Design and Analysis of Experiments

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Lecture - 37
Statistical Analysis of 2 k Factorial Design

Hello. We will continue 2 to the power k factorial design.

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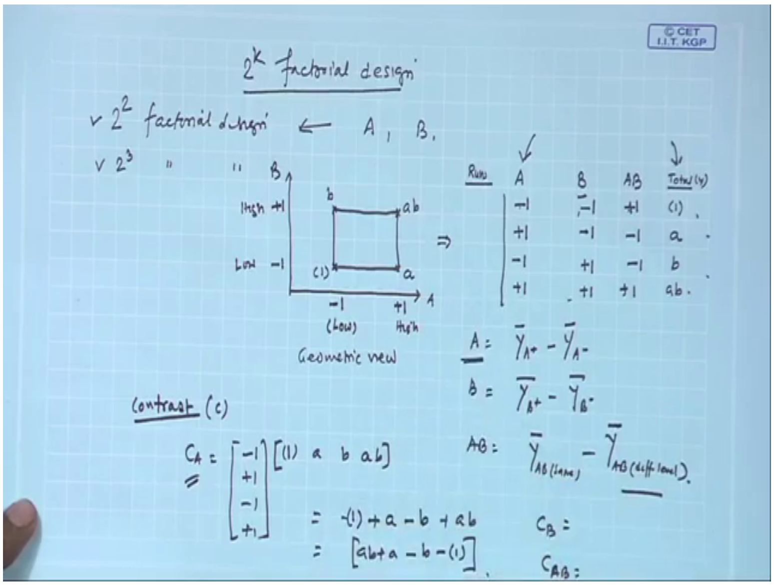

In last lecture, you have seen 2 to the power 2 factorial design. In this lecture we will see 2 to the power 3 factorial design. So, what have you seen in 2 to the power 2 factorial design? we have considered 2 factors, A with 2 levels, B with 2 levels, and then if A and B both have 2 levels, then we got 4 treatment combinations for A, minus, plus, B, minus, plus, or minus 1, plus 1, minus 1, plus 1 for which we have said low, high, low, and high and this is one factorial, factorial points, this is another factorial points, another factorial points, another factorial points, and from there we have seen the, this is basically the Geometric view.

From there we have seen the design matrix where, we have written A and B and A B. And our runs are experimental runs, and then we have given some notation for this point is 1, this point is a, this point is b, and this point ab. So, when A is minus, B is minus, this notation actually we say that total of y this is given as 1.

And then when this one plus 1, this one minus 1, this is given as a, and when this is minus 1, this is plus 1, this one termed as b, and this is plus 1, this also plus 1, this is termed ab. So, this 4 treatment combination and the interaction a b we found out by multiplying A column verses B column. If you multiply these 2 this will be plus, minus plus, plus minus minus, minus plus minus, plus plus plus.

So, the design matrix when you say, we will write basically here, minus 1, plus 1, minus 1, plus 1, minus 1, minus 1, plus 1, plus 1, plus 1, minus 1, minus 1, plus 1. And then we have computed that, A effect which we said that A plus Y bar when A is plus, minus, Y bar when A is at minus level, irrespective of the position of B factor. Similarly B effect we have computed Y bar B plus, minus, Y bar B minus and the A B interaction we have computed when we say that average of Y when A B both same level, minus, Y bar of A B at different level, this one we have seen. And we have said another thing that the Contrast, Contrast C when a particular column is multiplied by the total column then you will get the contrast.

So, for C A, the contrast will be minus 1, plus 1, minus 1, plus 1, this times 1, a, b, a b, you will get this one as minus 1, plus a, minus b, plus ab. Which is basically a b plus a minus b minus 1, this is your contrast of A. Similarly contrast of, contrast of B, C B and contrast of A B, all those things can be calculated by multiplying the respect, the product of this 2, this column mixed up with this column, this column mixed with this column, dot product will give you contrast B, this mixed up with this give you contrast C. And once you know the Contrast C, that mean you will see that this, these how to compute the A effect, B effect, A B effect those things we have discussed today.

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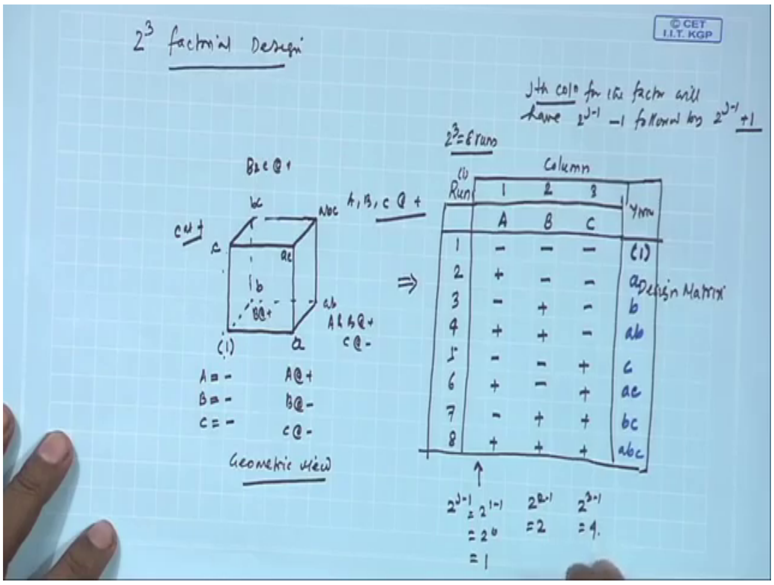

We will, we will see the same thing for 2 to the power 3 factorial design.

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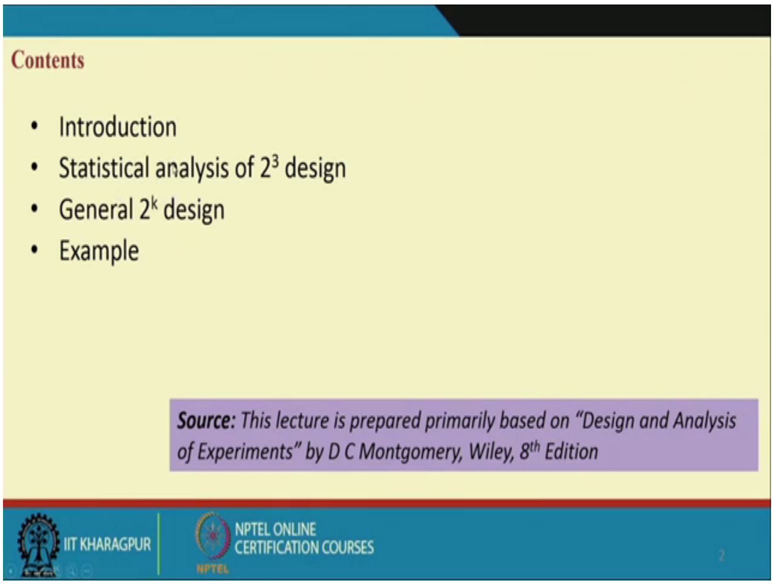

So, in this class what we will see, we see that, what are the, what is statistical analysis of 2 to the power design, then we will go for 2 to the power k general design. And the lecture is prepared based on chapter 6 of Design and Analysis of Experiment written by D C Montgomery.

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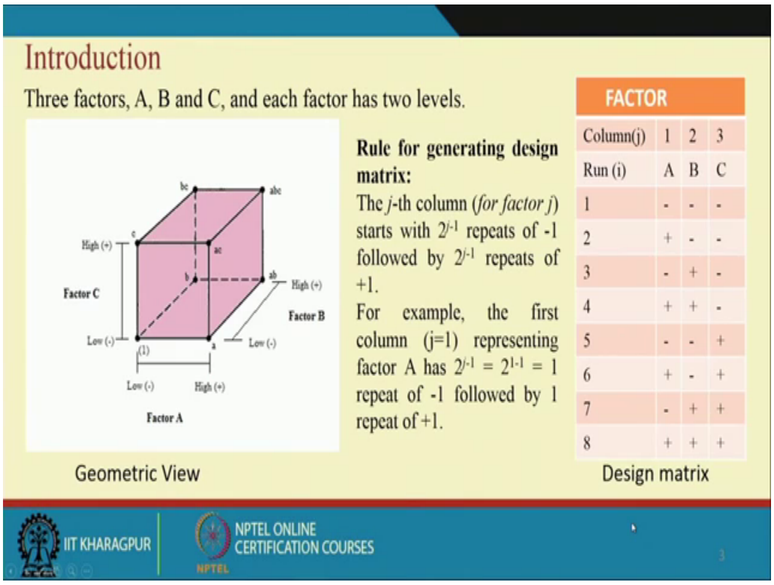

You see, this is a very important slide. When we have 3 factors, each at 2 levels, you will be getting a cube like demonstration, the geometric view you will become cube, and then from here, how do you get this design matrix A B C, that I will discuss first. Now you see the, see this one, the cube, this is a point represented by within bracket 1, that mean this is a point where all 3 are at low level, this is a point which represented by A, that mean A, this point is, that point where the factor A is at high level and other 2 factors at low level, this is a point, this point is represented by B, that means, factor B at high level factor A and C at low level, this is point represented by C, this is the factor point where C is at high level and others are low level. Now this is the point where A B, that mean A and B are at same level, A at high level and B at high level ok.

So, similarly B C, similarly A B C; so, how are you drawing, first is, you just think of that this axis is C, this axis is A, and there is another axis. So, this one is 1, that mean A at, A minus, B at minus, C at minus, low level. This is your a, means A at plus, B at minus, C at minus, this one is b where B at plus, rest minus, this is a b, so A and B at plus level, C at minus level. So, similarly this one is c and, so, if it is, if it is c this is b, this is b c, then what happen this one a, and this one is c, this is a c, and this one is a b c. Here what happen A B and C all at plus level, high levels. Here B and C at plus level means high level. Here only C at plus, at plus level mean C at high level, rest are low levels. So, this is your Geometric view. Now from this geometric view to we want to go to design matrix.

So, in the design matrix there will be row y runs, how many runs will be there? 2 to the power 3 equal to 8 runs. So, 1, 2, 3, 4, 5, 6, 7, 8, 8 runs and suppose if I say, run i and then column will be, there is a column, the column represent the factors, what are those factors? factor A, factor B, factor C.

So, we want to put, we want to put, against each of the runs what is the, what is our plus minus sign. This one, we will use the formula called the j th column, j th column starting from A means A B C for the, for the factors, j th column for the factor will have 2 to the power j minus 1, 2 to the power j minus 1, minus 1 followed by 2 to the power j minus 1 plus 1. So, we start with 2 to the power j minus 1 and then, then what happen? this plus minus will be there. Suppose if I say this is with first column, this is my second column, this is my third column. So, how many plus minus the repetition will be there? 2 to the power j minus 1 equal to 2 to the power 1 minus 1 equal to 2 to the power 0 equal to 1. So, 1 minus followed by 1 plus you repeat this, for the second column it will be 2 to the power 2 minus 1, so equal to 2.

So, 2 minus followed by 2 plus. what will be the third one? 2 to the power 3 minus 1, is basically 4. So, 4 minus by 4 plus this will be the repetition. So, this matrix is known as Design Matrix. So, now, if I want is, if I have the total of Y here, so Y total. So, this is 1, means all observations. So, this at low, it is 1. So, I will use another color. So, this is one. Now at the second run, A is at high so this is basically and B C at low level, a, third one B at high other 2 at low so b, fourth one A B at high C at low that is y a b, fifth one C at high others are low c, sixth one A C at high level positive and B at low a c, 7 th one B C at high level A at low level so it is b c, eighth one all 3 are at high level A B C ok.

So, 1 and 1, b, c this are basically representing the combinations, other hand also this will be represented by the total what is observed at that treatment combination with reference to the y. So, this is what is known as from Geometric view to our that design matrix.

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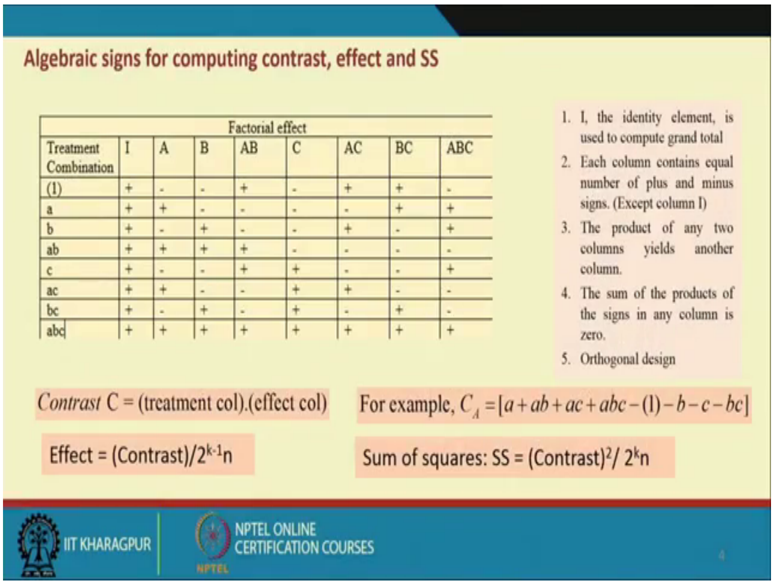

Now, what I will show you, I will show you another important table which is known as the how do estimating the effects and SS and contrast for different fact effects, factorial effects.

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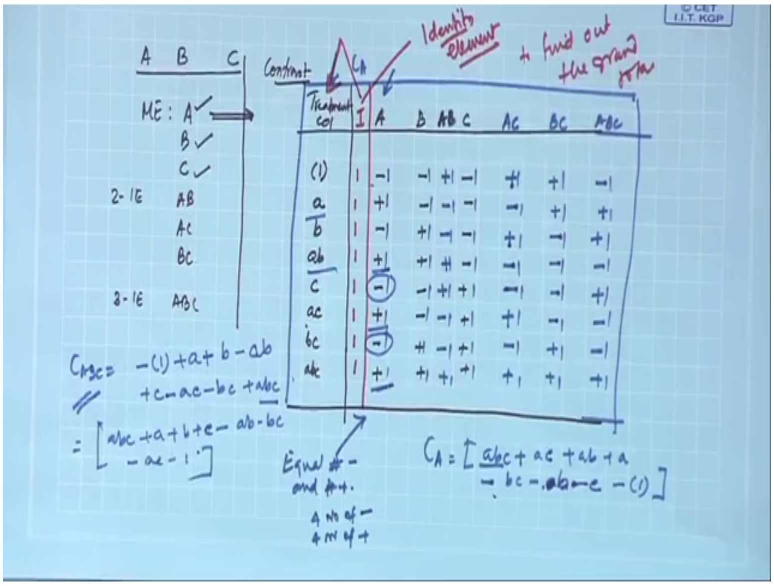

I have told you, in fact, we have discussed earlier that, if there are 3 factors A, B, C then the main effects will be 3, A, B and C. 2 way interaction effect will be, A B, A C, B C. And 3 way interaction will be A B C.

So, essentially we require knowing now, in order to estimate the effect A, B, C, A B, A C, B C, A B C, as well as the sum squares contribution of all those effects, we require to find out concept called contrast.

So, that mean we require to have contrast for A, contrast for B, contrast for C, similarly A B, A C, B C and A B C. So, in order to get these we use a table of algebraic signs, what is this algebraic sign this one, first we will write down the Treatment combination. And this also will be represented later by the total Treatment combination if you find see, you will get how many treatment combination? first is all low, a at high, b at high, a b both at high, c at high, a c at high, b c at high, and a b c at high, and what is the, what is the general rule you find out A B C from the design matrix, what you found out minus, plus, minus, plus, minus, plus, minus, plus, this is for a, for b minus, minus, plus, plus, minus, minus, plus, plus, for c minus, minus, minus, minus, plus, plus, plus, plus. Now what you require, when A, B, is there, you insert A B here.

So, I use another color. So A B here. So, what will be A B? A column multiplied by this, this is plus 1, minus 1, minus 1, plus plus plus 1, minus minus plus 1, mi plus minus minus 1, minus plus minus 1, plus plus 1. So, we will see what happen, what is happening here? if you see the column A, how many minus? there are 4 minus, 4 plus. So, Equal, Equal number of, Equal number of minus and plus. Now go to B, same. So, here it is basically 4 number of minus and 4 number of plus, we have 8 runs treatment combination. Now that will happen to A B also, you see plus, plus, plus, plus. So 4 plus, 4 minus, for C also 4 plus, 4 minus.

So, once you multiplied A and B, you will be getting A B column. Now you want to find out A C column, you want to find out B C column, and what more? you want to find out A B C column. How do you find out A C? multiply A verses C. So, minus minus plus, plus minus minus, minus minus plus, plus minus minus, minus plus minus, plus plus plus, minus plus minus, plus plus plus, 1, 2, 3, 4, the check point is equal number of plus and minus. So, find out B C. So, B C you multiplied B column verses C column, minus minus plus, minus minus plus, plus minus minus, plus minus minus, minus plus minus, minus plus minus, plus plus plus, plus plus plus.

Now if you want to get the C, A B C.So, take any one suppose B C into A; so, minus plus minus, plus plus plus, minus minus plus, plus minus minus, minus minus plus, plus minus minus, minus plus minus, plus plus plus, 1 2 3 4.

So, this is what is your algebraic signs, algebraic signs. And, and this one the treatment combination column, this also give you the Y total, this one basically the total of Y when all are that also. So, if you want to get contrast C A, you multiplied these 2, take dot product of these 2 column, how? you just put minus 1, plus 1, all put one here; so, 1, 1, 1, 1, minus minus 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, minus 1, 1, like this.

So, if you ask that, what is then contrast for A it will be just, a b c plus ac plus a b plus a minus, minus you see, what will happen? minus b c minus a c, b c minus, not a c, b c minus, c minus, sorry, this is a b c plus, this is plus, a c plus. So, a b plus then a plus, what is remaining? b c, b c minus then c minus b minus. So, you write b here, here also you write c then minus 1. So, this is the contrast for A. Now if you want to know the contrast for A B C this effect. So, that mean this will be minus 1 plus a plus b minus c, a b, this one minus a b plus c minus a c minus b c plus a b c. So, I can write like this a b c plus, plus a plus b plus c minus a b minus b c minus a c minus 1, this is contrast for A B C. Similarly Contrast for this thing also you can find out. Now the some of the basic or interesting thing about this matrix is there is 1 matrix here, which is known as I ok.

So, in this particular diagram we can add I matrix, sorry, I column where everything will be 1 here, all 8 of the rows will contain 1, this is known as Identity element, this is known as Identity element. And this one is used to find out the grand total, to find out the grand total. How? If you multiplied I verses this matrix this column verses this column in this matrix. So, if you take the dot product, what will happen? You will take sum of all those values 1, a, all those things. Earlier I said 1 represent the total of all Y values when experiment is conducted at this setting a, at for a setting like this so; that means, sum of all those things will give you the grand total.

So, again I told you that first one is this, second one is that each column contains equal number of plus and minus except column 1, this we have discussed. The product of any 2 columns, you will another column, A and B when you multiplied it gives you A B column, if you multiplied A and C you will get A C column, similarly if you multiplied A B bust and C you will be getting A B C column, suppose you multiplied A B with A what you will get? You will get B column, A B A, A into A square when you make this column square all will be positive. So, only B column will remain then. So, that is why any 2 column if you merge you will get another, if you multiply, if, you will get another column.

The fourth one is the sum of product of the signs in any column is 0; obviously, as there are equal number of positive and negative signs.

So, when you sum up they will give you the 0 value, and this is orthogonal design. So, if you, just you take dot product of these and these what you will get, suppose let me take the dot product of the first column A, A is minus 1, plus 1.

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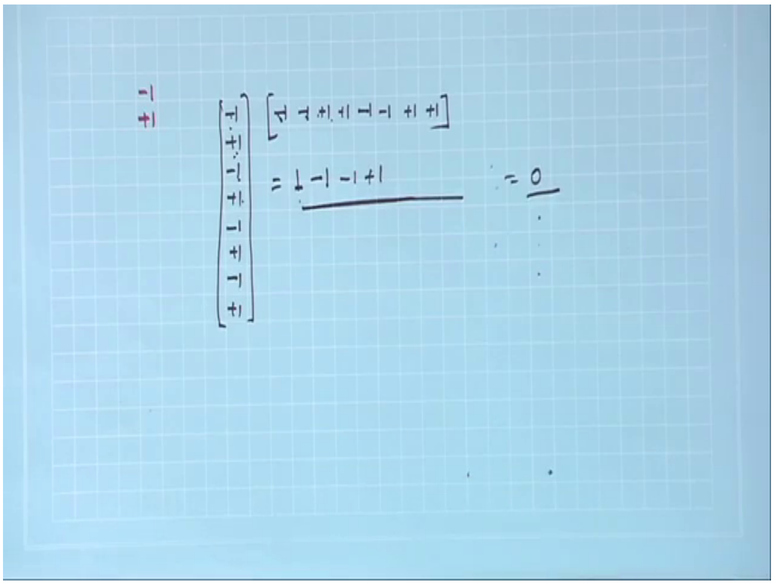

So, minus 1, plus 1, minus 1, plus 1, minus 1, plus 1, minus 1, plus 1, what is this is my A if I take, the dot product with the B one, minus, minus, minus 1, minus 1, plus 1, plus 1, then minus 1, minus 1, plus 1, plus 1, this is my B column minus plus minus B is minus minus all those things, if you multiply these what you are getting? This into this is 1, then this into these this is minus, this is minus, this into this plus 1, this into this minus 1. So, this into this minus 1, this into this plus 1; so, like these you will be getting equal number of plus and minus and finally, it will be 0. And we said that, if we multiplied by the 2 column the you will get another column.

So, sum of these will lead to 0. So, this is Orthogonal. So, now, once you have this then you have the contrast you can find out the effect.

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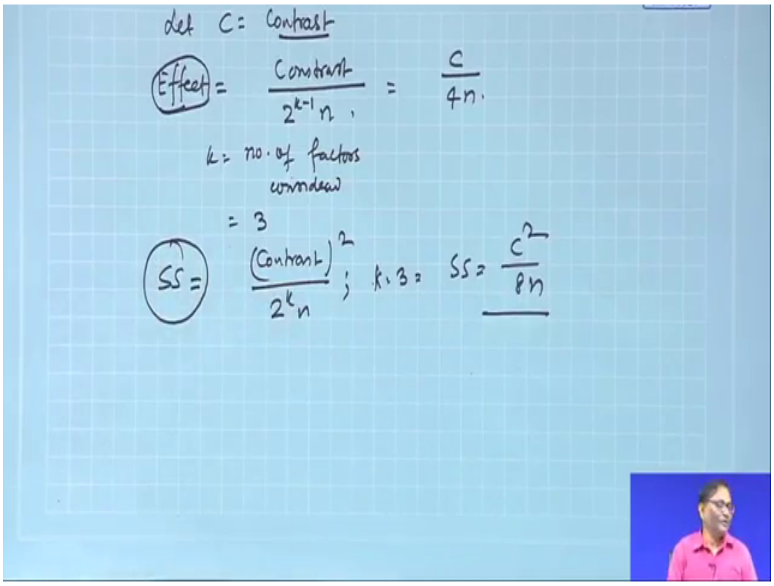

So, Effect will be, Effect equal to, if I say let, let C equal to contrast. Now C A is contrast to A, C B is contrast to B like these, then you are getting contrast by multiplying the treatment column into the respective column, then effect will be contrast square, sorry contrast not square, contrast divided by 2 to the power k minus 1 into n, where k is number of factors considered, if k equal to 3 then Effect equal to C by 2 to the power 3 minus 2 means 4 n. So, similarly sum squares will be contrast square divided by 2 to the power k into n. So, for 3, for k equal to 3, it will be, S S will be, C square by 8 n. Now you already have seen, how, what are the contrast values, you will be able to compute Effect as well as S S. So, this is what is given in this slide Effect is this like this.

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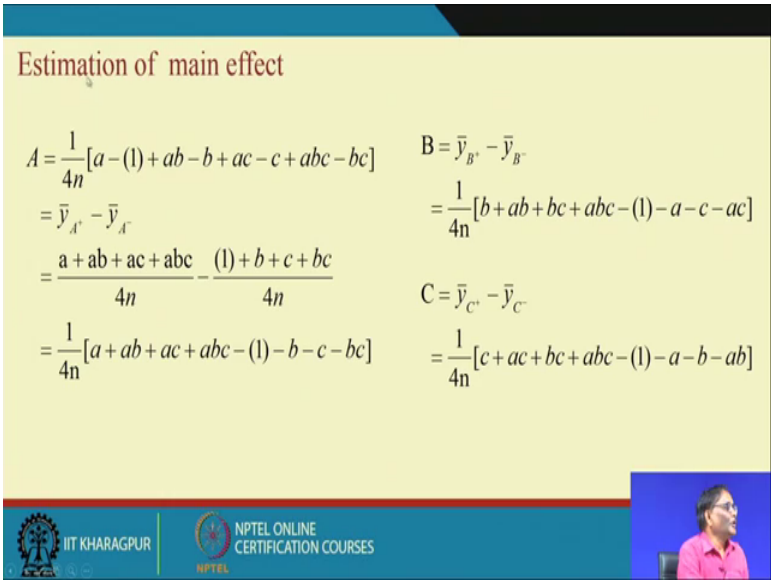

So, now main effect A, a contrast is a b c ,a plus a b plus a c plus a b c minus 1 minus b minus c minus b c.

So, these will be when divided by 4 n. So, it will give you main effect A, similarly main effect B, similarly main effect C. So, this, this the computation, the physical meaning is known to you, we have discussed earlier. Now here we have say, said that, once you have the matrix is representing the algebraic signs of the Effects, you will be able to find out the contrast. Now when you co, you find, you divide contrast by appropriate quantity like here 4 n when it is 3 factor case, if it is k factor case 2 to the power k minus 1 into n, then you will be getting the effects. So, A effect will be computed using this formula, B this formula, C this formula.

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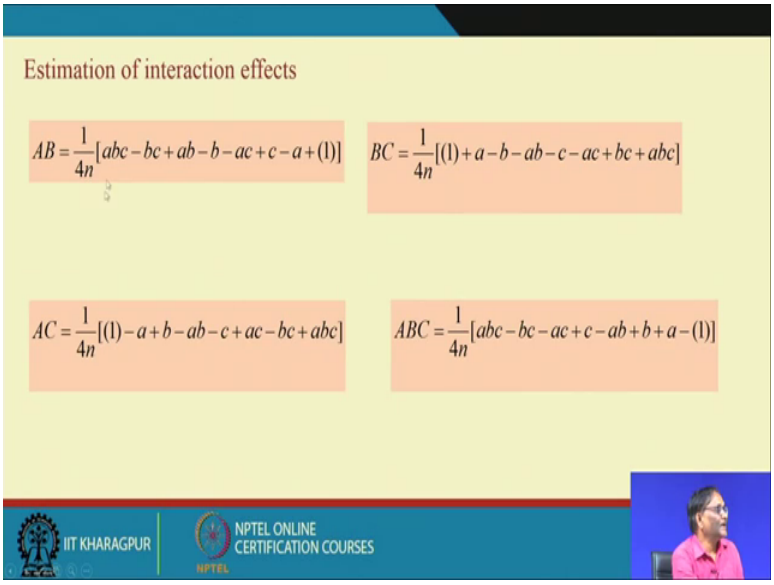

Similarly, A B will be computed using this formula, A C will be computed using this formula, B C using this formula, and A B C using this formula. So, it is easy one, you, you know that I have already explained how it is to be done.

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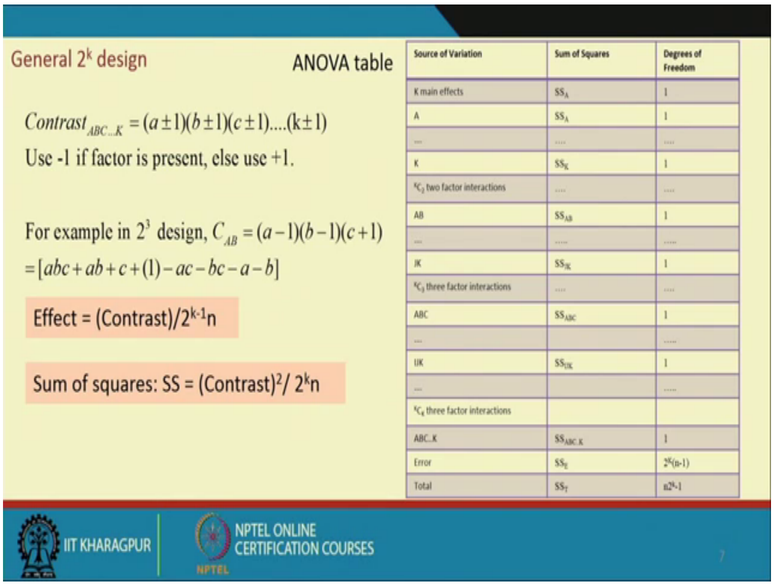

Now, so then what will be the case for general 2 to the power k factorial design for general design.

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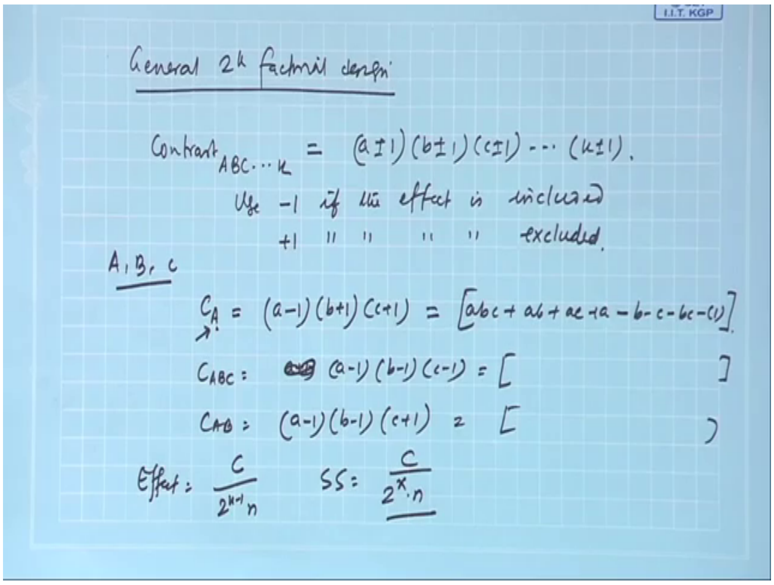

So, general 2 to the power k factorial design. So, how do estimate the contrast. So, contrast will be. So, there are k number of factors. So, contrast A B C dot dot dot k, if I say k factor this will be, a plus minus 1, b plus minus 1, c plus minus 1, dot dot dot then ultimately k plus minus 1. So, and then that use here, the Use minus 1 if the effect is included, is included, and Use plus 1 if the effect is excluded, what does it mean? the meaning is that, suppose I have 3 factors A B C, then I want to know the contrast for A, then what I will write? I write, a minus 1 because I have included a here, and b plus 1 and c plus 1 because b and c are not included here in the contrast calculation. So, these once you multiply and take the, you will find out this will be a b c plus a b plus a c plus a minus b minus c minus b c minus 1, it will be this 1, this is what your using earlier also. Suppose if you interested to know A B C, then here all a b c included. So, a plus, sorry a minus 1, b minus 1, and c minus 1, this will give you the desired contrast.

C, A B case, so a minus included minus 1, b included b minus 1, c not included so c plus 1, this will give the general contrast. And then Effect, Effect will be as I told you the contrast divided by 2 to the power k minus 1 into n. And S S will be contrast divided by 2 to the power k into n. So, you will be able to find out Contrast and all those things.

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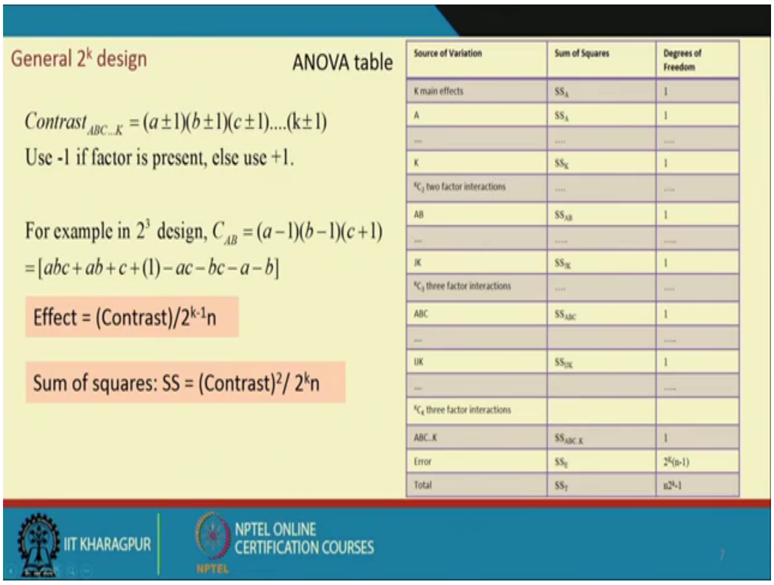

And then the general ANOVA table will be like this. All sources of variation, main effect, interaction effect, and then their sum square and degrees of freedom will be always 1, because there are 2 levels for each factor. So, for A, B, C, B C, A B all they will be only 1 degrees of freedom ok.

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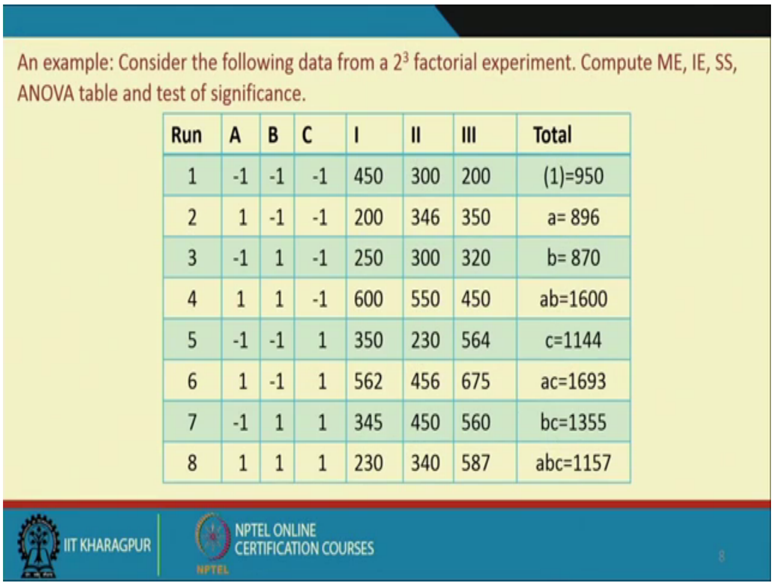

So, as a result, as a result what will happen?

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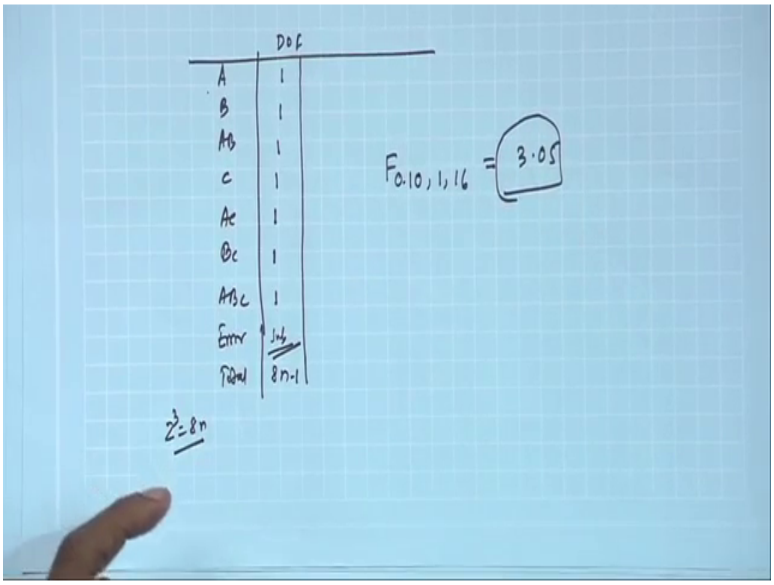

I will explain as your, suppose 3 factors A, B, A B, C, A C, B C, A B C . So, A having, so what is the degree of freedom now, A having 2 levels so, 2 minus 1, degree of freedom will be 1, B 1, A B , A minus 1 B minus 1, 1 into 1, 1, C 1. Same way it is 1, this also 1, A minus 1 B minus 1 C minus 1 all multiplication this will be 1 ok.

So, suppose you, your, have to then Error, then Total, suppose you have 2 to the power 3 equal to 8 settings, n replications, then what happen? 8 n minus 1 will be here the total degrees of freedom. Error degrees of freedom will be these by subtraction. So, that will find out, now let us see, see one example. Here 2 to the power 3 design A B C. So, the same nomenclature we have used here, and there are 3 replicates and this total is represented by the sum of these 3 will give you 950, this is the data. Now we will see how we have computed contrast and all those things.

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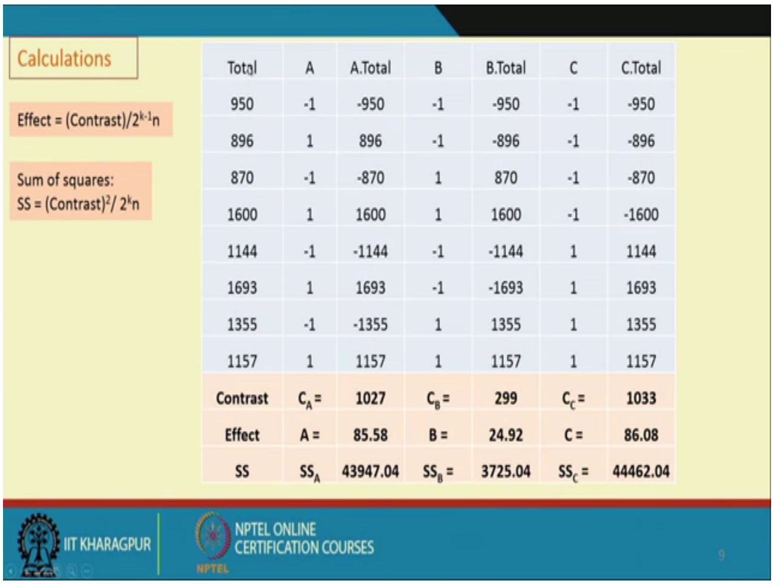

Total	A	A.Total	B	B.Total	C	C.Total
950	-1	-950	-1	-950	-1	-950
896	1	896	-1	-896	-1	-896
870	-1	-870	1	870	-1	-870
1600	1	1600	1	1600	-1	-1600
1144	-1	-1144	-1	-1144	1	1144
1693	1	1693	-1	-1693	1	1693
1355	-1	-1355	1	1355	1	1355
1157	1	1157	1	1157	1	1157
Contrast	C_A =	**1027**	C_B =	**299**	C_C =	**1033**
Effect	A =	**85.58**	B =	**24.92**	C =	**86.08**
SS	SS_A	**43947.04**	SS_B =	**3725.04**	SS_C =	**44462.04**

Here, this total I have written here. Now A column, plus minus sign, this minus 1 plus 1, is these when you multiply this by this, dot product, if you make, you get this one. When you take sum of all those things you are getting Contrast. Similarly for sum of these will give you Contrast B, some of this will give you Contrast C. Once you have contrast, then you find out Effect, Effect is contrast by what ? 2 to the power; that means, here 4 n.

So, what is n is equal to 3, 4 into 3, 12. So, 1027 by 12 is this, 299 by 12 is this, 1033 by 12 these. And what will be S S, S S will be contrast square by 8 n, 8 into 3, 24. So, 1027 square by 24 is this, 299 square by 24 is this, and this square by 24 is this.

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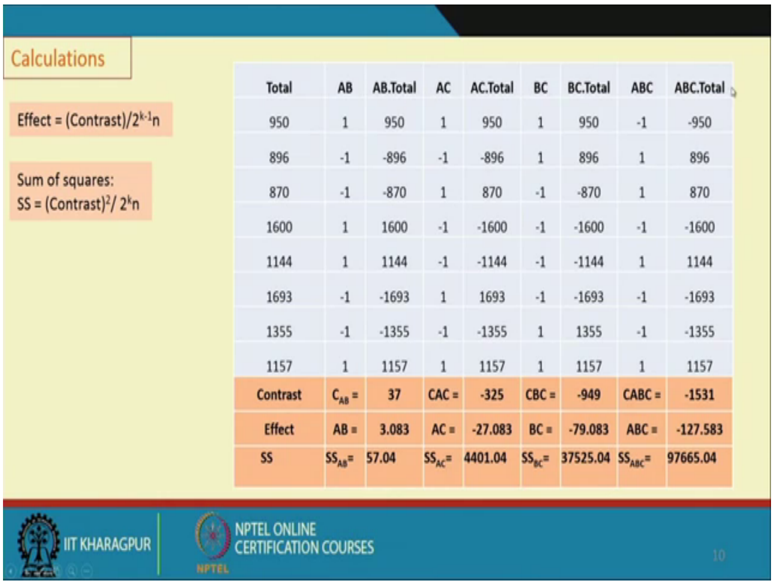

Total	AB	AB.Total	AC	AC.Total	BC	BC.Total	ABC	ABC.Total
950	1	950	1	950	1	950	-1	-950
896	-1	-896	-1	-896	1	896	1	896
870	-1	-870	1	870	-1	-870	1	870
1600	1	1600	-1	-1600	-1	-1600	-1	-1600
1144	1	1144	-1	-1144	-1	-1144	1	1144
1693	-1	-1693	1	1693	-1	-1693	-1	-1693
1355	-1	-1355	-1	-1355	1	1355	-1	-1355
1157	1	1157	1	1157	1	1157	1	1157
Contrast	C_{AB} =	37	CAC =	-325	CBC =	-949	CABC =	-1531
Effect	AB =	3.083	AC =	-27.083	BC =	-79.083	ABC =	-127.583
SS	SS_{AB}=	57.04	SS_{AC}=	4401.04	SS_{BC}=	37525.04	SS_{ABC}=	97665.04

So, this is A B C, in the same manner A B, A C, B C, A B C. So, A B C column is in the design matrix, algebraic matrix is this, then the totals are like these. So, multiplied by 2 you are getting this, and Contrast will be sum of these, you are getting contrast everywhere, you are getting Effect, you are getting S S. So, I have shown you how using contrast you will be able to find out all the Effects, all the S S reference, with reference to the main Effect with reference to 2 way interactions, with reference to 3 way interactions.

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Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	$F_{0.10,1,16}$
A	43947.04	1	43947.04	**3.04**	
B	3725.04	1	3725.04	0.26	
C	44462.04	1	44462.04	**3.07**	
AB	57.04	1	57.04	0.01	3.05
AC	4401.04	1	4401.04	0.30	
BC	37525.04	1	37525.04	2.59	
ABC	97665.05	1	97665.05	**6.74**	
Error	231752.70	16	14484.54		
Total	463534.96	23			

Once you have all those things calculation, you compute now, the ANOVA table. So, ANOVA table sources of variation there will be 7 sources, 8 sources of variation including Error, now the first one what is the S S, S S is 43947.04. So 43947.04, in the same manner all S S you write down, and what is the Degrees of Freedom for total? there are 24 observations, degree of freedom is 23. How many degrees are lost for the model parameters? 7, each having 1 Degrees of Freedom, so 7 degree lost for this.

So, how many remaining for error? 23 minus 7 is 16, then will find out mean square. So, here in fact for the model parameters, sum square and mean square will become same, because they are only 1 Degree of Freedom, but Error mean square will be reduced, because the error substantial degrees of freedom available for Error. And then find out the F 0, F 0 is nothing but mean square, that source or the Effect by the mean square error for example, that F 0 for A is 43947, this value, divided by this value will give you 3.04, in the same manner you see. So, now, what happen you see the table. Now we have considered 0.10 alpha value and 1, 16.

So, this value is 3.05, interestingly this is the value which will be compared for all the tabulated F 0 because, all the model parameters have 1 degree of freedom and all model parameters are compared with Error with 16 Degrees of Freedom here. Now you see that A C and A B C these are significant with reference to F 0, all though a value is 3.04 which is slightly less than 3.05, under such situation it is better to include this in to the model or keep this as a significant variable.

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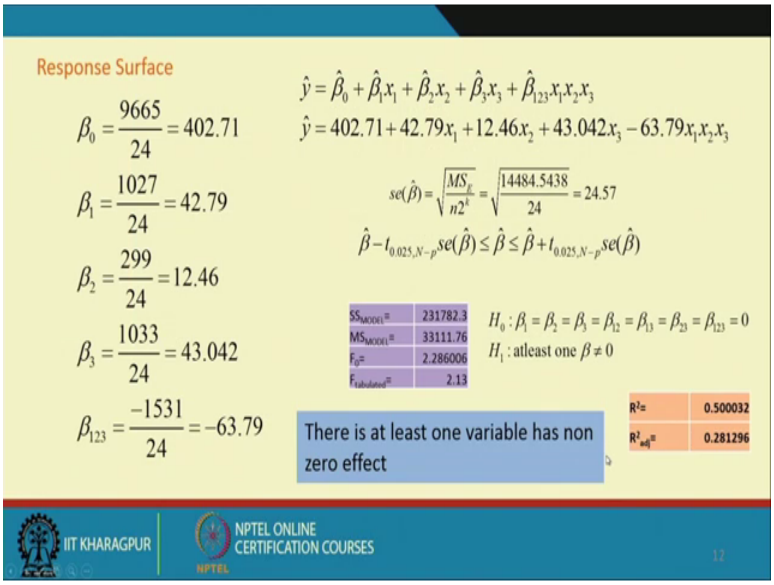

So, then how many we have all those things. So, as A B as significant, A and C significant. So, their interaction A C also should keep, in addition A B C is significant. So, what happen so we are also keeping B for further analysis because B, C interaction is also there?

So, that is why we want to find out a response surface keeping beta 0 that is the intercept beta 1, beta 2, beta 3, and beta 1 2 3, because this is what is the interaction significant. It is peculiar example data, the reason is that whether the second order interaction and not significant, but third order interaction become 1 2 3 beta, 1 2 3 becomes significant, it may be due to the way we have generated data, that arbitrariness this is there, but whatever may be the situation, the case such situation if arise, if we have to do in this manner and then what is our response surface? the response surface is the beta 0 plus beta 1 x 1, beta 2 x 2, plus beta 3 x 3, minus these. Because this is what is negatively, negatively influencing ok.

So, this is our response surface. Now using the response surface, you can find out the, you can find out the contour plot, also keeping money, keeping varying suppose x 1, x 2 keeping x 3 at certain level, let it be 1, you can see what is the variability, how Y is varying with reference to change in x 1 and x 2, or other way keeping x 1 at a certain value and you change x 2, x 3 or x 3 at a certain, x 2 at a certain value.

Change this, you will be getting the response surface and the contour plots. I told you about the use of contour plot earlier; it will be in the same way. Now here few more thing that standard error of beta, you just 1, 4 this is calculated and then beta, how do basically find out the Error estimates, these are also given and our ultimate aim is here in regression model to see that what is the models fit it is 0.50, it is not a good fit model ok.

So, what we have seen that using ANOVA, you have also, you have seen that are few sources of variation which are contributing, but if we consider the overall effect of the, of this model is concerned while explaining the variability of Y through regression, using r square you are getting 0.5, 50 percent, which mean able to 1, but whatever may be the thing, but this hypothesis is rejected here, that there is no effect of the parameters at, of the contour factors, it is not correct. There are some of the contour level factors and their interactions are significant, and use this result carefully and get the benefit like where to set the process finally, ok.

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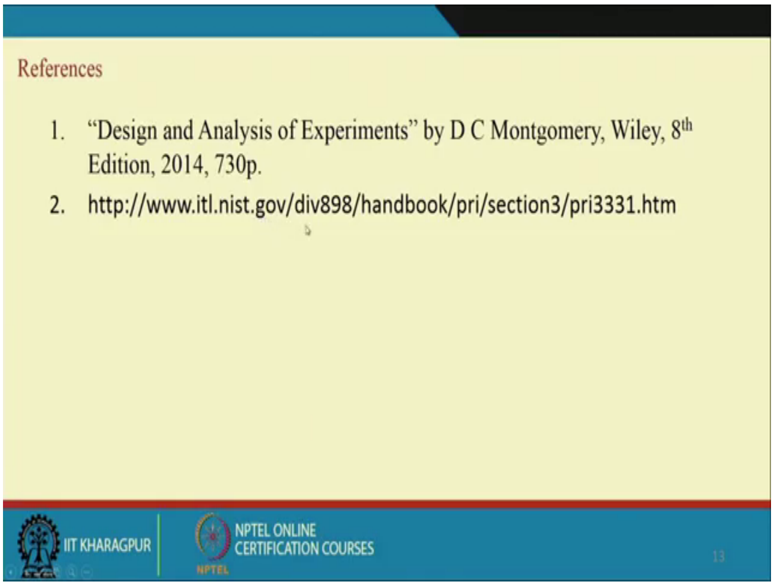

So, we have used Montgomery book, I told you, and also we have used another site that www itl dot nist dot gov, just we have gone through the site also, and we have read this, and got some, some benefits out of it.

Thank you, thank you very much, thanks a lot.