Course on Decision Modeling
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Mod08 Lecture39
Shortest Path Problems

So today we are going to begin on the network models, shortest path problems, right.

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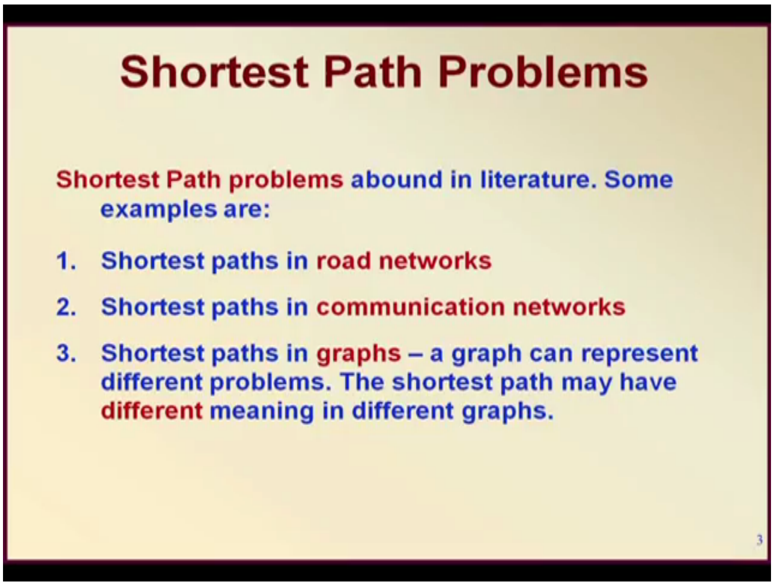

So shortest path problems are everywhere and you can see that you can get such problems in road networks, in communication networks, in different sort of graphs, the essential idea is that if we have a network the what is the shortest possible path from one node to another node and you can see, it is not always in the path has to mean the path alone, supposing we have a network and you know you have to transfer some material from let us say, 1 point to another point, right and what is the minimum possible cost, you know in this case, the path here, basically is represented by cost, so we are not really taking distance as a weight, but we take the cost as a weight.

So you know cost if you just replace in a distance by cost then that particular path which is the shortest path would then represent the path with the least cost, right. So these kind of problems therefore shortest path problems are available in the in the literature in a large number and the essential idea that how do we solve these problem.

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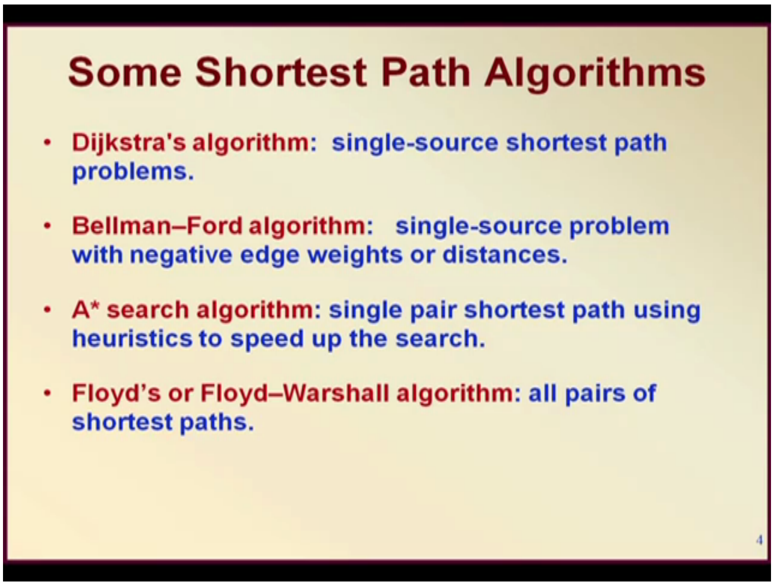

So there are different kinds of shortest path problems, algorithms there available, something the most popular one is known as the Dijkstras algorithm, you know it is a algorithm for a single source shortest path problem. there are certain situations where there are negative cost or negative distances, you know you may be thinking that how we can have negative distances or negative costs, particularly in when it comes to cost, sometimes what happens that if you go from a point a to point b, let us say you are going by know you are carrying your material by a truck and the truck is half full, so the remaining half you can you know give on higher to other carriers and you can charge some money and suppose, these charge is more than your original costs then you make a profit rather than there is a cost for going through this path. So in that sense for that particular point a to point b, the cost could be considered as negative. So what happens that Dijkstras algorithm does not work, if any of the distances are negative right for then there is a separate algorithm, you can call it the Bellman-Ford algorithm.

There are other kind of algorithm like A star search algorithm. These algorithms depend on heuristics, the certain rules through which the search is speed up. So kind of random search technique. So we are not going to discuss such kind of algorithms, but we shall discuss, the last one that is call the Floyds or Floyed-Warshall algorithm essential idea is to find out the shortest path from all pairs of you know the nodes. So not just from a to b, but from a to be, a to c, a to d, b to c and all such combinations we can find shortest path through the Floyds algorithm. It is a metrics based approach, right.

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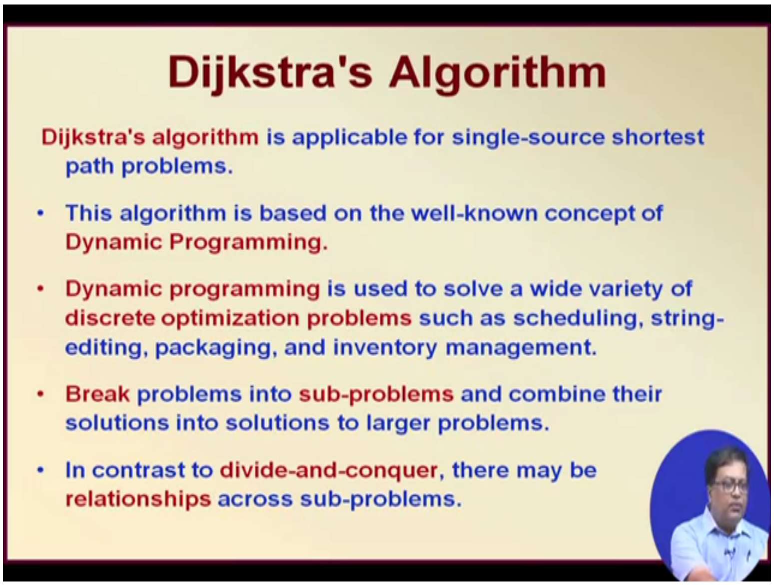

Now you see the Dijkstras algorithm, which will be our main topic, essentially depends on a particular principle that is called dynamic programming, right. What is dynamic programming? Dynamic programming is a method, you know by which you can solve optimization problems which as scheduling, string-editing, packing, inventory management and so on. The essential idea is that we have the total problem, break it into sub-problems right and through the sub-problems, use an optimality principle. What is that optimality principle? We are going to discuss next and through that optimality principle, if you apply in a judicious manner you can solve you know a large problem with the help of several such sub-problems.

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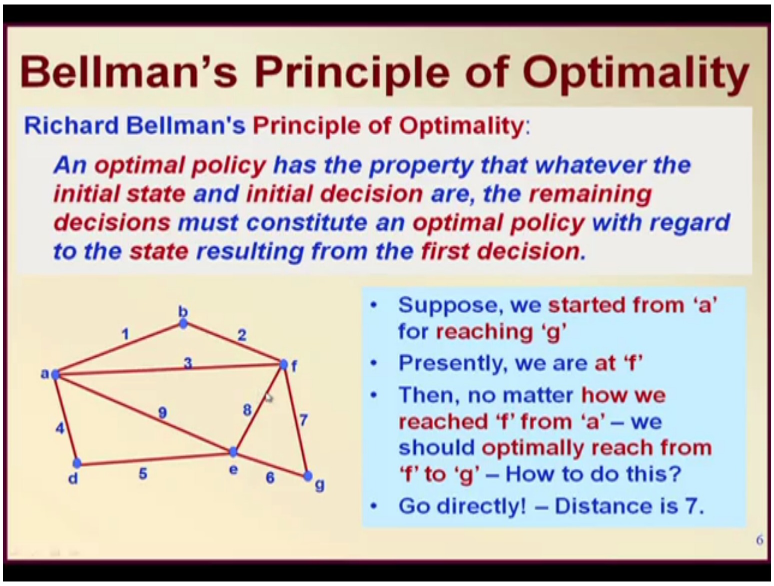

So there is a principle which is known as the Bellmans principle of optimality. What is Bellmans principle of optimality, let us read it first? An optimal policy has the property that whatever the initial state and initial decisions are the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. So you know what is exactly being said here.

Let us look at the diagram, so supposing you know we have these network and in this network we need to find out the shortest path from a to g. So we started a and we move to g, these are different cost and different paths or distances. Question is that how do you find out the a to g, right. Now you can divide into different sub-problems. So let us say that suppose we have already reached f, question is how did we reach f? Let us not question that at this point of time, having reached f irrespective of how we have reached f; you know we have to now find an optimal path from f to g. So the question is that if f is within the optimal path, you see f need not be within the optimal path, but assuming that f is an optimal path right. Question is having reached f what is the optimal way to reach g, it has to be found out that is what Bellmans principle says, right. Read once again an optimal policy has the property that whatever the initial state and decisions are remaining decisions must constitute an optimal policy. So having reached f we should optimally reach g, right.

Now let us invert the problem you see we have to go from a to g and we have reached f already, instead of starting from a let us start from g, the problem remains the same, we have to go from a to g and we have reached f that was what we said earlier, but now same problem we have to go from a to g and let us begin at g. you see from g you can reach 2 points e and f is it all right? So outo0f e and f you see the distances 7 and 6. So it is very clear that e is nearer to g and e is a more you know what is called promising candidate for finding the optimal path.

So let us say we choose g and we choose e, right that means e is the nearest point, if you are going to reach g. Now you see g to e, the optimal decision is therefore 6, it does not means that overall optimal path a to g will also have g to e, it does not mean that, it only means that if you have to go from g to e then that distance should be 6. Now let us look at another problem that if we have to reach from d to g, you see if you have to reach from d to g, then from d to g the you know if e is involve then from e to g, we can just have 6 nothing else and e is already optimal point as far as g is concerned.

Now out of all the several paths that we can have from d to e, you know you take all possible paths, if 5 is the minimum then we have reason to belief that from a to d being 5, g to d you can optimally reach at 11 total cost or total distance. So this is the basic idea how we actually look into Bellmans principle of optimality. Is it all right? so we start from a node, in this case the dista destination node call g make it say permanent and then look at the most optimal nearby point make it permanent also. So from the permanent that we have made that is e try to find out the next point which is nearer to e. So look at f from g to f distance is 7, so the current level of f is 7, but we cannot reach by any other path to f, right at a distance which is the you know lower than 7. So the question is that what is the other point which has got a value less than 7, if there are none then we can make f permanent anyhow more of these as we discuss.

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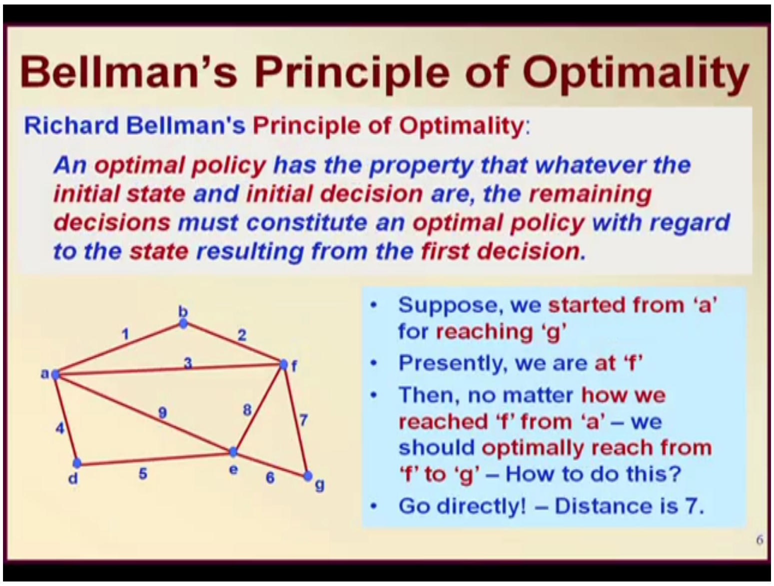

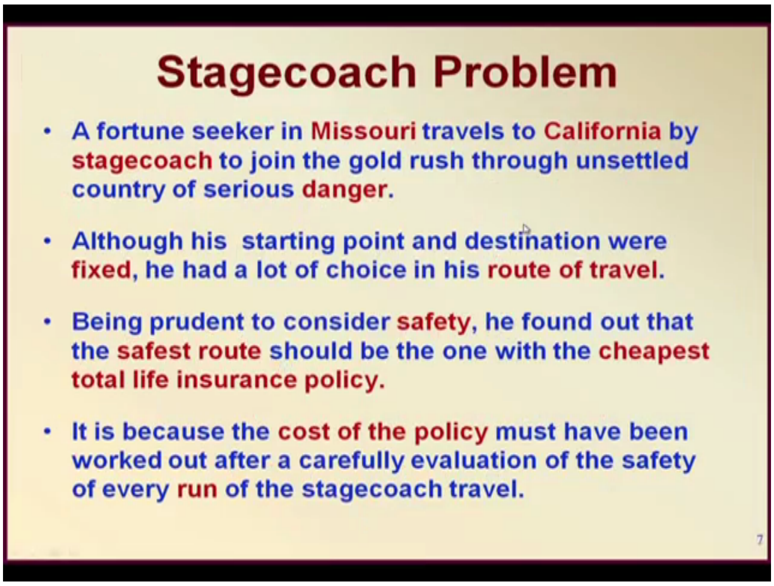

So let us look at you know one more problem and undressed how this Bellmans principle has originally been devised. So you know in the days when you know, there was gold rush (()) (11:08) and people in U.S. they all were trying to go through unsettled country and let say from Missouri to California and the idea was that what should be the stagecoach root. So what was done you know in a prudent way, it was decided that we go to the safest rot should be found out by the cheapest total life insurance policy. So what should be that I minimum cost path as reach from Missouri to California using stagecoach. So it was our original problem uh through which bellman has actually founded the optimality you know this principle, right.

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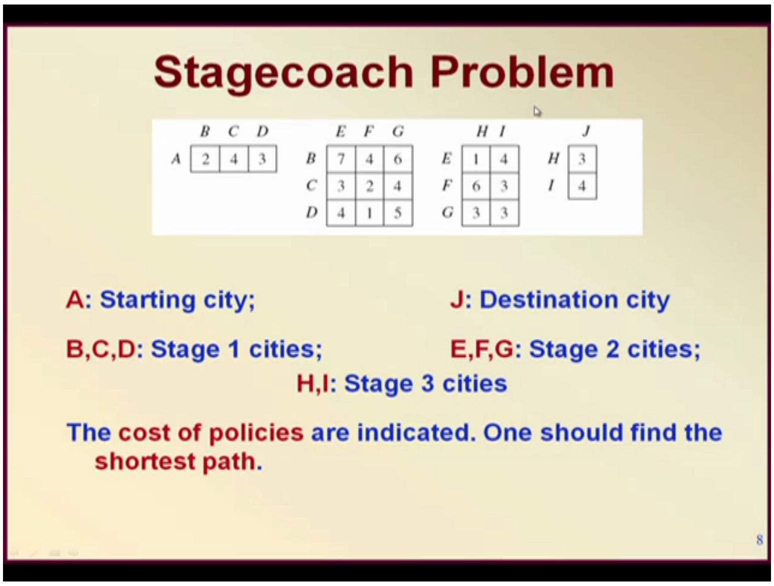

So now let us look at the stagecoach problem, the cities are divided into 4 stages, A is the starting city, then B,C,D, they are stage one cities, E,F,G stage two cities and H and I they are stage three cities.

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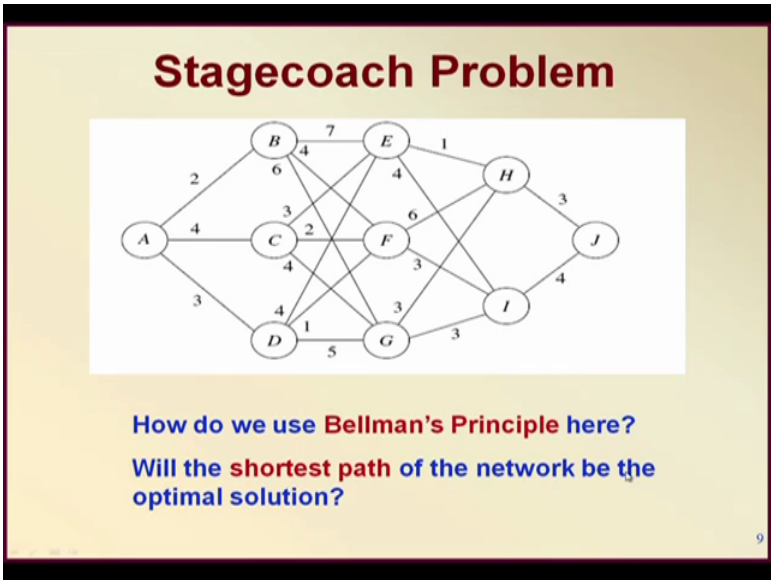

And how is the thing that looks. This is how it looks, right. We have to go from A to J. Now how do we use Bellmans principle here? How do we use very simpler that if we have to reach from A to J we and having reached let say H. how do we reach J? Now from H, there are several paths, right you can go by F to I to J or H is J directly. So what is the optimal possible path from H to J that is 3 only and 3 is a least distance from H to J out of all the points, which you can go from a point to J, in other words from J to H is 3, J to I is 4, J to E is say let us say

4, J to G is 7, J to F is let us say 7, etcetera 3 is the lowest. So H is you know least distance point from J. So in that sense we take J and H as our reference points and try to explore the rest of the network. How? Let us see, we will see very soon in Dijkstras algorithm, which actually uses the Bellmans principle. So will the shortest path of the network. Be the optimal solution. How do you know that that is going to be optimal, right?

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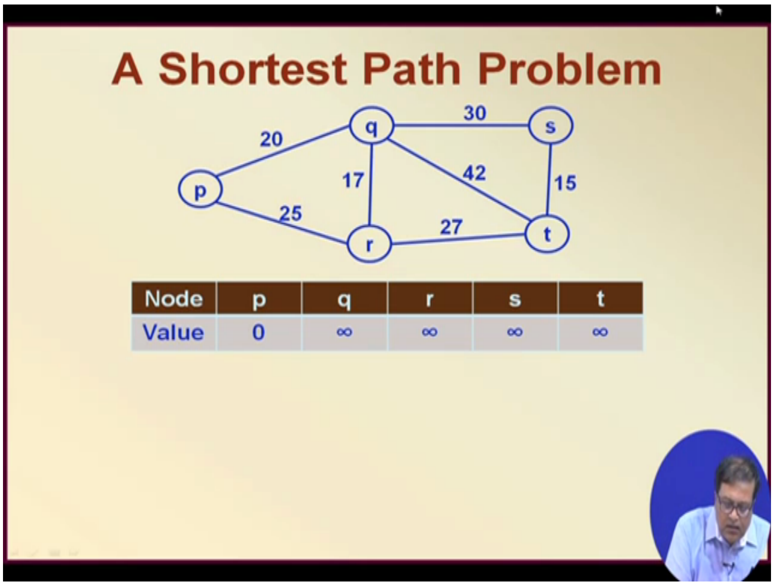

Now let us look at a particular shortest path problem, right. So we will do parallely also. Let us see, how this problem unfolds.

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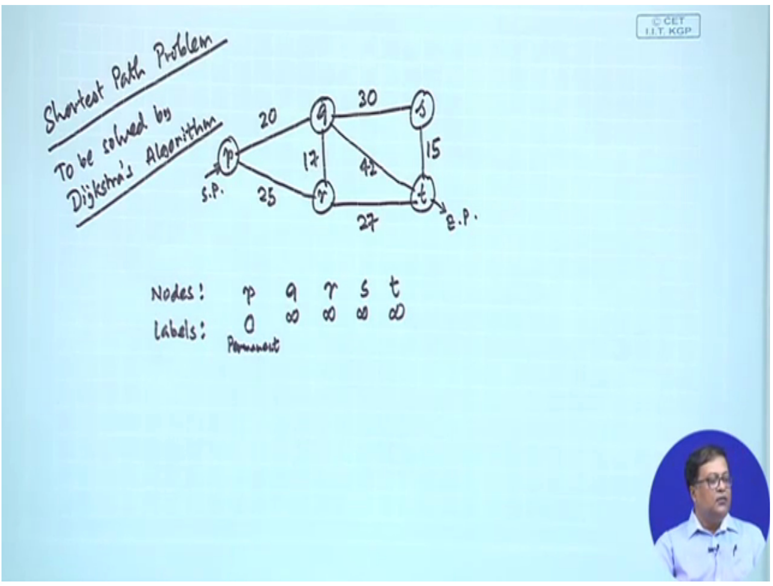

So let us say, we have 5 nodes p, q, t, r, s and t. So this is shortest path problem to be solved by Dijkstras algorithm, right. So from p to t we are moving in and distances are like this, p to q is 20, p to r is 25, q to r is 17, q to s is 30, q to t is 42, r to t is 27 and s to t is 15. So if you start from p you see, how we do initially we assign certain levels to this nodes. So these are our 5 nodes call it nodes and here, we put some levels, right. what are these levels I know at initially we put 0 to the starting point, this is a starting point, this is the ending point and we put these levels that means initially p to p distance is 0 and p to other points, the distances are infinity. So that is where we begin at this level and since, the p is the starting point and this distance is 0, so we make it permanent.

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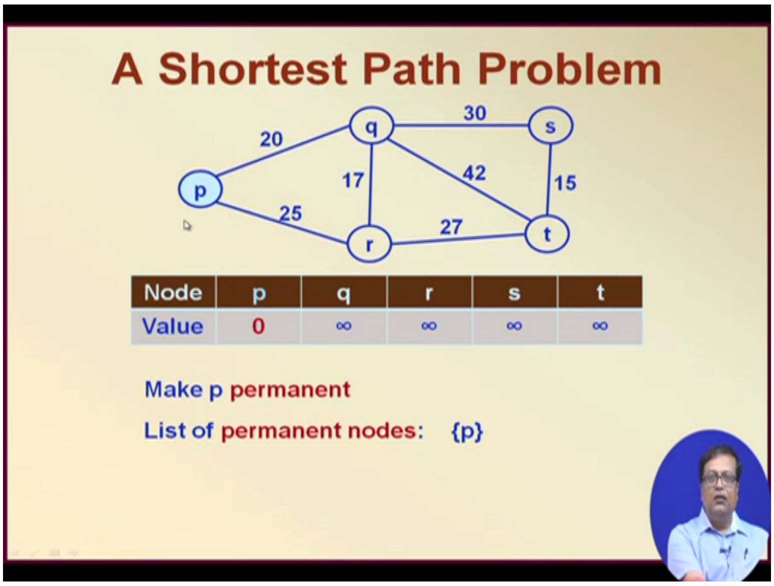

So look at the slide now exactly that is what we have done make p permanent, because p is the least distance point from p itself. So we highlight this particular value the level to 0 and p is permanent and these are the nodes which have been made permanent means the distance from these nodes to has already then finalized, right.

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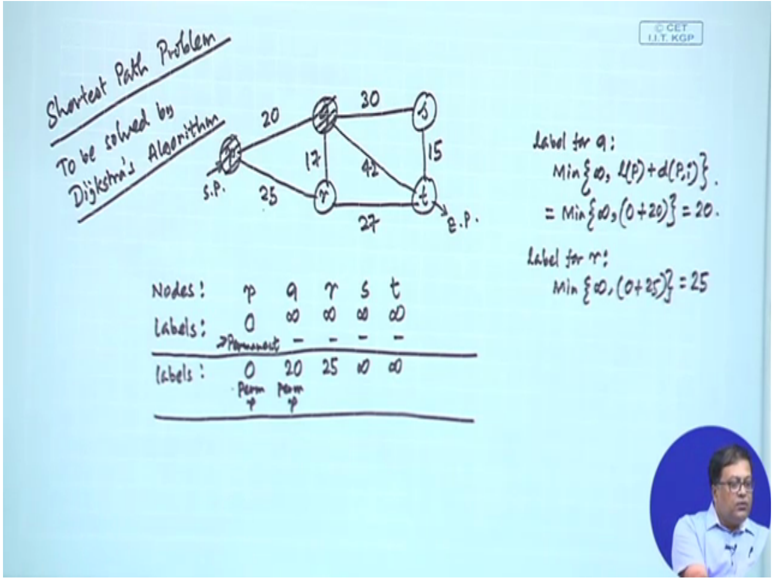

After p is made permanent you see, now for every point we are you see, this is one round is done. So we simply write p for permanent. Now at the second round, we see that out of the remaining 4 points what are some levels that we can now modify. So level for q will be now maximum of sorry, minimum of the current level that is infinity and the level of the last permanent mode right. What is that l(p) plus d(p, i) right. So you see, l(p) is the last permanent node level of last permanent node plus the distance from p to this point. So this would be minimum of infinity and zero plus 20, so this will be 20. What is the level for r? This will be again minimum of infinity that is a current one and zero plus distance that is 25. Why zero, because that is the level of the last permanent node. So if I do that then I get a value of 25. So at the next round we recomputed the levels as you know 0 any way this already available 20, 25, infinity infinity, right. So that is the next level at which we compute the levels.

Now you see p this is this was the last node made permanent. So this is already permanent. Now which one should be made permanent? Now you see, we have 20, 25, infinity and infinity as levels. The minimum amongs them should be made permanent. What does it mean? It means from p to q the least distance will be 20, right. It does not mean any other thing. So the only the permanent nodes will be available for us to understand. So now q is made permanent right. Why, because it is the minimum and made permanent from that also let us note. So this is made permanent from p, this is also made permanent from p, right. So made permanent from is also noted. So we have made p as permanent and also q as permanent. So these 2 nodes are made permanent, so they are highlighted.

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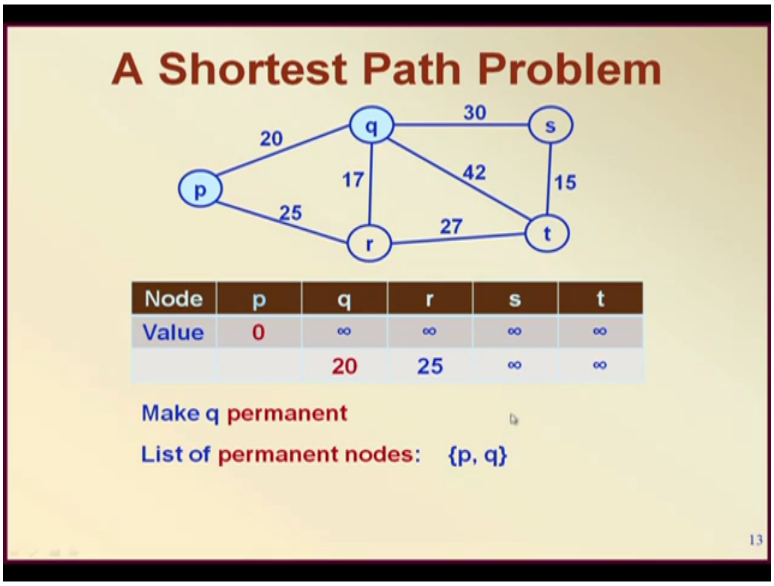

So look at the slides, now see these calculations are again shown for you that p was permanent, we compute the levels or the values for q and r for node q maximum of infinity and zero plus 20 that is 20, infinity and 0 plus 25 that is 25 and this values are obtained and now p is the last permanent node all right. So that last permanent values are taken here those distance is added and the minimum of those, sorry this is not max that should be minimum and minimum of that is computed all right. So now q is made permanent and least of permanent nodes are now p and q.

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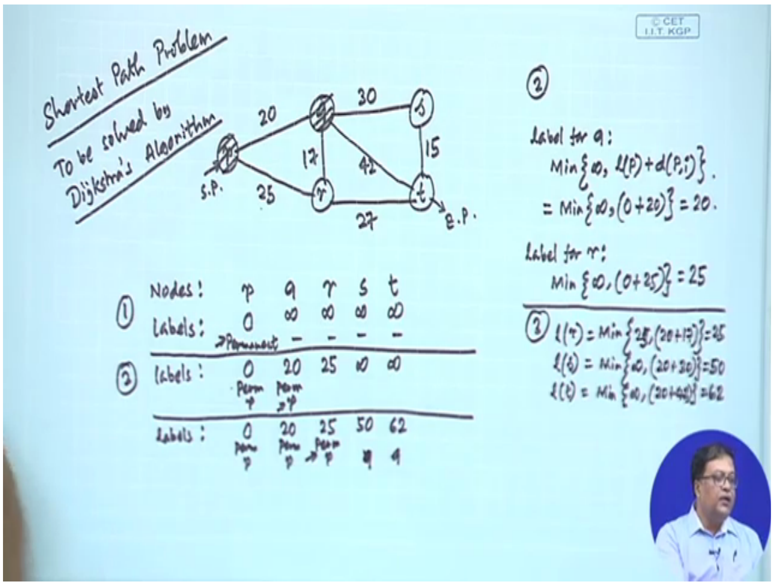

Now after that after we have found that is p and q as permanent. So once again q is the last permanent node, right. So we are here. So let us recomputed those levels once again, so if you recomputed those levels once again what should be our next levels. The next levels from q you see distance is there for r and s, no distance is there for t, is it all right? So you see what would be the new level for r, see this is already done. This is at say this is round one, this is round 2. So all these are round 2 calculations.

Now we are at round 3we. So round 3, what will be the level of r? It should be minimum of current level that is 20 sorry, 25 that is the level now and 20 plus that is a last node to be made permanent plus 17. So it should be now which one is minimum out of 25 and 37. It should be 25 itself. So no change happens as far as level of R is concerned. What about level of s? Level of s will now become minimum of current level of s is infinity and q that is last node it should be 20 plus 30. So it should be 50 and can we calculate t, t also we can calculate, level of t that should be minimum of infinity and 20 plus 42, so 62.

So we have already computed all these levels, so 0, 20, 25, 50 and 62. So which one is the minimum? You see these 2 are already made permanent. So you see, these 2 are we need not worry, because they are already made permanent and made permanent from p and 25 is also made from p, they are all made from p now. Now 50 is sorry, 50 is made from q and these are also q. So which one should be made permanent 25, because this is the least. So now this is the last node to have made permanent all right.

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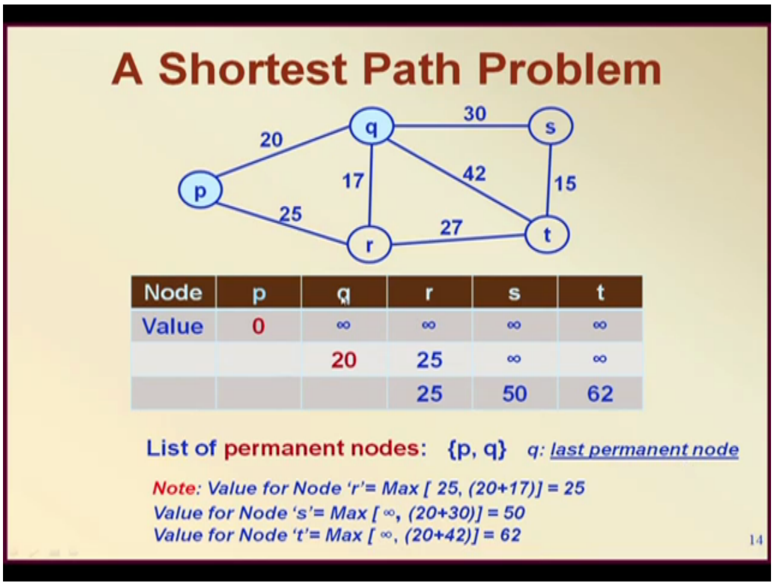

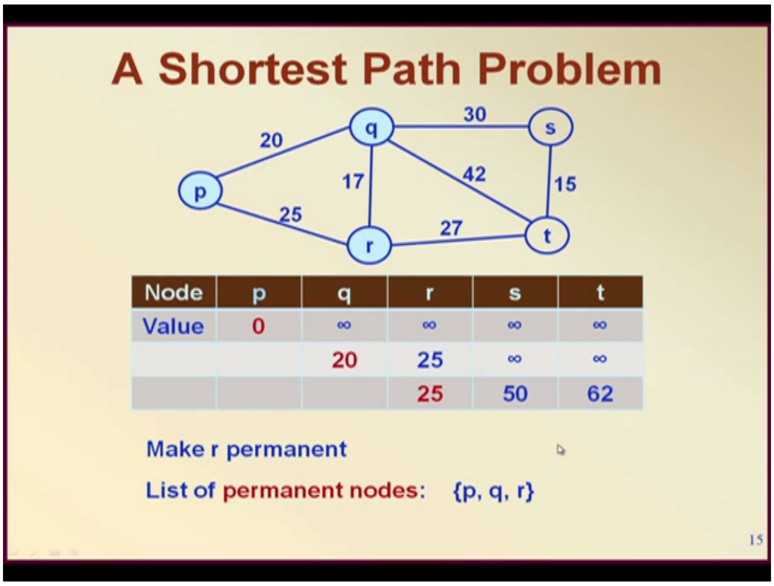

So now look at this slide once again. So p and q were permanent and we calculate all these values, q is the last permanent node and here we made r also as permanent. So as you made r as permanent and these are the values 25, 50 and 62, right.

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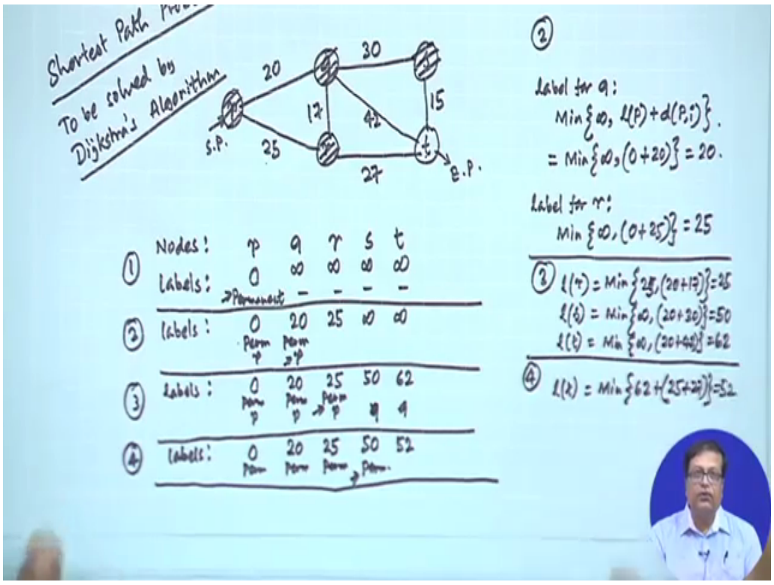

Now once you make r permanent, this is the level 3. Now we come to next level that is level 4. So we have to take the distances from r. So what are the levels? Now levels are 0, 20, 25 no change here. Can you change 50 s, because no path from r to s. So r to s no path no direct path, so obviously no change happens in s. what about t at this level that is at the 4th round you know l(t) equal to minimum of current level 62, last node to have made permanent is r, r is also made permanent. So 62 plus 25 that is a last node level r that is 25 and the distance that is 27. So which one is minimum out of 62 and 22 5 and 27, it becomes 52. So now we have l(t) coming out to be 52. So now which one should be made permanent very clear, then we have to make now 50 is lower, so that should be made permanent. So s is now made permanent and last node to have made permanent is this. So we have done level 4.

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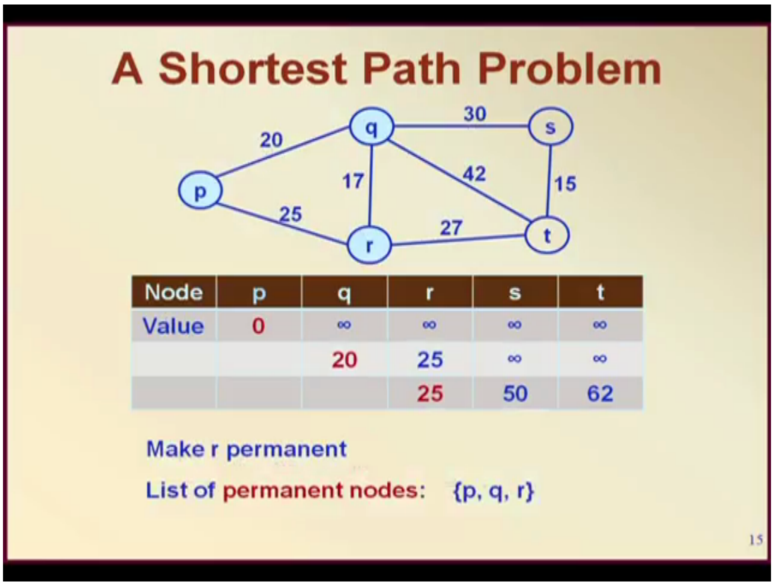

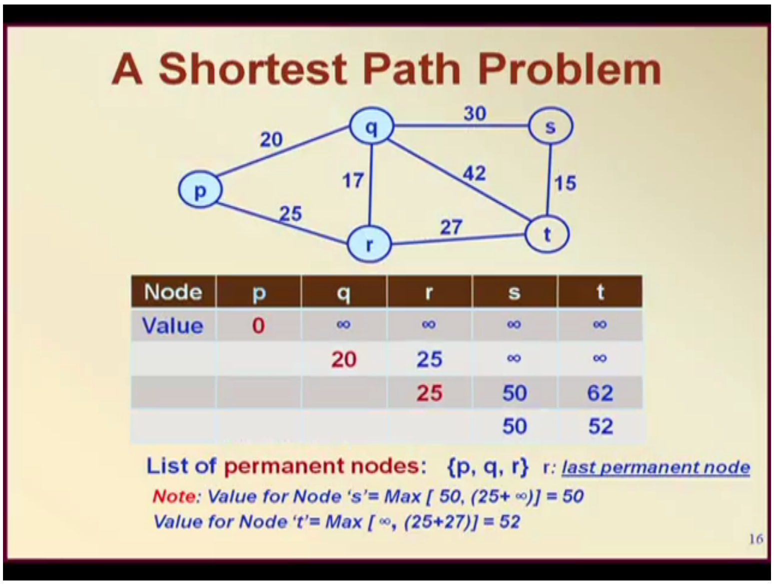

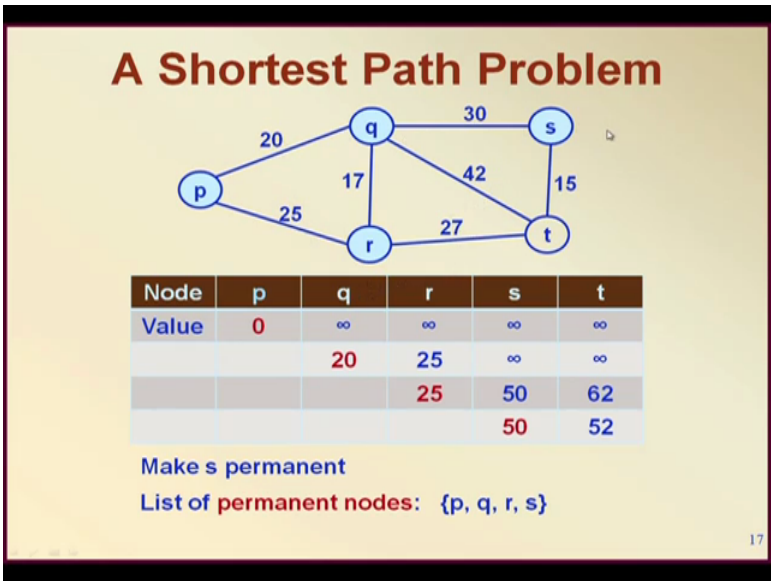

Now let us look at this also, see this computation and here you know s is now also made permanent and these are the 50 and 52. So s is now permanent.

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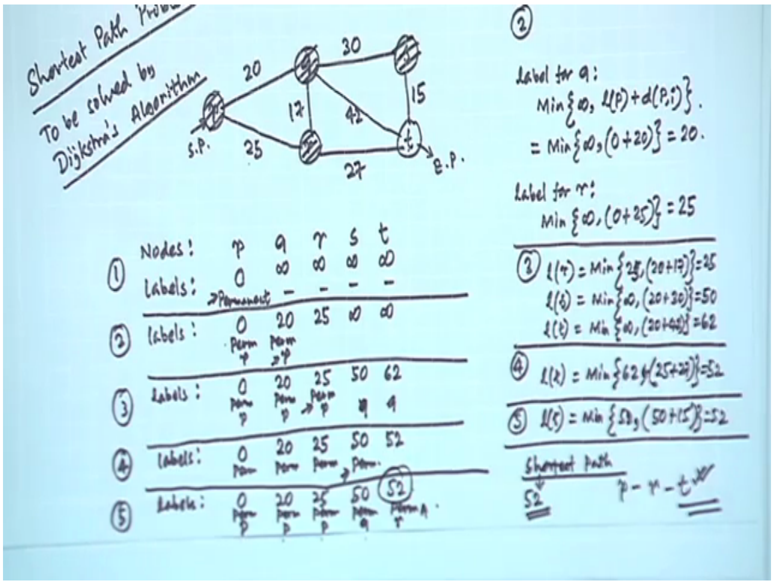

So now we comes to the 5th level, let us say what are the levels now? The levels are 0, 20, 25 and 50. So only change we can now compute l(t) once again, because what is the last node to have made permanent s, it should be minimum of your current value 52 and, then last node to have made permanent 50 plus the distance that is 15. So out of 52 and 65 which one is minimum? 52 so no changes, so we have 52 itself and since all are other nodes are now permanent. So now t is also made permanent right. So t is permanent and then our problem is solved.

And we have now look here then, this is made permanent from r, right 50 this is made permanent from q, 25 is made permanent from p, this is from p, this is from p. So shortest path if you have to find then you have to see, see t is made permanent from r. So we write t here and then write r, r is made permanent from p, so that solve. So this is a shortest path. So you see, so made permanent from p,p,p,q,r, these are going to give us an idea of what is the shortest path the p-r-t and what is the shortest path you know distance means 52. So look at this 52 fine. So that is about how we use Dijkstras algorithm to solve shortest path problem, right.

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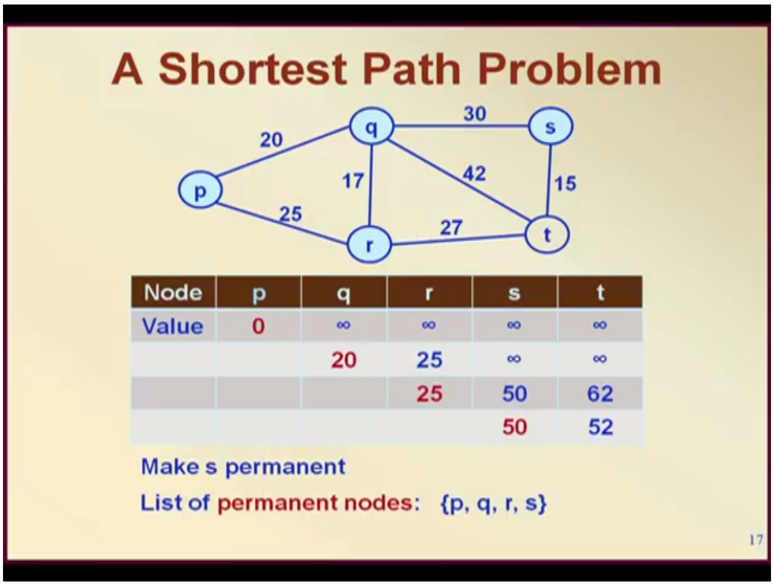

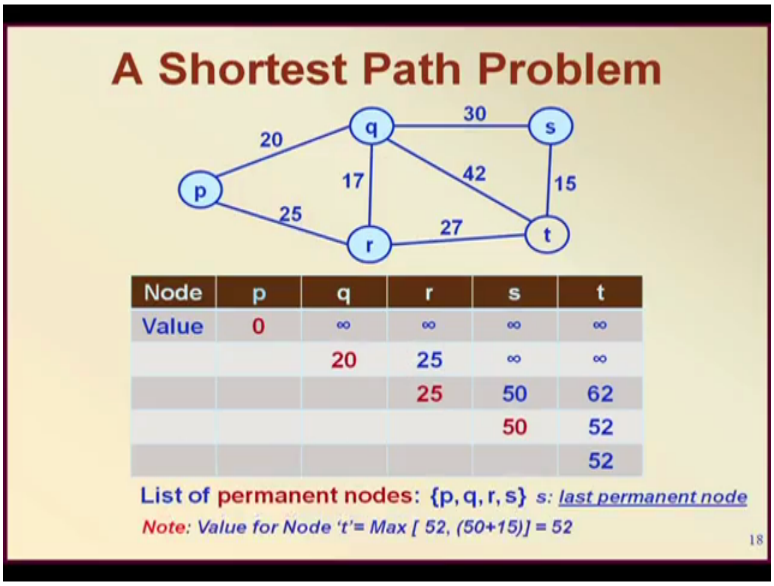

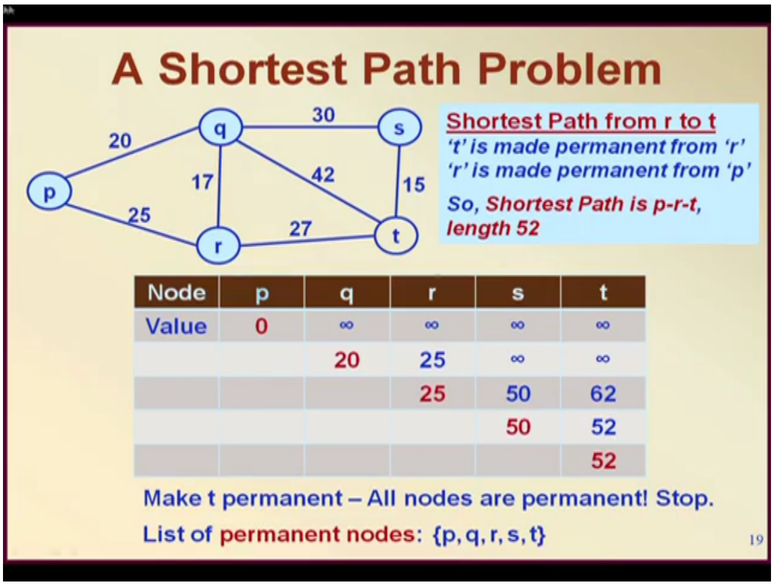

We will do one more problem but currently, we stop here and see this is our final solution that t is made permanent from r, r is made permanent from p, so shortest path is p-r-t and length is 52, right. So thank you very much.