Design and Analysis of Experiments

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Lecture - 17
Two –Way ANOVA

Welcome. Today we will discuss Two and Multi-way ANOVA

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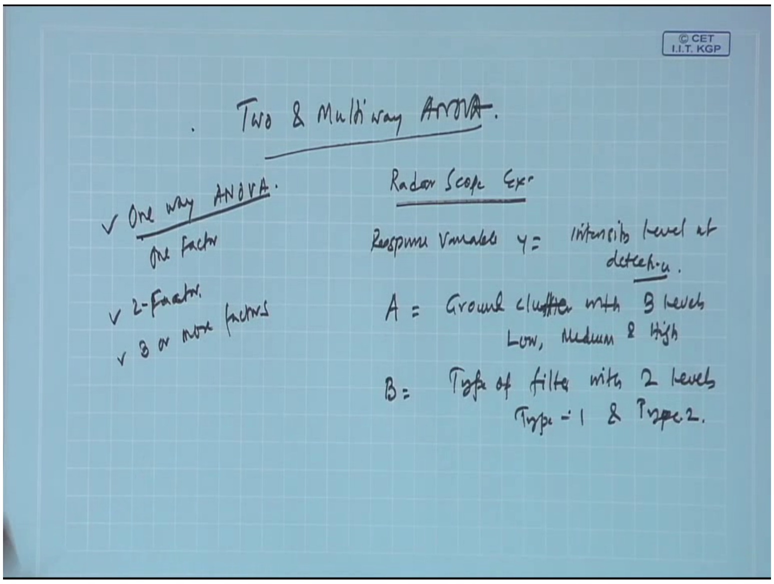

I hope that you have understood one way ANOVA. So, I also equally hope that you will be able to work with two and multi-way ANOVA. This way, one way means one factor two-way means 2 factor, multi-way means 3 or more factor. So, what is that two-way? If you consider radarscope example; so the response variable y that is the intensity level at detection, detection of targets. And if you recall we say that ground one factor A is ground clutter with three levels low, medium and high. And type of filter B; type of filter with two levels filter type 1 and filter type 2.

So, what I can say that if you do a random experiment actually you are having a situation like this.

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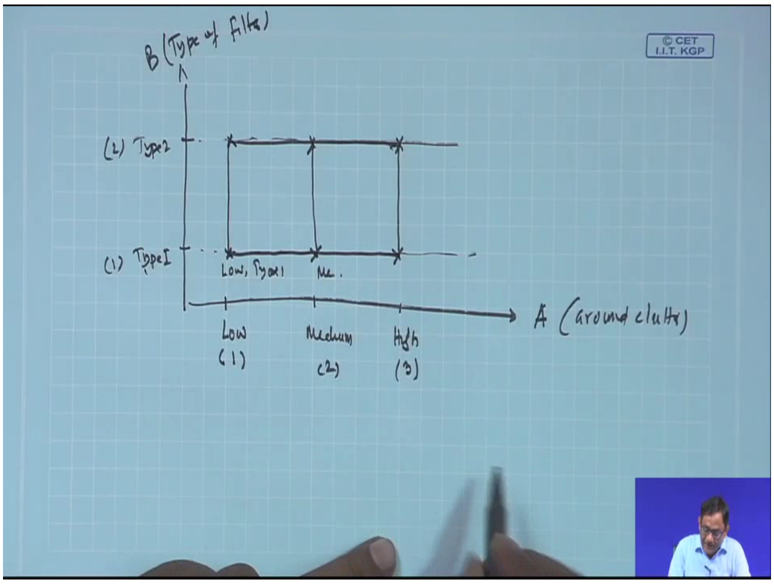

This side A ground clutter, this said B type of filter. So, you have three levels of ground clutter level low I can say 1; medium maybe 2; and high 3. And you have two levels of; so type 1 filter and another one is type 2 filter.

So, this is level 1 this is level 2. So, you have how many experimental settings. You have six number of experimental settings or treatment combinations. This is one, this one is another one, this one another one, another one, another one, another one. So, here low low; low and type 1, here medium will type 1 medium the type 1 like this. What does it mean?

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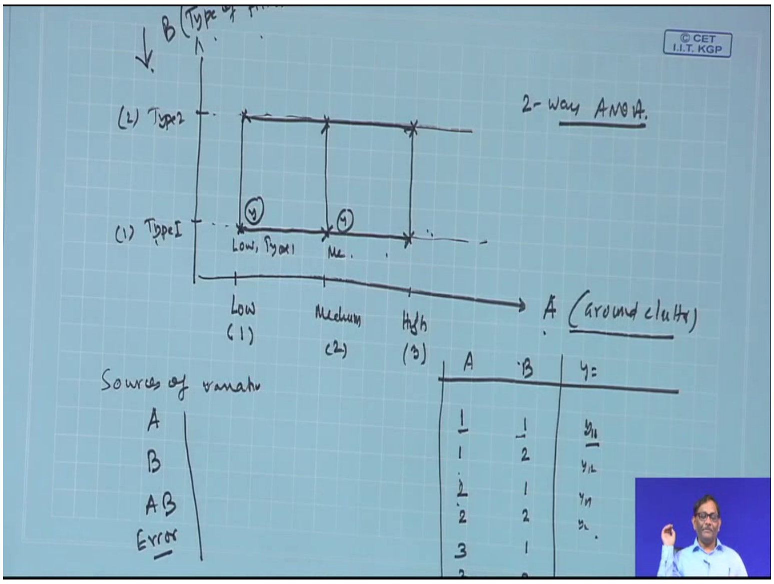

You are conducting an experiment in such a manner that suppose A factor level 1 then B factor level 1, then you are getting y value here the intensity level at, so intensity level at. So, this is maybe y 11 if I go for.

So again this is 1, this may be 2, and this is 2 1 2 2 3 1 3 2; so you will be getting several. So, what happened here this means y A 1, B 1, 11, 12, 21, 22 so like this. But when we I will show you the data I will we will not represent in this manner there will be replication one more things will be there one more direction will come.

So anyhow, then in this case this is a 2 factor experiment. Now if you want to develop a ANOVA model for this it will be a 2 factor ANOVA model. And when you analyze the data, you have to analyze considering that 2 factor and their interactions are there. So, what does it; does it mean that why here the response values y values here also y values will be there all those response values not only depends on the ground clutter it may depend on the type of filter. In addition it may depend on the interaction between ground clutter and type of filter that interaction between A and B.

So, effectively sources of variation will be not only A it will be B and their interaction AB and plus there will be a random source that is error. So, accordingly the observations will be partitioned reference to A factor, with reference to B factor, with reference to their interactions and obviously, error will be there. So, this kind of data will be analyzed using two-way ANOVA.

So, we will come to some other example, but for the time being you please understand two-way mean there are 2 factors and each of the factors has at least 2 or more levels. So that means, there will be A level for A factor B level for B factor and there will be a into B AB treatment combinations, and in each combination there will be n number of experiment; obviously, in the random order it will be done. So, in total you will be having AB into n number of experimental runs or the object data will be AB n.

So, let us see; what will be the model for two-way ANOVA. You see the basic 2 factor model.

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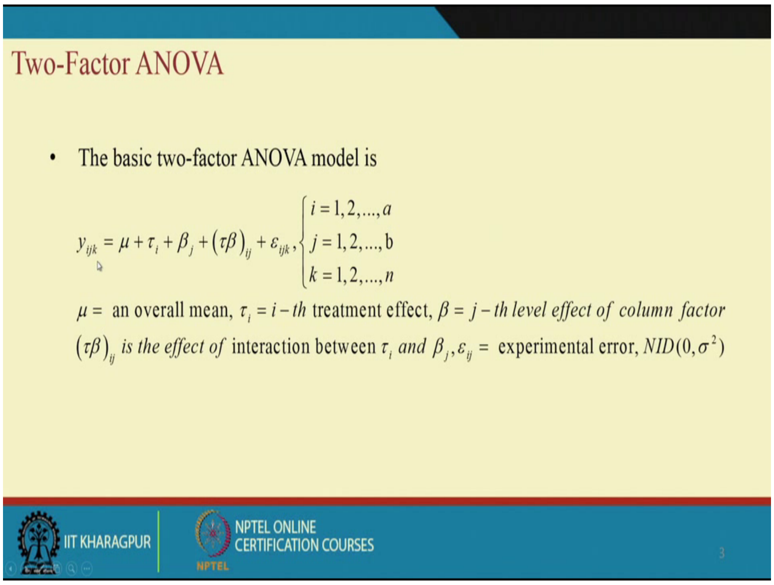

So, here again what we will do; we will see the table first.

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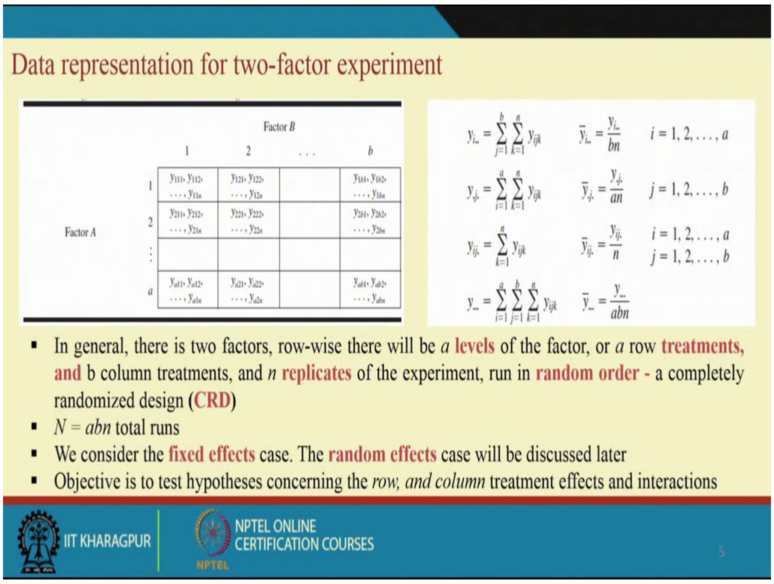

Now see this table. Factor A, factor B: A has A number of levels, B has been number of levels, and the each cell here in this table is A treatment combination; so A at level 1 B at level 1 this treatment combination.

Similarly A at 1 B 2, so there will be AB treatment combination unique independent treatment combinations. And another one interesting one is that in each of the combination you find out that there are n number of experimental runs; n number of data available. So, as a result we index this data like this.

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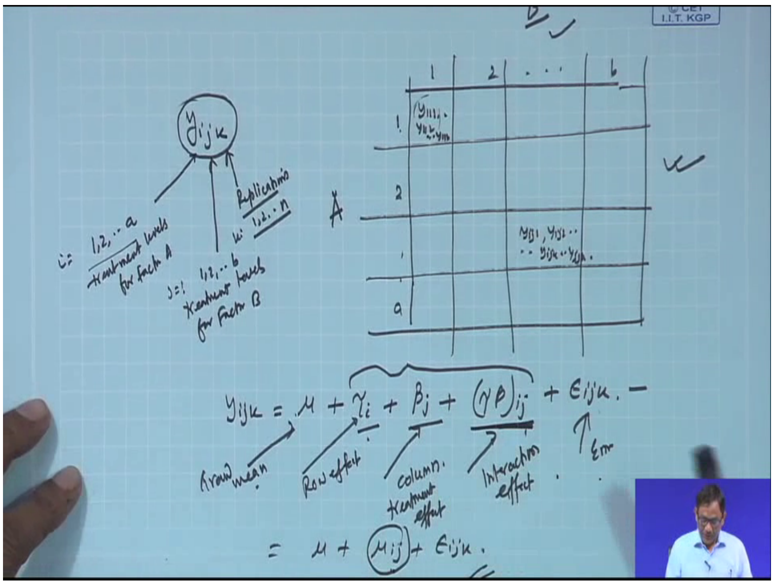

Y ijk: so i stands for that 1, 2, a; that means, this is the treatment level for factor A, j stands for 1, 2, b treatment levels for factor B, k is replications. So, it will be 1, 2, n; n number of applications will be there.

So, this is the general data and our table is like this: 1, 2 like dot dot dot a and here 1, 2 dot dot dot b. And here y 11 as I shown you earlier also that y 11, but and that 11 level there will be n number of experimental run; so y 111, y 112, like this y 11n. So, in general y ij1, y ij2 like this, y ijk, y ijn.

So, there will be general observation y ijk. You can partition this y ijk equal to grand mean mu plus that is A factor affect B factor affect also. So, tau i is the factor A f treatment A f effect, beta j for B treatment effect plus tau beta ij that is the interaction effect plus epsilon ijk. So, this is my ANOVA model.

So, you can understand now i from 1, 2, a; j from 1, 2, B; k from 1, 2, n and this is your model. So, I can say other way also this is row effect from considering this kind of table this is column effect. So, other way row treatment effect, column treatment effect that is interaction effect this is error and this is grand mean, ok. So, now you may be interested to add this together, so you can write this one equal to mu plus mu ij plus epsilon ijk; mu plus mu ij. So, this mu ij is row column and treatment this all those things. So mu ij here, mu ij every cell this image every cell there will be a mean value. So, y ijk, so the general value will be grand mean plus mean of this plus error of this; so where mu ij equal to tau i plus beta j plus tau beta ij; ij a tau beta j.

So, let us see in the table again. Now, there will be if you take sum total across rows that will be y i dot dot this one. If you take sum total across columns that will be y dot j dot. If you take sum total of observation in a cell that is y ij dot; if you tell all the grand total considering AB and this is the case. And you can calculate average also. If you compute average or across rows then the row basically you see there are B number of columns and under every cell there is n number of data points so that why it will be divided by B n. For column average, the column total will be divided by n. For cell average, the cell total will be divided by cell num number n. And for overall average, the overall total will be divided by total number of observations, ok.

In this case also there will be random fixed effect models and random effect models, we are considering fixed effect model. And please keep in mind that we are basically interested to know whether the row treatment and column treatment effects present and their interaction if it is there or not; So that means, we want to know whether the factory affecting the response variable factor B affecting the response variable, and their interaction also playing a role in the observed response values. So, that will be tested.

So, as a result here there will be three kinds of three different null hypothesis and we will be testing it.

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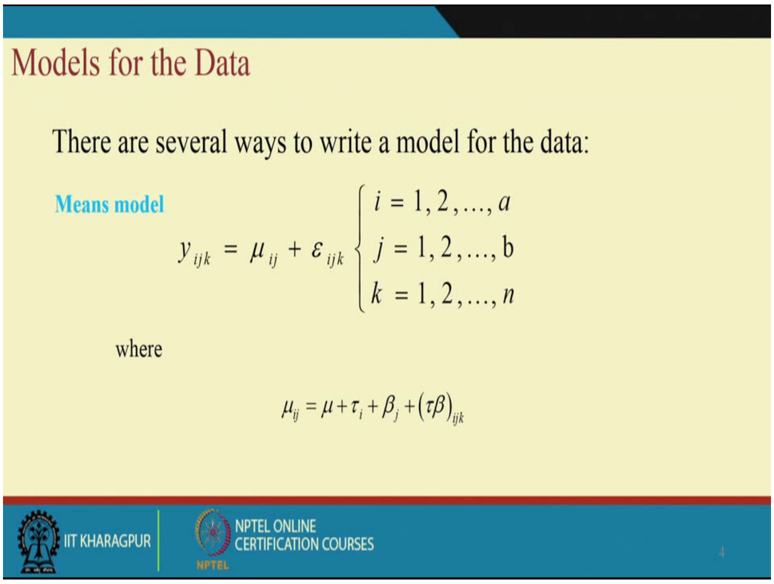

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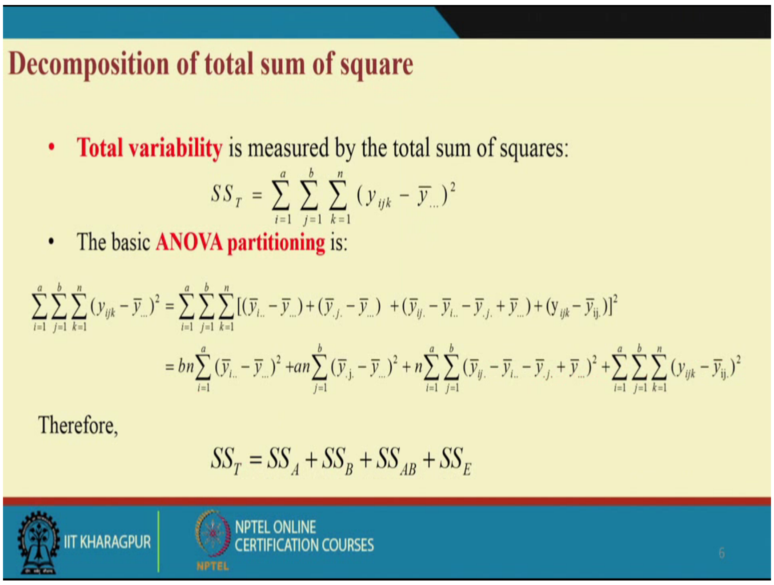

Now that test will be done using analysis of variance. So, the similar type of calculation we will be doing. First partitioning the general observations into overall average, into row average, into column average, and into cell average plus errors and then making their sum square total as the treatment color row treatment sum square, column treatment sum square, then interaction sum square and error sum square.

So, this is what we will be doing. In one way ANOVA the sources of variability a variability are the factor itself, and the error in two-way ANOVA sources of variability are factor A, factor B, their interaction and error; four sources will be there.

Now see the partitioning. As I told you let me; what is the general y ijk. That is the general observation.

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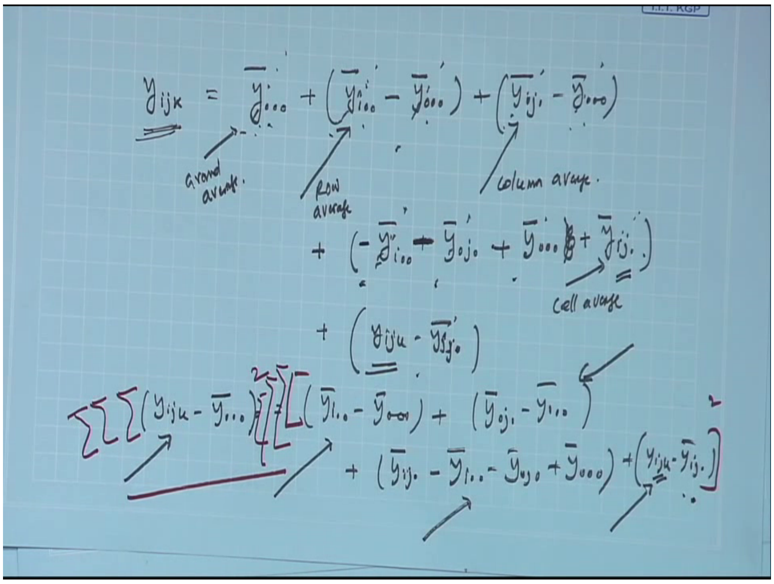

So, this can be written like this y dot dot dot bar, this is the grand average. So, you can write now y i dot dot bar, what is this? This is row average. I can subtract this using this one grand average also. Plus I will go to dot j dot bar minus y bar dot dot dot, so this is my column average. Otherwise, you can say row means factor A average factor B level average, so plus if you see that here what happened this is one time plus minus this but these two time you have already taken. So, if I write like this y i dot dot bar plus y dot j dot bar then, I write this one minus this; minus this because this is got plus consider plus consider plus y dot dot dot this. So this, this; and I add one more term here y ij dot this is my cell average.

Then, what is left here? This, this one will be canceled out by this one sorry; this will be cancelled by this one, y i dot dot dot this will be canceled out by this one, y this will be canceled out by this one, then this will be canceled out by this one, so remaining this, this also to be canceled out. So, if I write y ijk minus y ij dot bar then this will be canceled by this then y ijk equal to y ijk. So, essentially what you have done then? You have written a general observation in this way ij k minus i dot dot dot bar, this is the deviation; individual observation deviation from the grand mean. I can write this equal to i dot dot bar minus i triple dot bar plus y dot j dot bar minus triple dot bar plus i can write y ij dot dot bar minus y i dot dot bar minus y dot j dot bar plus y i triple dot bar, then plus I can write y ijk minus y ij dot bar.

So, this is the general observed deviation for individual observation. This is the deviation of the row average from grand mean, grand average, this is the deviation from column average from the grand average, and this is the basically the deviation in the cell from cell average to the grand average and other things because you are combiningly; and this is what is the deviation from every observation to itself average. Now, this is the partition.

Now what you can do, you can square it then take sum triple sum will be there here also tipple sum and you will ultimately get this kind of results. So, let us see the slide here. In the slide itself, that means the left hand side you see the left hand side that is the individual observation is subtracted by the grand average and when we take the square and this is known as sum square total. And this quantity when you base expand it and then take the appropriate sums you will find out the inter sum of the cross term products will be vanished.

So what will remain here would generally, it will remain only this B n into these, A n into these, n into this, and this. So, this if you carefully look the first one you see that it is basically talking about the difference row level row difference my average deep from the grand average, here column average from the grand average, here cell lever is from the rests, and here basically the individual observation from the cell average.

So, this term is known as the row effect; row sum square row or sum square A this is sum square B, this one is sum square AB, and this one is sum square S E.

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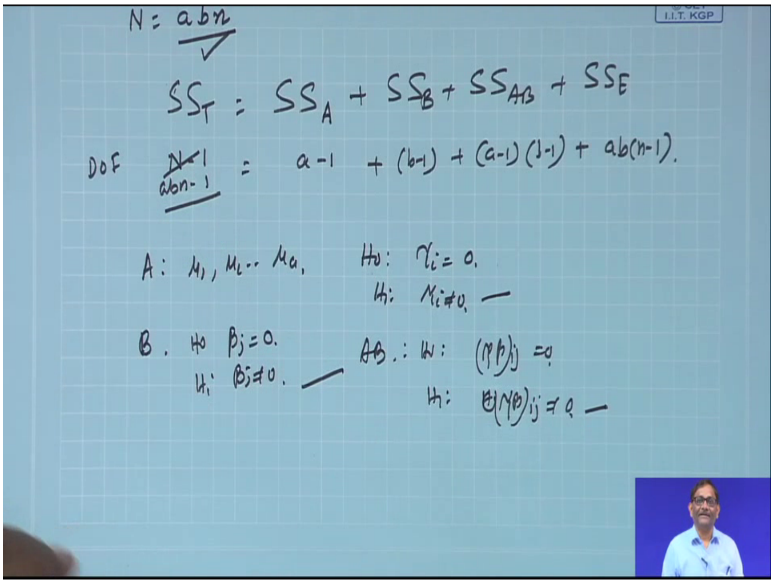

So, the variation part which is basically sum square total is now partition into sum square factor A sum square B sum square AB and sum square E. Like in 1 way ANOVA you have seen SS T equal to SS A plus SS E, here two more terms coming because the B source of interaction and; your B source and interaction source are coming.

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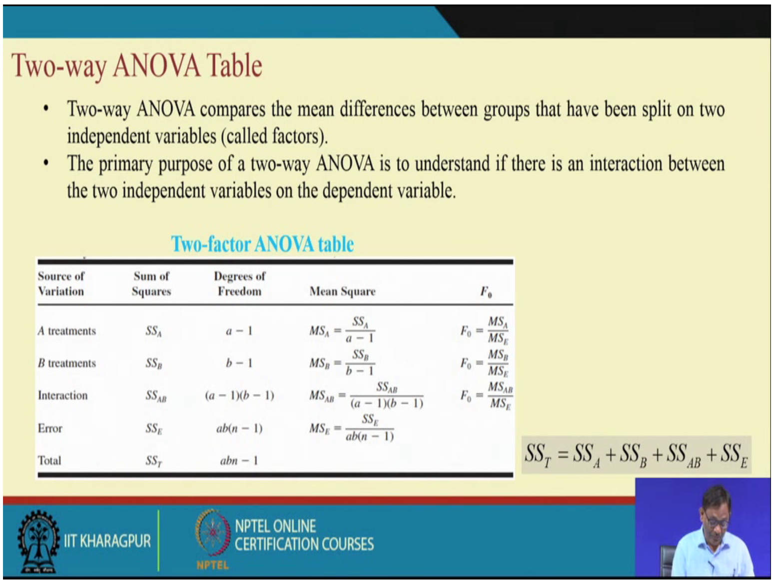

So, here what happened ultimately you will find out that the in earlier case like this that when we talk about the one way ANOVA we say that this if; what is N here total number of observation here will be ab into n. So, the degree of freedom here will be N minus 1, and this will be a minus 1, this will be B minus 1 plus this will be a minus 1 into B minus 1, and this last one will be your ab into n minus 1. So, the degree of freedom is also partitioned and if you add all those things this is nothing but ab N minus 1 because n equal to ab n. So, other way this can be written like abn minus 1.

From here you can develop the ANOVA table, see the table here. So, if you see the table sources of variation A treatment B treatment interaction error SS A, SS B, SS AB, SS E, SS T; these are the degrees of freedom. Compute MS A, MS B, MS AB, MS E. And then what is this? The SS A by their respective degree of freedom; so, what do you want to test whether A treatment effect is there or not, you create a statistics value for A that is MS A by MS E. Similarly we want to test whether B treatment effects are there or not, so compute F 0 for B MS B by MS E. Similarly you want to know the interaction effects are there or not, so compute F 0 for MS AB by MS E.

So then, all those things will be tested using appropriate F statistics. So, what will be the degree of freedom for this first F 0? That F statistics there will be numerator a minus 1 and denominator ab into N minus 1. The second one: b minus 1 ab into N minus 1. Third one: a of minus 1 B minus 1 into ab into N minus 1, that way.

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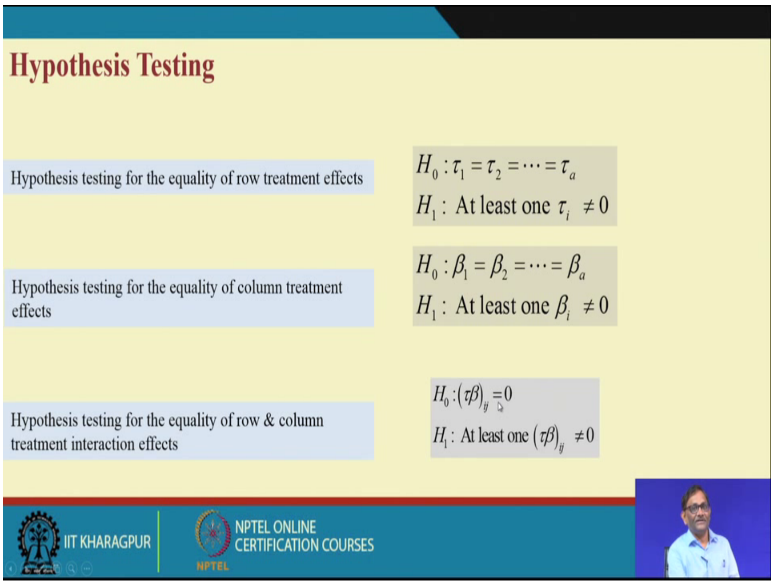

What are the hypothesis? You see the hypothesis, hypothesis testing for equality of row treatment effects; that mean all treatment are equal here, I mean at least one treatment. So, you just write down this one that tau i equal to 0 and this is basically mu 1 equal to;

mu related to mu 1 to mu a. This is similarly beta j equal to 0, at least one of the beta j not equal to 0 and this one is like this. So, I am writing this one carefully.

What happened here? When I am talking about A treatment, so I have mu 1 to mu 2 to mu a levels or H 0 I am creating like this tau i equal to 0; H 1 tau i not equal to 0 or other way mu 1 equal to mu 2 like the other one. Similarly for beta B you are creating that beta j equal to 0 beta j not equal to 0. Similarly for AB what your H 0, this is H 0 and H 1 and this H 0 is tau beta ij equal to 0 and H 1 tau beta ij not equal 0. Obviously for these, these, and this case at least one of the treatment or interaction is not 0. That is what is the test.

So, now you see the computation part.

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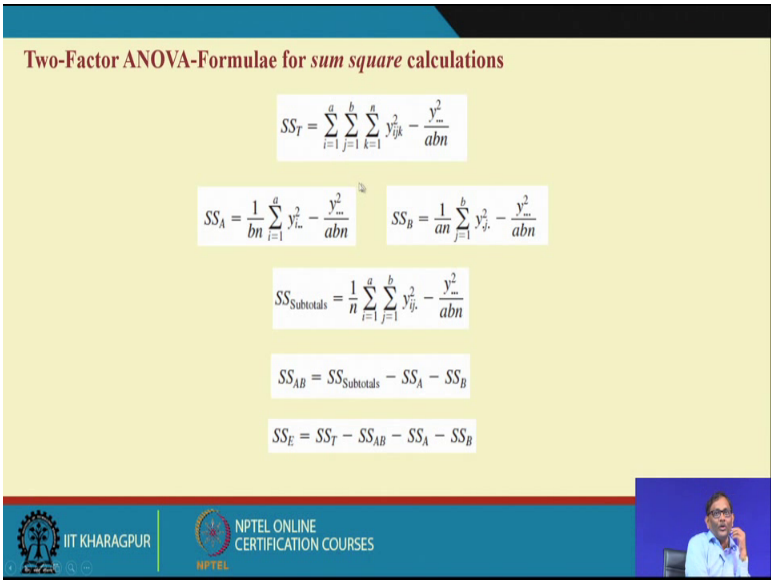

You calculate how do compute manually or at a eg computation for equal samples are SS T here sum total of all the observations squares minus grand total square by abn. For SS A sum total of row total squares divided by bn, and in my end of the correction factor we subtracted, similarly SS B. Now another concept here is SS B SS subtotals; SS subtotal means the cell k for every cell we are interested in.

In the cell means the every observation is y ij and k is changing. So, k is 1 to n not k ij. Now if you make for every cell the total then this is basically ij dot; that square when you sum up across all the cells you will be getting this quantity and it will be divided by n and subtracted by the substitution factor you will get subtotal.

Then what is SS AB? SS AB is subtotal minus SS A SS B something like this. So, let me repeat the calculation.

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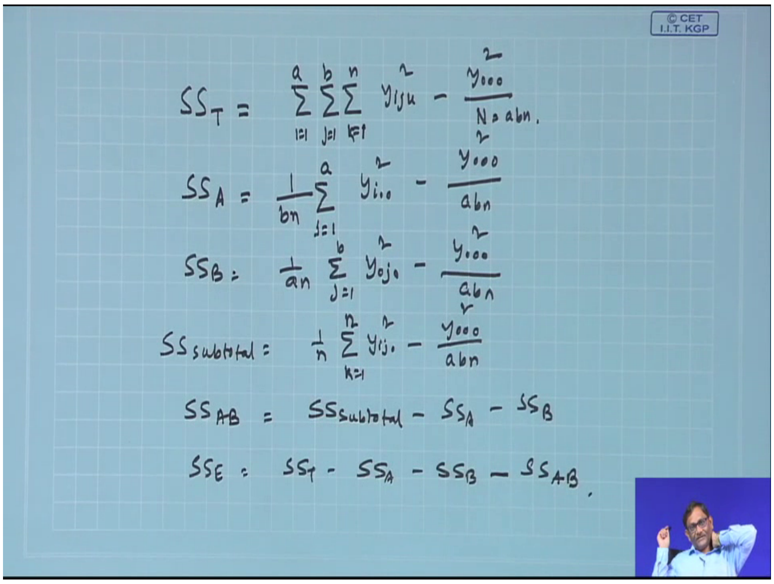

The calculation is SS total equal to three sums i equal to suppose you write i equal to 1 to a, j equal to 1 to b, k equal to 1 to n, then y ijk square you have seen this one. Minus this is the grand total square by N; N equal to abn. Suppose you want to know SS A, what do you require? You require I equal to 1 to a, then y i dot dot square minus y dot dot dot square by abn, but here you have to divide it by 1 by B n.

Similarly, SS B you will find out 1 by a n j equal to 1 to b y dot j dot square minus y dot dot dot square by abn. Everywhere you see y grand total square by abn is subtracted. Then you are creating SS subtotal which is basically 1 by n sum total y ij dot square k equal to 1 to n minus y dot dot dot square by abn. Then you are creating find out interaction which is SS subtotal minus SS A minus SS B.

Then you find out SS E which is SS T minus SS A minus SS B minus SS AB. This is the computation part for sum squares. And this you will be using ANOVA table. Now see the table SS A, SS B, SS AB, SS E, SS T all those the computation I have shown and now you will be able to find out this.

So, let us see one example.

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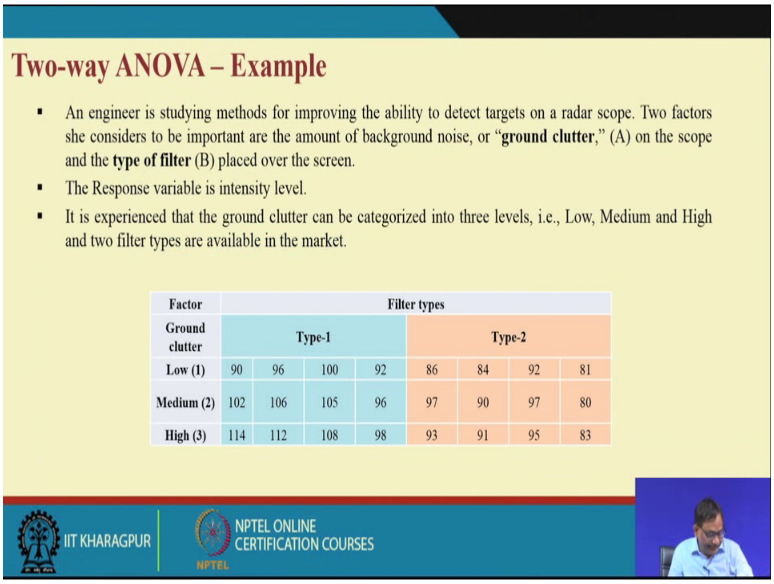

Now example is this. Now, what happened we introduced that filter type; please remember I told in the beginning that these data; data this is experimental data is taken from one of the example of montgomery's book. And here what happened not only this ground clutter and filter type, but there is another one operated different kind of operator. So, it is a 2 factor with a blocking that experiment was done and the same data he is taken in. I, first what happened we ignored the filter types an operator in one way ANOVA case and we found there is no effect.

Now, when two-way effect is coming that the operator effect is ignored. And we are assuming that thus this type of data we will get even if operator effect we will not consider. So, as a result this is the type 1 and type 2 filter type; this we have data points and here we have data points for different on clutter level.

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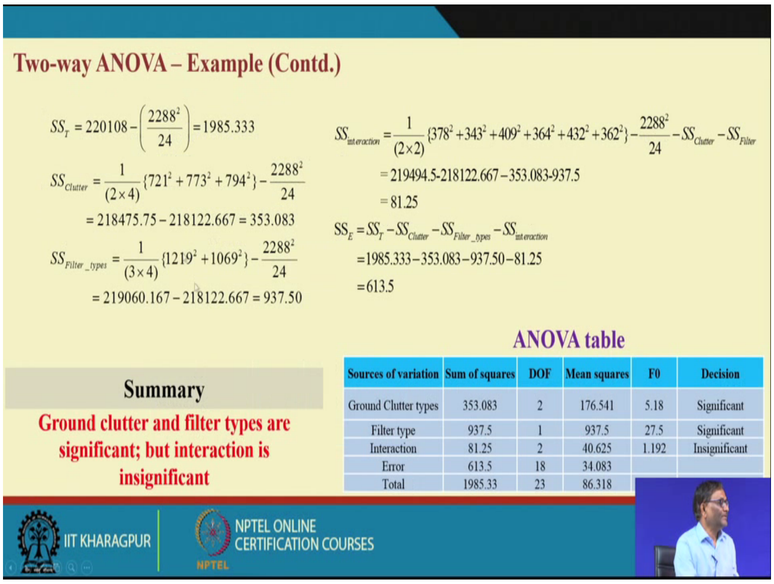

And what we have done basically, we have computed all SS T is a squatter, SS filter type, SS interactions using the formula I have already given to you. What are the formula? Let us see this formula; these are the formulas; so what I have done. So these are the SS T, SS A, SS B, SS subtotal these are the formula. Using this formula you have found out a SS T for that example and similarly up to a SS E. Come to the slides now, what happened SS T value is 1985.3, this is clutter value is this, this is filter type is this like this. And then, then using ANOVA we found out the F 0 values 5.18 to determine these.

Now, you know that this 5.18 value is little higher value. Now when we compare with the tabulated value considering the respective degrees of freedom 2 and 18 and that value is less than this for alpha equal to 0.05. And obviously then, they 1 and 18 also less than this 27.5; so fill the clutter type and filter type becomes significant.

That means, there is difference in the response value variable value which is here in the; that is the intensity level at the time of detection they differ if the ground clutter level changes from the low to medium to high. We do not know whether it is low medium high (Refer Time: 34:12), but there is a at least if you change from low to medium or low to high or medium to high in one of the case there will be difference. That is also true for filter type; that means, if you choose use filter type 1 the intensity at level at target detection will be different if you use filter type 2. But in one factor case when we have considered the same data with only ground clutter and as if there is no other factors affecting and the data what we have got actually same if I do that one factor level experiment. There then we have analyzed and we found that there is no significant difference for different ground clutter level when he introduced the type 1 and type 2 that filtered type then there is a difference. So, this is the beauty of introducing more factors.

And another interesting thing is that the interaction between the ground clutter and filter type is insignificant. There is no significant, means there is no dependency between these 2 factors. So, fantastic; so what we have discussed then in this lecture? In this lecture we have I conclude them all two ANOVA part.

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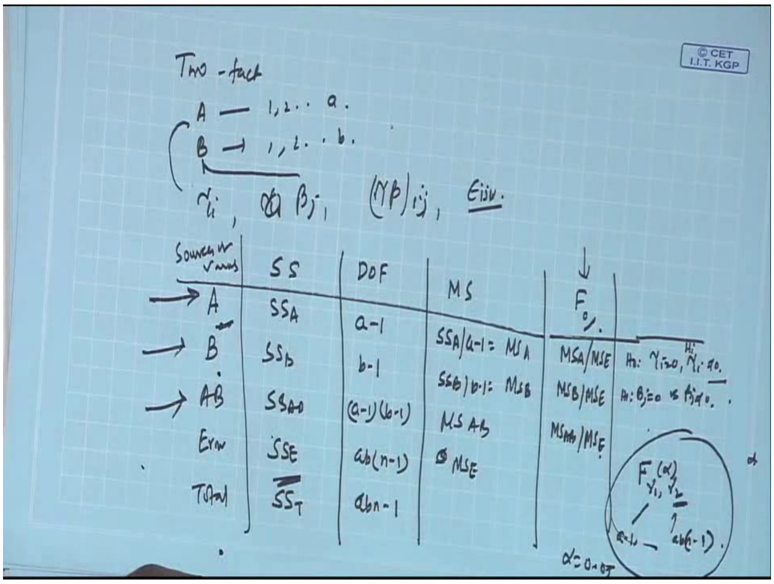

Now that in two-way ANOVA there will be two factors: there will be factor A with 1, 2, a level; factor B 1, 2, b level; and the model will have tau i that is for factor A there will be the treatment tau beta j that will be for factor B and there will be their interaction tau beta ij and obviously there will be error epsilon ij k.

So, there are how many sources of variability? Sources of variability AB interaction and error and finally total. This is basically sources of variability. Then we have computed SS, so that is for total it is SS T, for error it is SS E, for interaction it is SS AB, for factor B and for factor A like this. Then you compute a degree of freedom a minus 1 for A, b minus 1 for B, a minus 1 into b minus 1 for AB, then your ab into n minus 1 for error, and abn minus 1 for total.

Then what you have done? You have computed MS. MS is nothing but SS A by a minus 1 this is for MS A, similarly MS B will be SS B by b minus 1; so MS B. That means, SS by its degree of freedom. And similarly you will calculate MS AB, you will come calculate MS E, you do not require to compute as nothing not required for other things.

So, then you create an F statistics or F 0; that means, when hypothesis is true then this is nothing but MS A for treatment A it is MS A by MS E; for treatment B it is MS B by MS E; for treatment AB it is MS AB by MS E. And what you are doing you are testing hypothesis like this: H 0 in this case that tau i equal to 0 versus tau i H 1 tau i not equal to 0 for at least one i.

Similarly, here H 0 beta j equal to 0 versus beta j not equal to 0 for 1 and like this, and similarly your AB also interaction. Then what happened? There will be your threshold value which is F alpha numerator degree of freedom and denominator degree of freedom. For the first case factor A case this one is a minus nu 1 is a minus 1 and nu 2 will be always same because this is the arrow degrees of freedom. So, this is ab n minus 1.

Now if any of the computed statistics say F 0 value is greater than this value F alpha this; usually alpha we will considered at 0.05, then if this is the case then that corresponding treatment effect is there or interaction if it is there. If this is less than this the effect is not there. And with an example we have shown; the radarscope example we have shown that the ground clutter and it will filter type FFT is there, but interesting effect is not there. So, thank you very much for your patient hearing.

Thanks a lot.