Design and Analysis of Experiments

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Lecture – 34
General Full Factorial Design

Welcome. Now we will discuss General Full Factorial Design.

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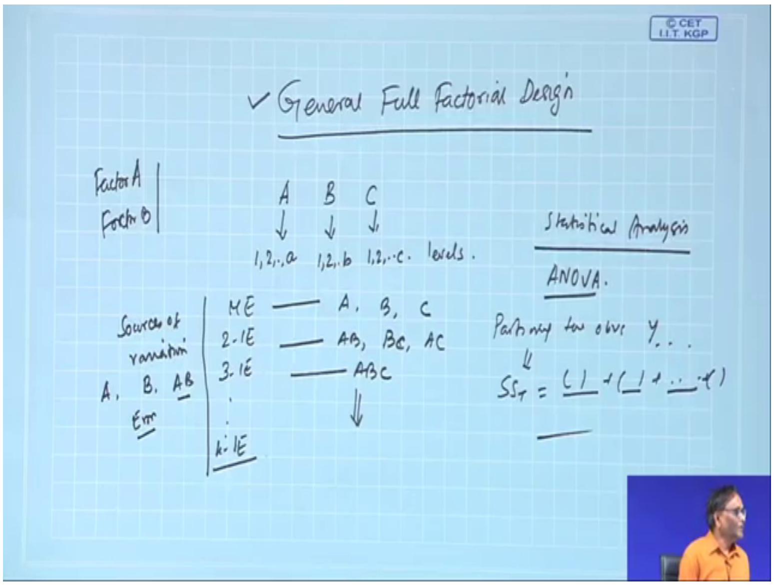

We have discussed full factorial design with reference to 2 factors. So, far, factor A and factor B. So, today we will spend some time on general factorial, General Full Factorial Design. And we will be considering 3 factors A, B and C. A with 1, 2 total a levels; B with b levels and C with c levels. So, will discuss with 3 factors and it can be generalised to k number of factors.

The fundamentals remain same; only difference when you go from 2 to 3 factors or 3 to more factors is that number of effects will change, will increase in terms of main effects, in terms of 2 way interaction effects, in terms of 3 way interaction effects, like this in terms of k way interaction effect. As a result also, this is this is the result of increasing sources of variation, sources of variation.

When we have 2 factors we have, 2 sources A, B and their interactions, AB and it is their interaction AB and error. When we have 3 factors, we have A, B, C, AB, BC, CA, ABC;

these are the sources of variability; so 3 factors and their 2 way interaction and 3 way interaction.

So, in that manner if you have k way in General Full Factorial Design, so there will be number of sources of variability will be more and accordingly the effects parameters will also be more. And the fixed effect model ready to bond which will be a lengthy one. And the general analysis the statistical analysis will be the same. Like the, we will ultimately interested to know to find out the ANOVA table with the increase sources of variation.

So, we will start with same way that partitioning the observation, partitioning the observation, y with appropriate i, j, k like this. And then partitioning the sum square total into individual source wise, source wise we go on partitioning. So, this is what we will learn here also.

The formula will be also similar with for every show every effect, we will be estimating the SS, every main effect every interaction effects, whether it is 2 way or multi way and the error s also will be computed, that degrees of freedom will be computed, then their mean square level will be computed, f will be computed and finally, using the appropriate f statistics will be seeing that which of the effects are significant, which of the effect are not significant. And we will see the tutorial kind of thing that is what example with excel I will show you. And please keep in mind that the theoretical basis is similar to 2 fact factor factorial designs, only the addition is the number of effects, number of SS and the resulting computation.

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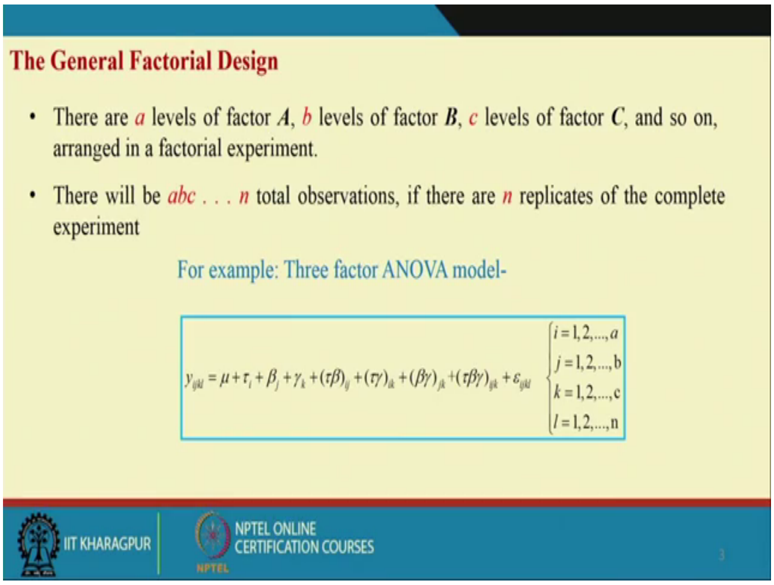

So, in 3 factor factorial design, the fixed effect model will look like this. So, you please see that what I have written for a 3 factor factorial in a general observation is y i j k l.

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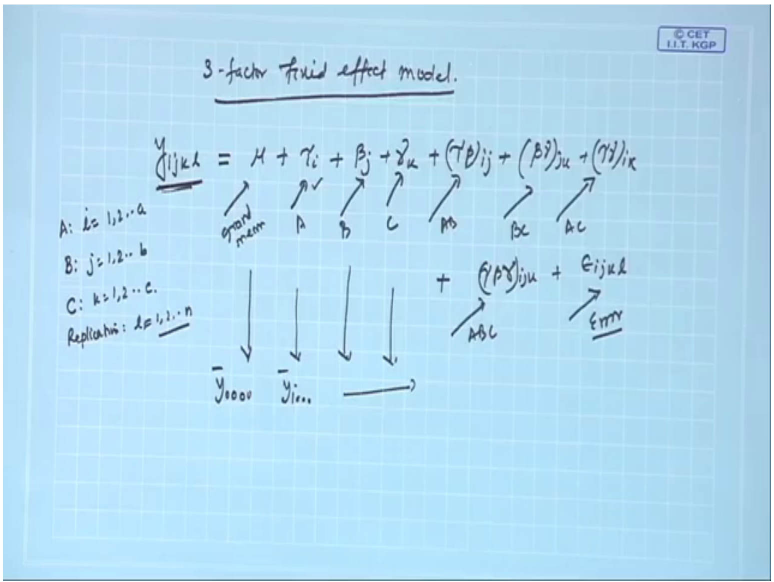

So, i from 1 to a, this is with reference to factor A. With reference to factor B, j changing from 1 to b; with reference to factor c, k changing from 1 to c. But we have n replicates, with reference to replication l change from 1 to n. So, accordingly if I say i j k l, this is that is the experimental that observation or the y value, when a is at ith level, k is at jth, b at jth level, c at kth level and replication and with l, lth replication. So, this one can be having grand mean that is mu plus tau i, that is the factor a effect, beta j factor, b effect gamma k factor c effect then their interaction is tau beta i j, interaction between A. So, this is your grand mean I say, a effect, this is related to b, this is related to c, this is related to AB, then you will write down beta gamma j k plus tau gamma i k. So, this will be your beta gamma mean BC effect, this will be AC effect plus tau beta gamma i j k that is ABC effect plus error epsilon i j k l, this is your Error.

Now, this is your 3 factor fixed effect model, 3 factor fixed effect model. So, you require when you collect data, after experiment you have every data and the resultant that what you require; you require to do parameter estimate with reference to all those parameters with reference to mu it will be y i j k l that bar, with reference to tau i that mu that is y i a 1, 2, 3 this bar. So, something like this we will finding out for all others the way we have done earlier. And then, we the parameters will be estimated and also in SS computation, we will make the square of all those things.

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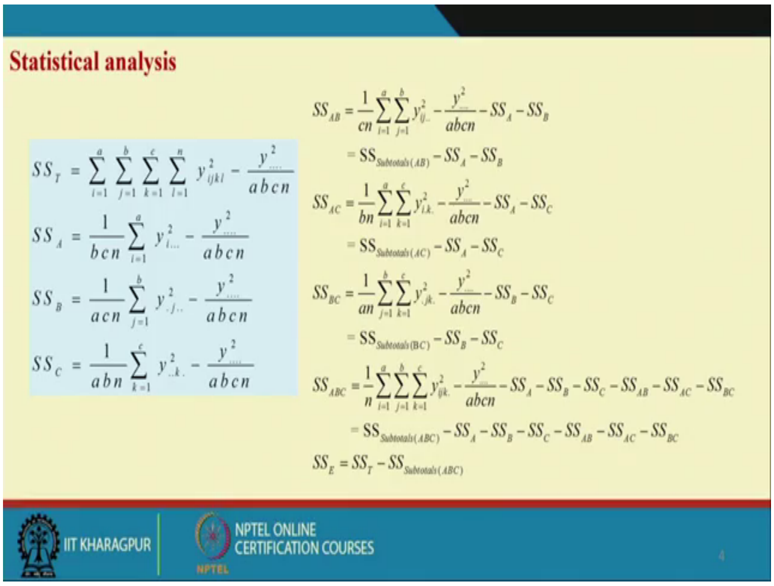

Now if you want to calculate, suppose you are interested to know what is SS t that is Sum Square Total.

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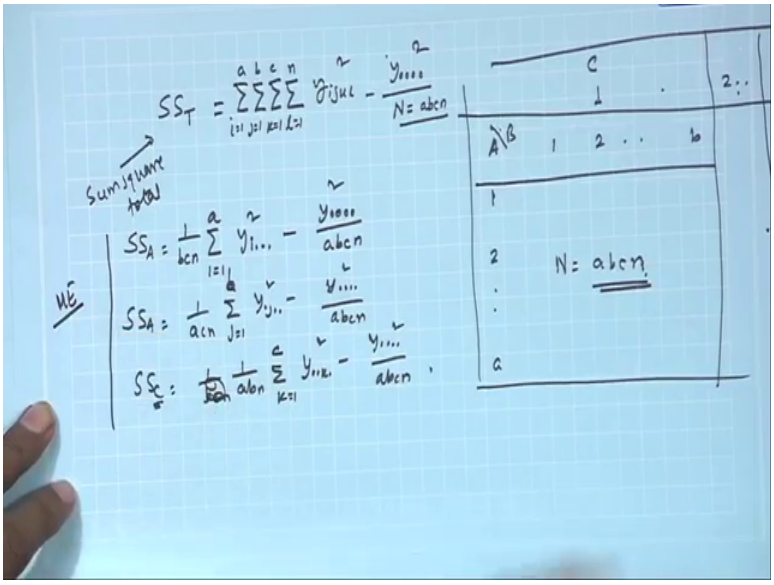

So what is, what we know here in sum square total. So, if you see the data metrics here, data metrics will be A will be suppose 1, 2 like a level, then if I say B will be 1, 2 then b level. So, suppose B and there will be another C and that will also be 1, 2 like this levels. So, your data metrics will be having for every c, there will be every a and b, every level for these. Similarly, for c equal to 2, the same thing will be repeating. So, ultimately what happened, for SS t you will compute that to consider all the values. So, N is abc into small n. Now you know what is abcn.

Then SS t will be having 4 sums, one is l equal to 1 to n, k equal to 1 to c, j equal to 1 to b, i equal to 1 to a; then you find out i j k l this square. So, minus we have seen the correction part these square by N, why 4 dot square by N? Because 5 for, one for a, one for b, one for c, one for replications. So, this is our formula for SS t and this is nothing but a b c into n. So, then what will happen for SS A, SS A means there are a row total, 1 first row, second row like this.

So, all row totals will be considered, it will be sum will be i equal to 1 to a then y i triple dot square. Now this is best on that bcn, replication. So, what will happen 1 by bcn minus y triple 4 dots 4 naught by abcn. So, what will happen to then SS B, 1 by acn, j equal to 1 to a, 1 to b, y dot j dot square minus y double triple 4 dots square by abcn and similarly SS C will be 1 by bcn, 1 by abn, not bcn, 1 by abn will this is C, sum total of k equal to 1 to c the y dot k dot square minus y 4 dot square by abcn. Now comes the, these are the main contribution of the main effect or SS for the main effects. So, what you require, now you require suppose contribution for the 2 way interaction effects.

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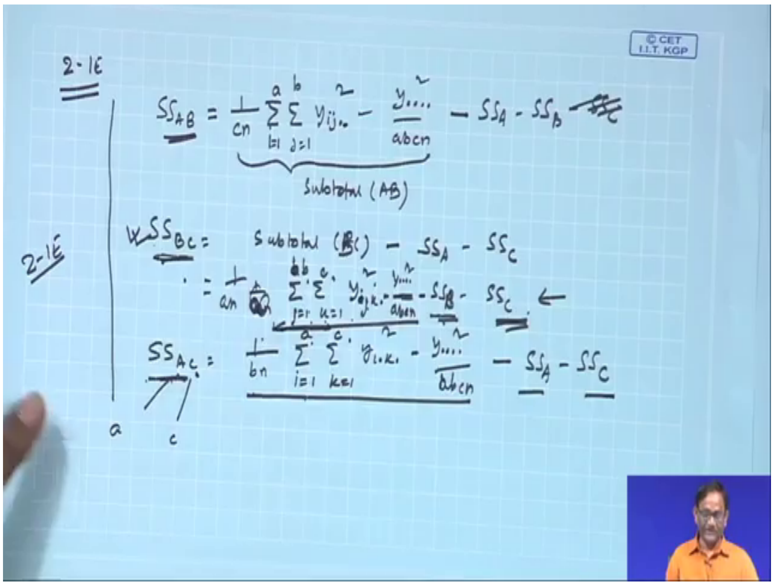

So, 2 way interaction effects when you compute, we say that the sum square is SS A B. Here A and B both are there. So, you consider i equal to 1 to a for a, j equal to 1 to b for b, y i j dot dot; this square and this will be divided by cn, because the other all values c n values are considered here then minus y 4 dot square by abcn. These gives you subtotal, this gives you subtotal AB.

Then this one will be subtracted S with SS B, SS A, SS B, sorry SS A and SS B because SS AB we are computing. So, similarly for SS BC, you find out the subtotal AC, then you subtract it by SS A and SS C. Now what is subtotal AC, 1 by bn sum of i equal to 1 to a, k equal to 1 to c, y i dot k dot square minus SS A minus SS C. Similarly, for SS AC will be SS AC, SS BC, 1 by an; I have done the mistake. 1 by SS AB is 1 by cn, SS AC is SS BC will be 1 by an and SS AC will be a and c term will be out.

So, that i and that a this a will not be there, that c will not be there, then b will be there, b n sum total of SS AC means i equal to 1 to a and k equal to 1 to c. So, that mean if I write SS BC here, this is I equal this will be j equal to 1 to b and k equal to 1 to c, then this will be dot j dot square minus SS B minus SS C, this correction you make. You are writing SS BC here not AC. So, SS BC this will be 1 by a n j equal to 1 to b, k equal to 1 to c, y dot j k dot square minus SS B, SS C.

Now, I am writing here SS AC. So, that why 1 by bn, i equal to 1 to a, k equal to 1 to c, then y i dot k dot square minus y 4 dot square by abcn minus SS A minus SS C. So, as such here also minus y 4 dot square by abcn will be there. So, SS subtotal BC, SS subtotal BC minus SS B minus SS C. Here SS subtotal AC minus SS A minus SS C. So, that mean these are 2 way, 2 interaction 2 way interaction and their sum square computed. So, and then finally, then there is another source.

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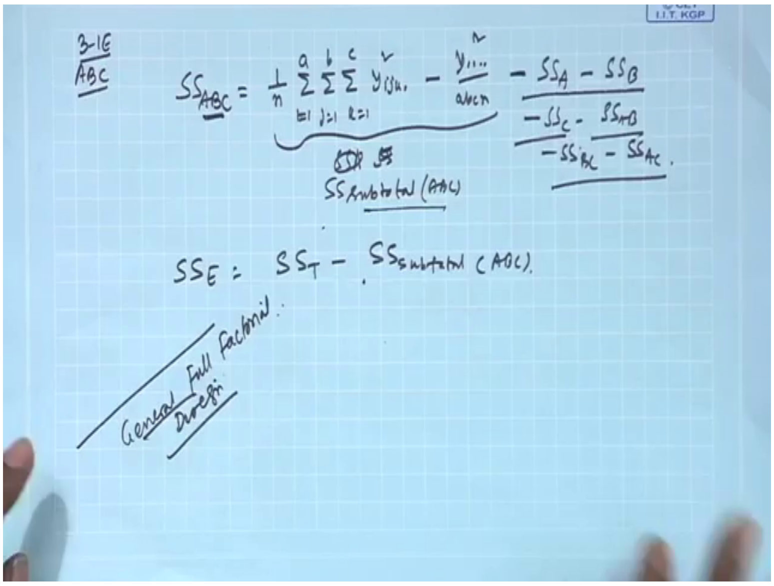

Which is 3 way interaction ABC, 3 interaction ABC. So, you have to compute SS ABC. So, SS ABC, when you compute these will be, now ABC is taken they will be varying. So, you are having 1 by n, j is not varying then i j k, i equal to 1 to a, j equal to 1 to b, k equal to 1 to c, y i j k dot square minus y 4 dot square by abcn. This is nothing but SS subtotal. SS subtotal called ABC. Then it will minus the main effects and 2 way interaction effects, SS A, SS B, SS C, SS AB minus SS BC minus SS AC. So, that mean that SS subtotal ABC minus 3 that SS A, SS B, SS C, this one the main effect contribution and second order s contribution will be subtracted. Then SS error will be SS total minus SS subtotal ABC.

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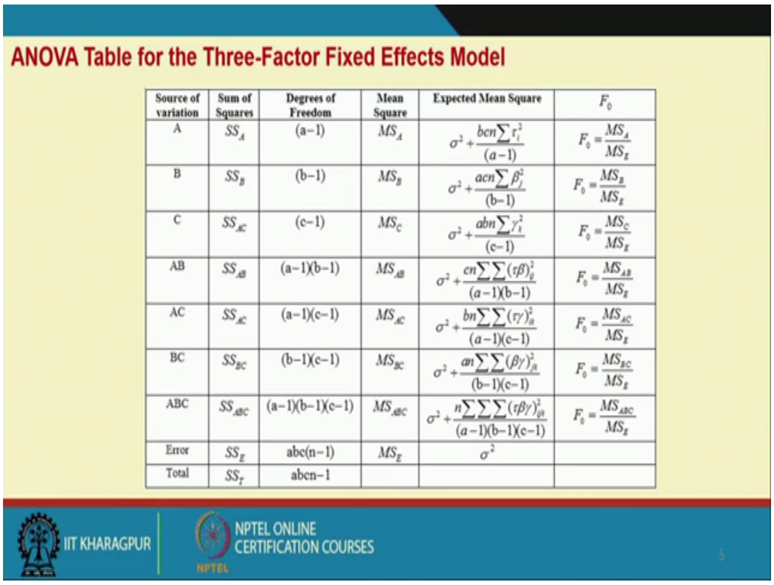

ANOVA Table for the Three-Factor Fixed Effects Model

Source of variation	Sum of Squares	Degrees of Freedom	Mean Square	Expected Mean Square	F_0
A	SS_A	(a−1)	MS_A	$\sigma^2 + \frac{bcn\sum\tau_i^2}{(a-1)}$	$F_0 = \frac{MS_A}{MS_E}$
B	SS_B	(b−1)	MS_B	$\sigma^2 + \frac{acn\sum\beta_j^2}{(b-1)}$	$F_0 = \frac{MS_B}{MS_E}$
C	SS_{AC}	(c−1)	MS_C	$\sigma^2 + \frac{abn\sum\gamma_k^2}{(c-1)}$	$F_0 = \frac{MS_C}{MS_E}$
AB	SS_{AB}	(a−1)(b−1)	MS_{AB}	$\sigma^2 + \frac{cn\sum\sum(\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
AC	SS_{AC}	(a−1)(c−1)	MS_{AC}	$\sigma^2 + \frac{bn\sum\sum(\tau\gamma)_{ik}^2}{(a-1)(c-1)}$	$F_0 = \frac{MS_{AC}}{MS_E}$
BC	SS_{BC}	(b−1)(c−1)	MS_{BC}	$\sigma^2 + \frac{an\sum\sum(\beta\gamma)_{jk}^2}{(b-1)(c-1)}$	$F_0 = \frac{MS_{BC}}{MS_E}$
ABC	SS_{ABC}	(a−1)(b−1)(c−1)	MS_{ABC}	$\sigma^2 + \frac{n\sum\sum\sum(\tau\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$	$F_0 = \frac{MS_{ABC}}{MS_E}$
Error	SS_E	abc(n−1)	MS_E	σ^2	
Total	SS_T	abcn−1			

And you know that they are resulting Anova table will be then after this the resultant Anova table will be like this. You see the source of variation A, B, C, AB, AC, BC, ABC, then error.

Sum squares, SS A, SS B, SS C, this will be SS C, then SS AB, then SS AC, then SS BC, and SS ABC. So, make a correction here. This is a C this is a type of. Then degrees of freedom we have a level so for factor A. So, that is why a minus 1, b minus 1, c minus 1, a minus 1 into b minus 1, a minus 1 into c minus 1, b minus 1 into c minus 1 and like this; these are these are known to you.

Now why it should be computed? And then finally, you will then you compute mean square mean SS A by degree of freedom. So, SS by degree of freedom will give you mean square corresponding mean square computed. Now every mean square of the effect is compared with the mean square error where mean square error is MS C which is sigma square. And then f value is computed by comparing every effect mean square divided by mean MS C that will give you the f 0 value, this f 0 value will be compared with the threshold value. So, that is it is.

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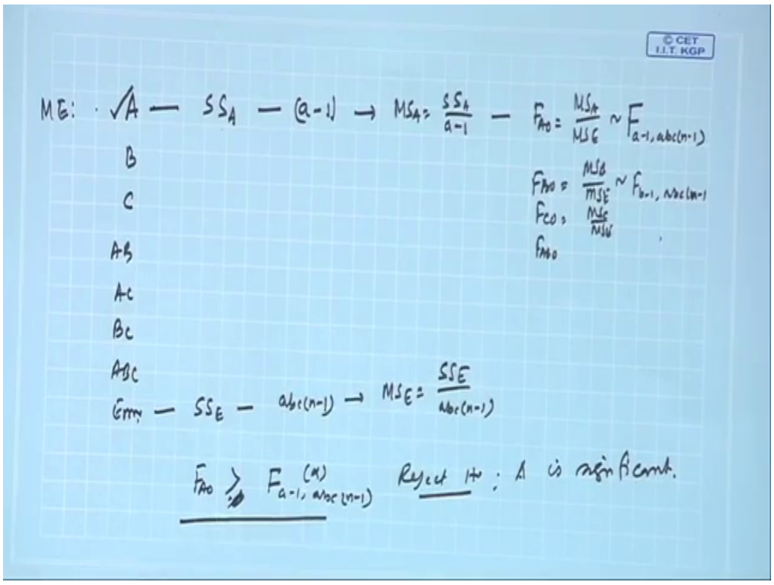

So, one example I am giving. Suppose you are you are interested to know whether factor that mean A, that main effect A is effect having influence or not. Then you are computing it is SS A, you are computing it is degree of freedom is a minus 1, then what you are computing it MS A which is basically SS A by a minus 1. So, similarly B, similarly C, similarly AB, similarly AC, similarly BC, similarly ABC, then similarly error. Now for error you are writing SS error. Then you know the degree of freedom is abc into n minus 1, abc into n minus 1.

So, it is MS E you are computing which is SS E by abc into n minus 1. Then now you are computing F from there you are computing F A0 which is MS A divided by MS E. So, similarly FA0, F B0 MS B by MS E, F C0 will be MS C by MS E. F AB will be F AB0, 0 for null hypothesis same manner. Then your these will F A0 will be distributed as F degree of freedom a minus 1, abcn into n minus 1. So, and see this one will be F B minus 1, abc into n minus 1 like this. Now suppose we are interest to for example, talk about A, then what you do F a minus 1 abc into n minus 1 alpha; this value will be compared with F A0; may be F A0 greater than equal to this or let us be write down greater than greater than these reject H0. So, A is significant. That mean main effect A is significant. Same manner other effects will be considered.

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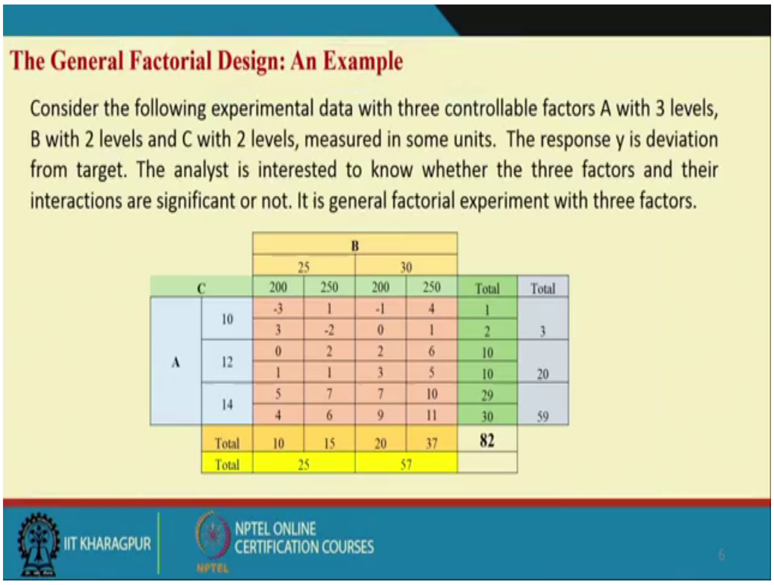

		B					
		25		30			
C		200	250	200	250	Total	Total
A	10	-3	1	-1	4	1	
		3	-2	0	1	2	3
	12	0	2	2	6	10	
		1	1	3	5	10	20
	14	5	7	7	10	29	
		4	6	9	11	30	59
	Total	10	15	20	37	82	
	Total	25		57			

So, let us see one example here there are 3 factors A, B and C. A with 3 level is 10, 12, 14. B with 2 level 20, 30. C with 2 level 200 and 250. So, our example, we have 3 factors A: 3 levels, B: 2 levels, C: 2 levels.

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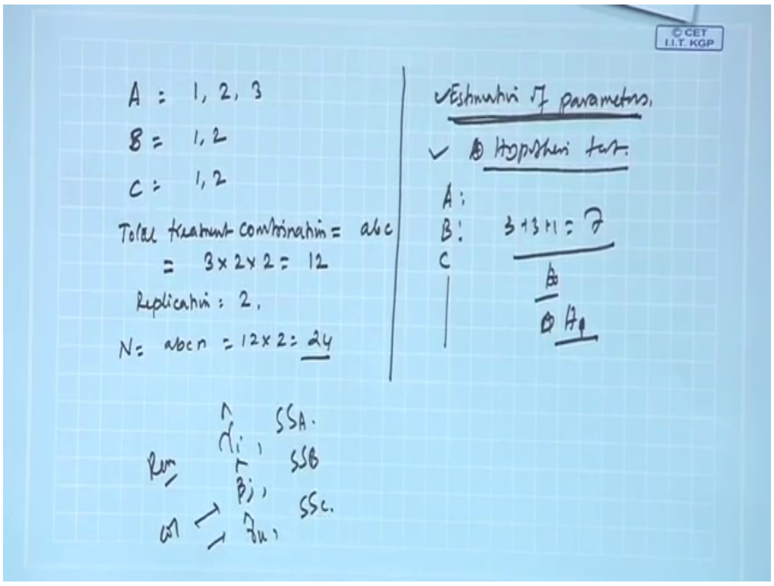

So, we have total treatment combination equal to abc equal to 3 cross 2 cross 2 equal to 12. If you see the data, see the data suppose A is 10, B is 25, C is 200, then we have 2 observations here minus 33. So, that mean we have replication equal to 2. So, then what is the total observation n this is abcn equal to 12 cross 2 equal to 24; we have 24 data points if you see this 1, 2, 3, 4, 6 into 1, 2, 3, 4 that 24 data points we have. So, what we want basically? Suppose our job will be of 2 forms. One is estimation of parameters and second one is that Hypothesis test.

So, Estimation of parameter, I have discussed earlier in the same manner we will it will be computed and hypothesis test there will be hypothesis related to each effect with related to A, with related to B, with related to C, like this there are AB, BC, CA, 3 plus 3 plus 1, 3 plus 3 plus 1, 7 hypothesis H 0 and 7 h naught alternate hypothesis also H naught and H 1. So, those every hypothesis will be tested and it will be tested using the using this kind of things.

So, I will show you how the SS for different thing will be computed here. You see that if you consider A, the row total is 3 with respect to 10 and 20, 59 grand total is 82. So, you can find out 82, the average will be 82 by 24 and here also you will find out the average that is 3 by 2 plus 2 plus 2 plus 2, 8; so 3 by 8. So, then the parameter estimate effect of a equal to 10 that level is this 3 by 8 minus 82 by 24.

So, in this manner you will find out the different level effects, different levels. Similarly for b 25, 57 is computed, you compute the average, subtract from them grand mean you will get the B effect. And for C is 200 and 250. So, 200 plus 200, this 200, this 200 value will give you 10 plus 20, 30 and for at 250 it will be 15 plus 37, 52. So 30, 52 that average you calculate minus the grand average. And in the same then manner the AB, BC will be computed.

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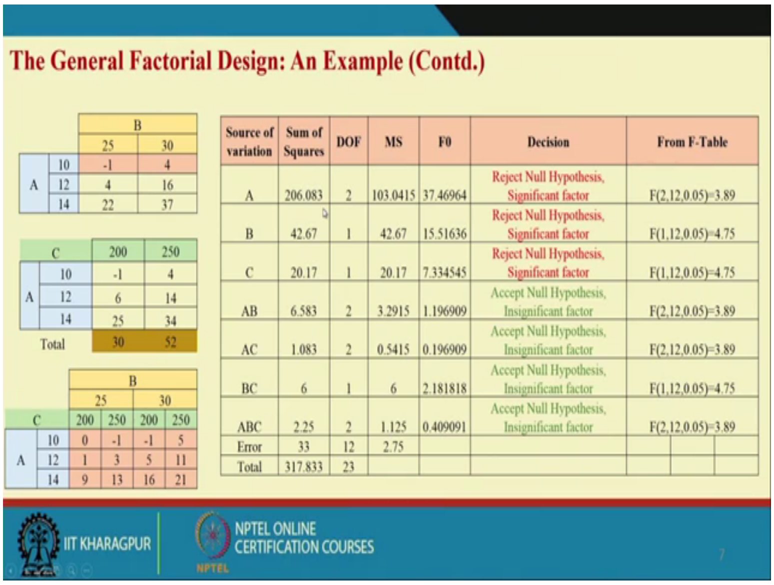

Now for SS computation, the SS values if you see that A, B, C all those these are the values sum square values, how we have computed those sum square values. So, suppose we want to A, B, C parties we have seen, suppose we want to compute AB, AC, BC, now when you are talking about AB, you are creating that creating the cell total when A equal to 10 and B equal to 20, what is the cell total? A equal to 10 and b equal to 25 then there are how many data points, 4 data points, minus 3, 3, 1, 2 minus 2. So, it will be minus 1, sum of these will be minus 1. So, in this manner minus 1 is coming.

Similarly, suppose A equal to 14 and B equal to 30, it is 37; how it is coming you see. A equal to 14 and B equal to 30, b equal to 30 then 14, 30 mean this 4 observation 7 plus 17 teen plus 11, 28 plus 9, 37, so 37. In the same manner, AC the totals, the corresponding special totals and ABC, ABC were the only the n will be replication will be added. So, minus 3 minus 3, 0 and like this.

So, that is what is the different table you are creating from the main table, from the main table you are finding out the row total for computation of tau i and SS A that is row total, then column total will give you beta j, estimate of beta j and SS B and in column total again there is an C factor C factors is also there. So, that will give if I come from column again, that is all tau, beta, then gamma k and SS C for computation of an AB, BC, ABC, similarly BC, BC interaction. So, what will happen, you are creating separate tables. And from here using simple that, arithmetic operations, you will be able to compute all SS A, SS B, SS C. So, what I will do, I will show you excel sheet here.

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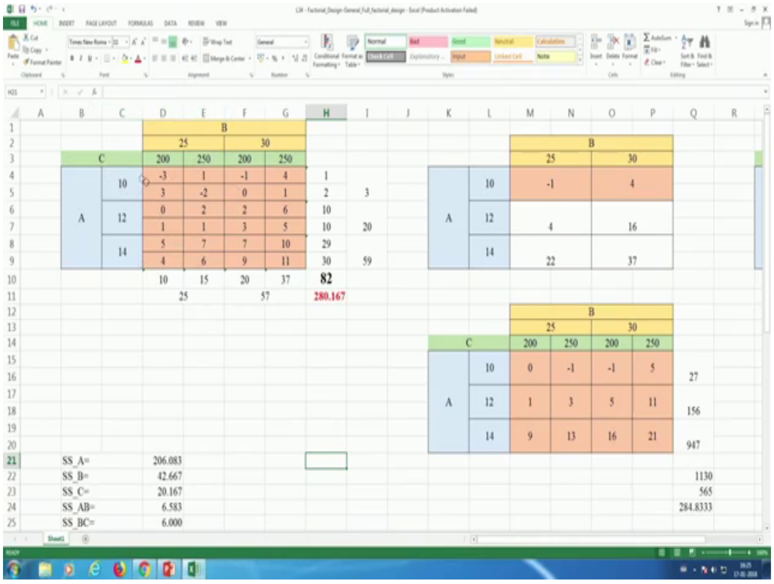

And will show you 1 or 2 calculation. For example, you say this table, here total is 3 how this total is coming you see that sum total of all with respect to 10. Sum of d 4 to g 5 all those things. Similarly if you want this with this cell you are getting this totals. Now the grand total, you see everything is considered here. So, then y 4 dot square is this, you see what we have is there h 10 square by 24 because this is basically the grand total. So, grand total square by 24 is giving this.

Now, when we are going to compute SS A, what is the formula you are using? What is the formula you are using? So, SS A formula or SS T formula, suppose SS A formula is 1 by bcn into this the rows, row total square and sum of them and this. So, this one, I will show you. Suppose SS I row total I am clicking here, what you have 1 by 1 by bcn. So, b equal to 2 c equal to 2 and n equal to 2.

So, 1 by 8 into this square rows, this row square plus, this row square plus, this is the square is the first part this minus that correction one this. Now let us see that in the same manner BC, will consider suppose I want to know this one interaction AB. So, I am clicking here and how we have computed you please see. So, what is interaction AB, interaction AB formula is this. 1 by cn sum total of this square minus this correction factor minus the individual that main effect sum square. So, 1 by cn is 4 and then y i j square. So, where is this, you just see it is coming here. It is here 1, 2, 3, 4, 5 this 6 cell, this total values are squares minus H 11. Now what is where is H where is H 11, this one because this is the correction factor y dot dot dot square by abcn minus d 21, d 21 is SS A and d 22 is SS B ok.

Suppose you want to see the SS ABC that is the 3 way interaction. I have what I have created you see. What is the formula for 3 way interactions? The formula for 3 way interaction is s subtotal ABC minus all the factors and this is your subtotal 1 by n. So, 1 by 2, then where is this 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, because ABC equal to 12, this 12 cell values are, totals are, we have created different cells. So, will get total square and minus obviously, this one because that is what is coming here minus you see SS A, SS B, SS C, SS AB, SS BC, SS AC, all these things are considered here up to d 26, d 26 is SS AC.

So, this is giving you the way you can compute. So, that mean within a table, within the data table by using the arithmetic summation and arithmetic operations, you will be able to find out not the parameters tau i, beta j, gamma k, tau beta i j, all those things plus SS A, SS B all the sum square also you will be able to compute and then you will be you will be and you can develop the Anova table. So, this is what is our Anova table; finally. So, finally anova table, the degrees of freedom are also given and MS is calculated, F 0 calculated and then found out that with reference to the threshold value that ABC effect are significant, but interaction effect are not significant. So, none of the interactions are that values are F 0 values are more than what is the threshold or tabulated value.

So, we can say that the 3 factors effecting the y but they are, they as they have do not have the correlation or dependents relationship with any of the factors, I cannot say dependent. I can say that they are not correlated, so that means, the joint effect is not there. So, their interaction are not contributing towards y, as per as the model and the data is considered.

So, then what they you, I think you now know the General Full Factorial Design and how it is to be done. So, I think we have spent lot of time on this full factorial design; particularly we started with introduction, then we have shown you the 2 factor, then we have seen you have seen the with the with replicate, single replicate, now the general full factorial design. So, the remaining things may be how do you use blocking, because there will be blocking when which instructor which is to be blocked. So, how do how do blocking with General Full Factorial Design or with full factorial design how do you use blocking, that will be discussed in next class.

Thank you very much.