted earlier and you have seen and what is to be done, how.

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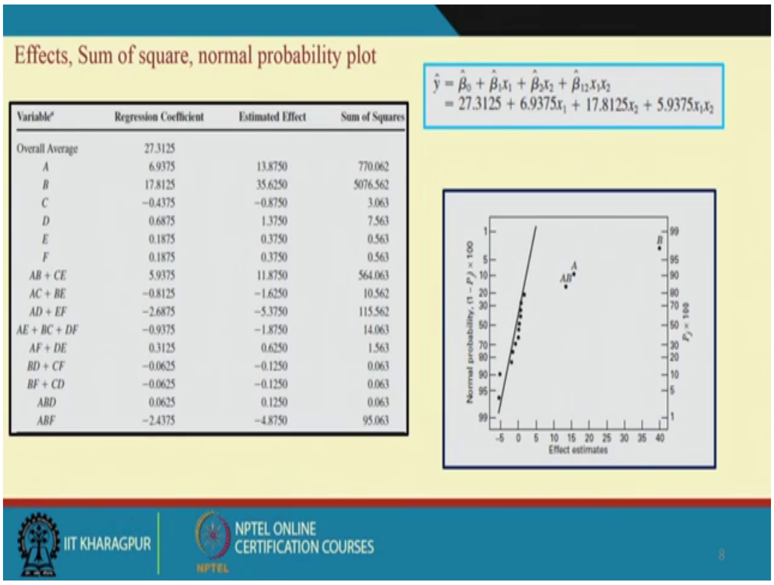

Variable*	Regression Coefficient	Estimated Effect	Sum of Squares
Overall Average	27.3125		
A	6.9375	13.8750	770.062
B	17.8125	35.6250	5076.562
C	−0.4375	−0.8750	3.063
D	0.6875	1.3750	7.563
E	0.1875	0.3750	0.563
F	0.1875	0.3750	0.563
AB + CE	5.9375	11.8750	564.063
AC + BE	−0.8125	−1.6250	10.562
AD + EF	−2.6875	−5.3750	115.562
AE + BC + DF	−0.9375	−1.8750	14.063
AF + DE	0.3125	0.6250	1.563
BD + CF	−0.0625	−0.1250	0.063
BF + CD	−0.0625	−0.1250	0.063
ABD	0.0625	0.1250	0.063
ABF	−2.4375	−4.8750	95.063

So, in the same formula using the similar formula you have computed all those estimated effect and their regression coefficients and their sum of squares. So, here what is assumed that third and fourth order interactions third and higher order interactions are not significant.

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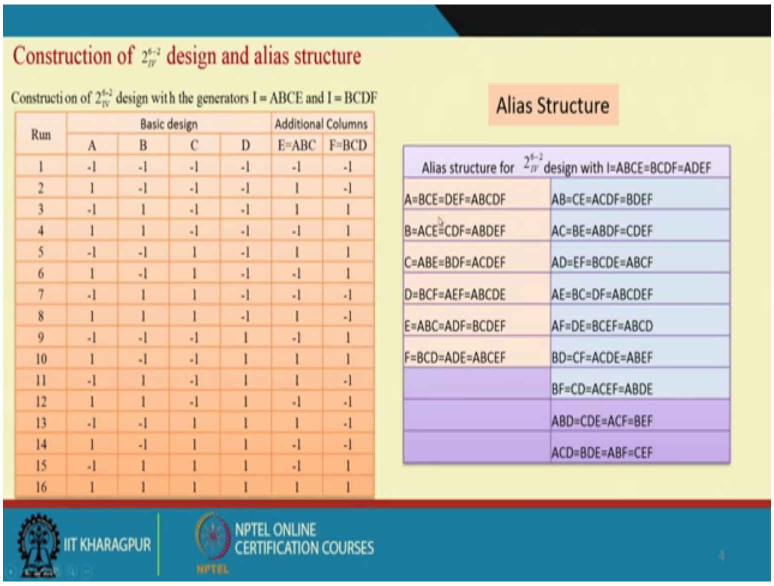

Run	Basic design				Additional Columns	
	A	B	C	D	E=ABC	F=BCD
1	-1	-1	-1	-1	-1	-1
2	1	-1	-1	-1	1	-1
3	-1	1	-1	-1	1	1
4	1	1	-1	-1	-1	1
5	-1	-1	1	-1	1	1
6	1	-1	1	-1	-1	1
7	-1	1	1	-1	-1	-1
8	1	1	1	-1	1	-1
9	-1	-1	-1	1	-1	1
10	1	-1	-1	1	1	1
11	-1	1	-1	1	1	-1
12	1	1	-1	1	-1	-1
13	-1	-1	1	1	1	-1
14	1	-1	1	1	-1	-1
15	-1	1	1	1	-1	1
16	1	1	1	1	1	1

Alias structure for 2_{IV}^{6-2} design with I=ABCE=BCDF=ADEF	
A=BCE=DEF=ABCDF	AB=CE=ACDF=BDEF
B=ACE=CDF=ABDEF	AC=BE=ABDF=CDEF
C=ABE=BDF=ACDEF	AD=EF=BCDE=ABCF
D=BCF=AEF=ABCDE	AE=BC=DF=ABCDEF
E=ABC=ADF=BCDEF	AF=DE=BCEF=ABCD
F=BCD=ADE=ABCEF	BD=CF=ACDE=ABEF
	BF=CD=ACEF=ABDE
	ABD=CDE=ACF=BEF
	ACD=BDE=ABF=CEF

So, as a result what happened if you see the alias structure under the this design then a all main effects are indirect aliased with third order higher order. So, if the third and higher order interactions if we neglect so, from analysis we can write that these are the these are the effect estimate the regression coefficient for only the main effects.

But, second order interactions they are aliased with another second order interactions. So, you cannot remove them and so, that means, these are the things given here and ABDEA with the two third of the interactions are given and which are aliased with other third order interactions also.

Now, if you carefully observe the regression coefficient or the effect you will find out that AB, AB effect are significant and other effects are not significant because the A is 6 B is 17.8 and C others are very small and A B plus C is 5.93. So, and that also there if you see their sum square the contributions is also more for a B and A B plus C, but as C is C is not significant and we and also E is not significant.

So, we are assuming that the effects C also will be not significant and as a result AB plus CB we are we are considering that that may be the effect of AB only because A and B are significant there. So, now, other the same thing can be achieved from this normal probability plot of the effects. So, if you do this you see that all the other effects are along the straight line, but only AB and AB they are away from the straight line. So, these effects are different or significantly higher than the remaining effects.

So, using this you will be in a position to find out this find out this regression equation that why E is beta 0 beta 1 x 1 beta 2 x 2 beta 1 2 is x 1 x 2 and then beta 0 is the average of all the y values and beta one is basically 6.93 beta 2 is 17.8125 and beta 1 2 is 5.93 and that is what we have given here, ok. So, in this manner you can discard many of the effects and you are you are getting a direction or further research as well as you have screen out that is why your screen out many of the effects,.

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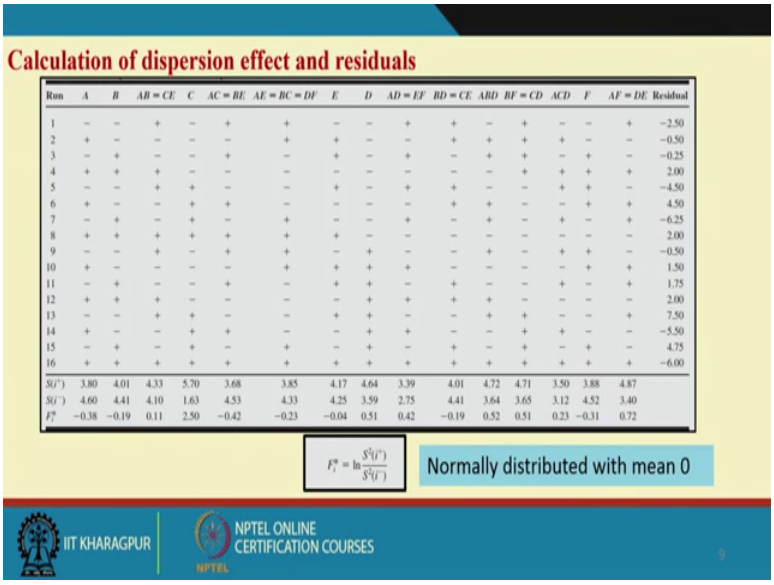

So, now with this another important aspect the dispersion effects. So, the dispersion effects here is so, what you require to know that you require to know whether the different factors at a low level and at high level the are they are producing the same dispersion effects or not. So, in order to do so, what you require to do you require to find out the residuals.

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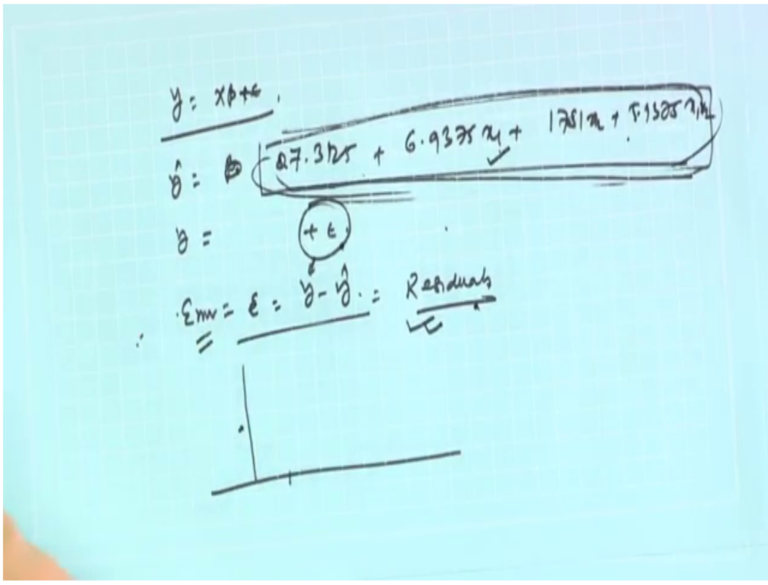

If I go back you all of you know y equal to x beta plus epsilon, now in this example in this example y star is basically beta 0 or straightaway I can write 27.3125 plus 6.9375 x 1 plus 17.81 x 2 plus 5.9375 x 1 x 2 and what is y? y is this plus epsilon. So, the error then the error equal to if I write E or epsilon this is y minus y caps that is y minus y caps.

So, y values are known y cap we will be finding out using this equation then all the error or otherwise these are also known as the residuals when it is estimated and then it is known as residuals. So, residuals should be known. Now, if you see these this equation this is a fixed part we have been knowing x 1, x 2 this is a fixed value. So, if when you plot for a certain such combination of fixed value then and then y part y part the pattern part is captured here, but the variability part variable part will go to here.

So, that is why all the variability of y is not captured here if the mean is captured here the variability part is captured there. So, now, if we have the residual now using the analyzing the residual values; we will be able to know the dispersion effects at different level of x x plus x minus kind of things.

So, that is what is done here for all the factors. So, the residuals are calculated for all the observations and, then for a plus and a minus the standard deviation of the residuals are calculated for example, in this case S i plus 3.80 it simply shows that simply shows that when if we consider the data on the all the even observation 2, 4 all those things for the residuals and then calculate the standard deviation you will get this value and similarly for a S i minus.

So, this is what actually been calculated for all the effects and then if you see the values you will find out there are certain cases where that the way if the factor is at high level the residual is different when the factor is at low level such cases there we say there is dispersion effects.

So, in order to estimate this a quantity F i star is created which is log of that standard variance log of the ratio of the variance when the factor is at quality high level and factor is at low level and this log of this ratio is normally distributed with mean 0. Now, if you if you do the standard deviation or the variance plot in the normal probability plot you will find out that those which where the dispersion effect is present they will be away from the straight line

For example, if you consider C here this C factors this C factor you see if C is at positively high level then the dispersion is 5.70 when it is at low level it is 1.63 this is quite different high difference and the ratio is 2.50 which will be significantly higher than the low difference case.

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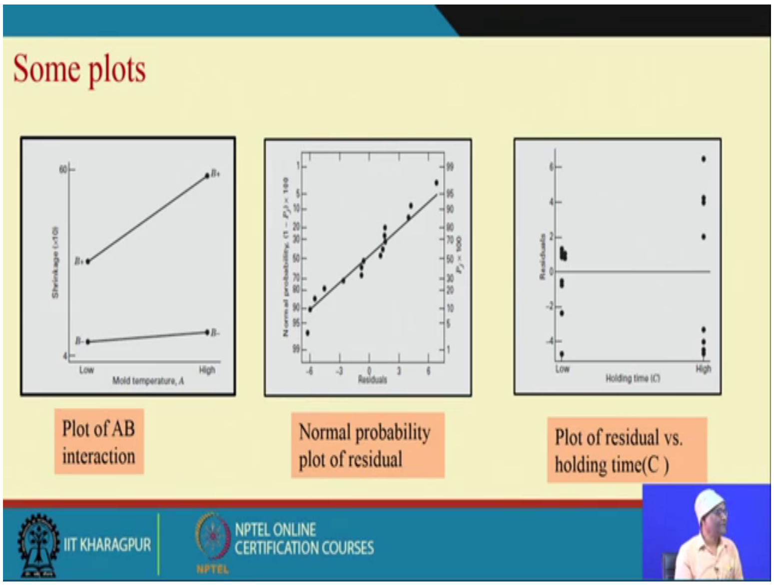

So, that is what is seen in the third figure here you see when C is at low level the dispersion effect is low, but when it is at high level the dispersion it will be very very high getting me. So, these are some residual plots now if you want to interpret the first plot if you see that the mole temperature effect with B when B is at negative and what is B? B, if you see B is the screw speed and temperature is A.

So, screw speed if it is at low level screw speed is at low level you see then your mole temperature effect is negligible there is almost no change, but if your screw speed is at high level then the mold temperature has it is cross effect when you change it from low to high. This is basically the interaction plot, this way you have been. So, that mean what do you want to keep? You want to if you want shrinkage to be minimized you will keep both mold temperature and screw speed at the low level.

What does is the second figure say? Second figure talks about the normality assumption whether if the residuals are normal then they will definitely follow the straight line and this is the procedure we have discussed much earlier and you know how to compute the probability property plot. How to draw the probability property plot,. So, here it is almost normal and the third figure says that variability dispersion effects in C at and in low and dispersion effect when C at is at high is different.

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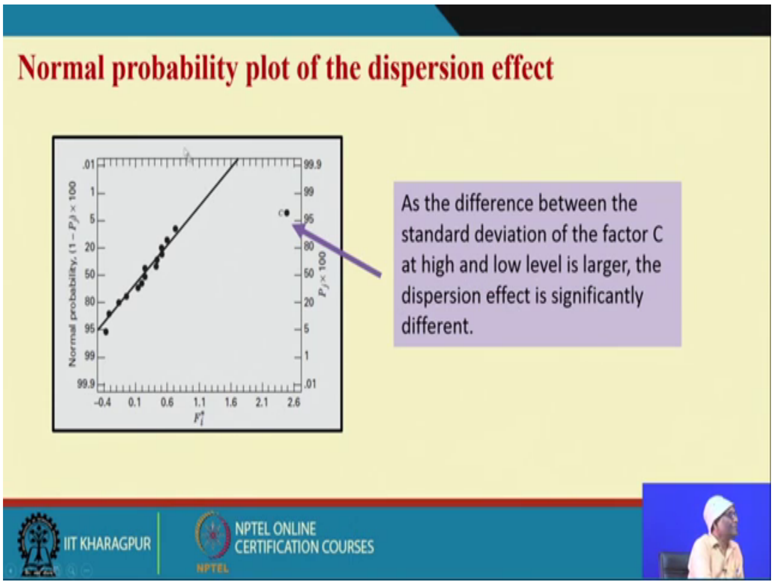

And, that is what is seen from the probability plot of the dispersion effects F i you see that all others values are following the straight line, but C is very far away hence the difference between the standard deviation of the factory C at high and low level is larger the dispersion effect is significantly different for C high and C low effects.

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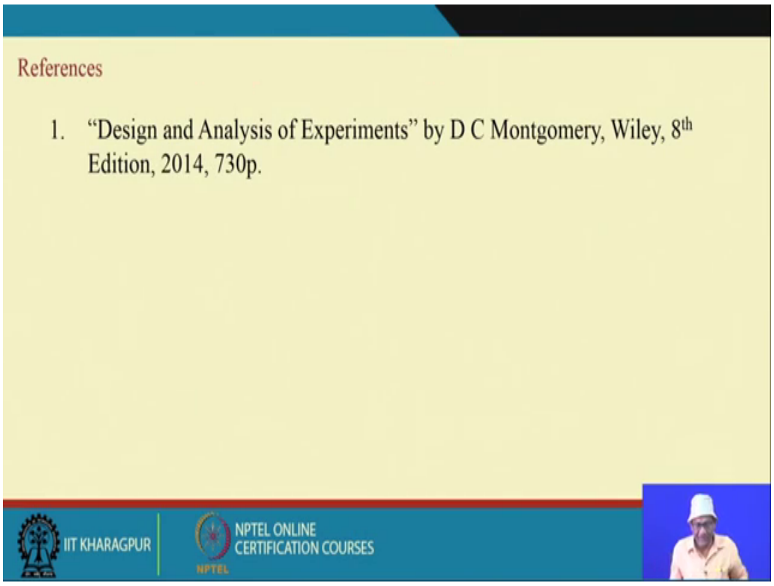

So, this is the, these are the ways from interpretation point of view.

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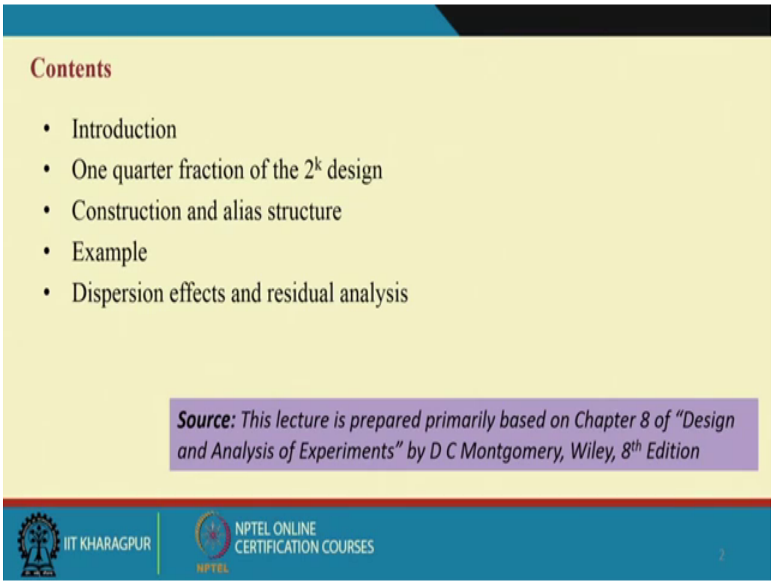

So, conclude to this x x n, so, what I mean to say here is that that fractional factorial design is very very important design for designed experiments because there are a lot of many factors many a times and you cannot go for all factors even at true levels and we have discussed earlier 2 to the power k minus 1 mean one half fractional factorial design with different resolution.

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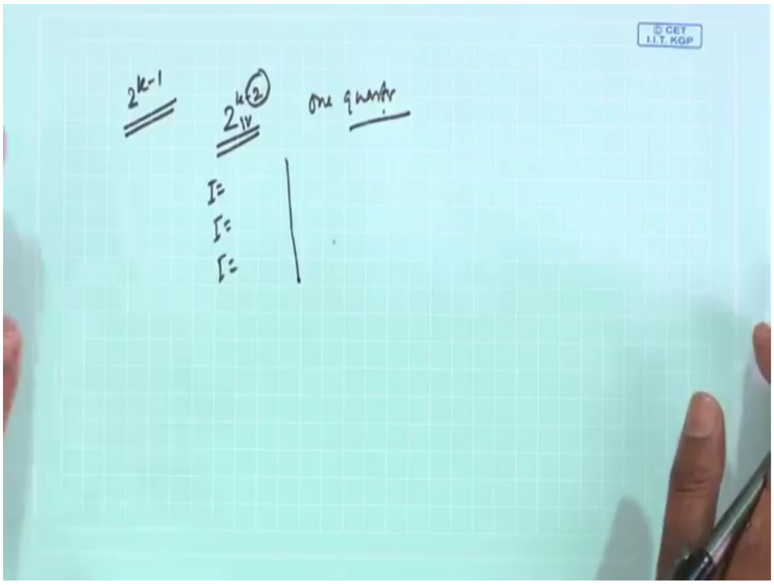

Today, I have given you example of 2 to the power k minus 2 resolution 4 and k minus 2 means 1 quarter fractional factorial design maybe elaborately and you have seen many things.

First one is your defining relations it depends on that what is the fractional level say if P equal to 2 there will be 2 generators and it will lead to three defining relations. So, there will be principal fractions where there will be there will be alternate fractions, if you use principal fractions your alias structure will be one way, if you use alternate fraction the alias structure will be different whatever maybe the thing, but you must answer must be able to interpret and the results whether you are using principal fraction or alternate fractions.

Now, when you go for fractional factorial design that the beauty is that the higher order interactions are usually negligible. So, at lower level if your resolution is quite good you will get lower level lower level reasonable good estimate values and you will find out in the 2 to the power k minus 2 fractional factorial design. So, there will be a single replicate full factorial design with involving k minus 2 factors or less depending on resolution and also 1 half of k minus 2 fractional factorial design for those factors which are not included involved in the defining relations.

And, then what we have given you we have given you that how to find out the significant effects and then from there how to develop the regression equation once you have the regression equation you can find out the error terms which are basically known as residuals here. Now, the residual has. So, many useful properties to test the assumptions, you can test normality assumptions, now you can test the dispersion effects also and we have introduced a concept called F star which is basically a log of that variance.

When the factor is at high level by minus the log of the variance in the factor of at a low level which is basically log of ratio of the two and this is basically can calculated using the using the residuals because the you have seen from the regression equation that the pattern part that we will talk about the mean of y values not above not anything about the dispersion of y values. And, dispersion effect is translated or transferred to the residuals. So, the that is why the residual is used to check the check the dispersion effects.

And, with this example we have seen that there is one factor C which was at low and high level there are lot of dispersion difference in this percent and this is this is this is a significant one and these things must be known because the assumptions are violated in many if such things are happening.

Thank you very much. Have a nice day.