Course on Decision Modeling
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Lecture No 31
Introduction to Graph Theory

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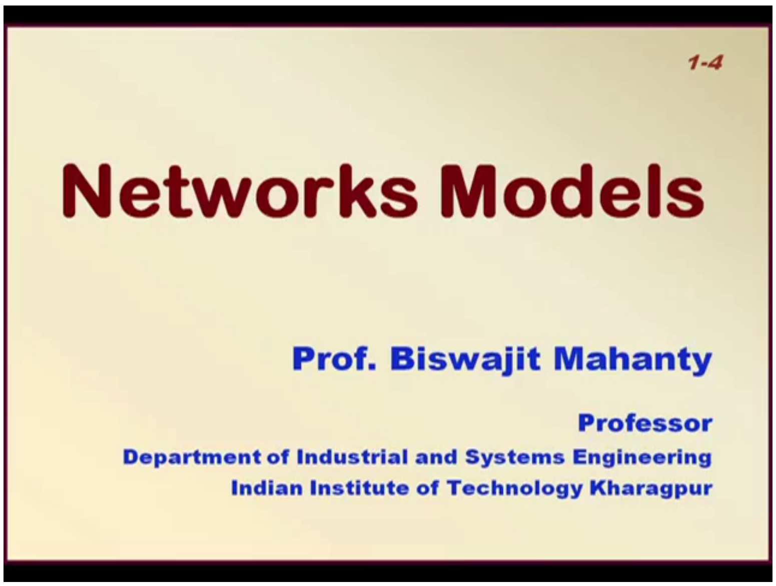

Today we are going to begin a new chapter that is on the network models and in this network models basically we are going to discuss first of all different graph theoretic concepts and after the graph theoretic concepts are discussed specifically what is a graph and different components of graph? And thereafter the tree and specifically the spanning tree concepts and finally after all this basic graph theoretic concepts and their uses you know we shall specifically take up 2 kinds of models, one is the flow model and the other one is the shortest path model, right?

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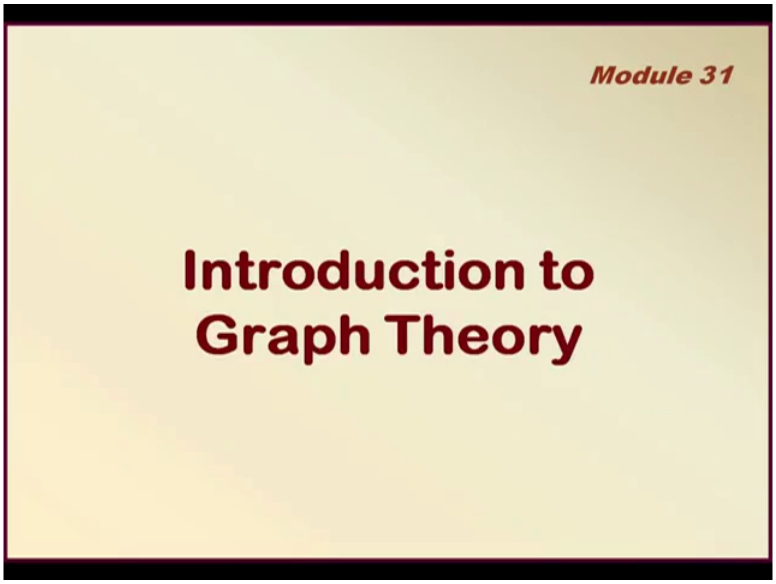

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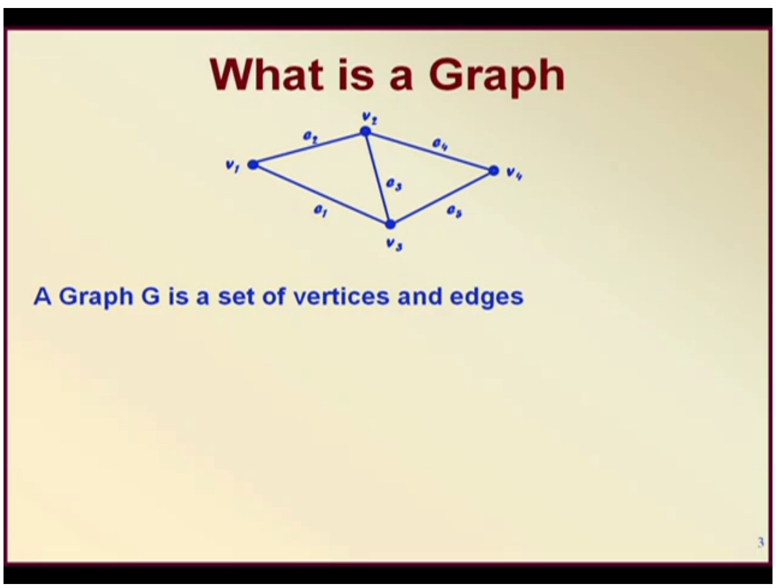

Now the basic graph theoretic concepts that is our first thing the introduction to graph theory which is basically used in the network models, so basically the first and foremost topic will be first of all, what a graph is, right? As you can see in this particular diagram you know graph G is a set of vertices and edges, right?

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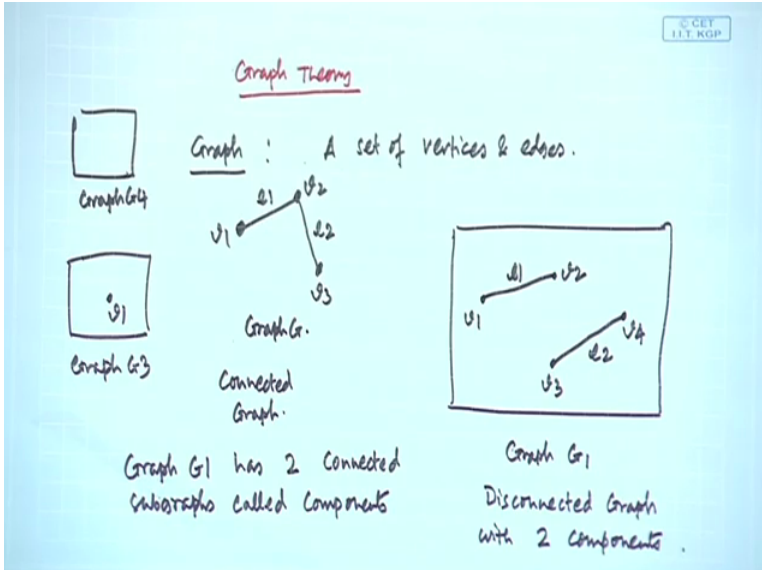

So if I look at what is a graph? So if I look at this essential concept of graph theory then we say that a graph is a set of vertices and edges. So this is a vertex and this is an age, here is another vertex, so V1, V2 and this is an edge e1. Let's take another vertex v3 then we have another edge e2. So is this a graph? Answer is yes. Maybe let us call this as graph G. Now within this space let us draw another graph, suppose I have one, 2, 3, 4 vertices I connect these 2 and I connect these 2. So I have V1, V2, V3, V4, e1, e2 look at the whole thing, is it a graph? The answer is yes. Yes this is a graph.

So this is a graph and this is also a graph, so some of you might be thinking that you know graph has to be what is known as connected all the time. So this is a graph definitely but not connected, right? So it's a disconnected graph, a disconnect graph has got components, right? So this is a corrected graph and this is a disconnected graph with 2 components, each component is a connected graph.

So in a sense, look here this graph has got 2 subgraphs. Graph G1 has 2 connected subgraphs called components, graph G1 has got 2 connected subgraphs called components. Now within these I will draw one more graph, graph G3 is it a graph? Just one vertex, the answer is yes it's a graph, right? It is a graph of a single vertex, it is not necessary that graphs must have edges not required.

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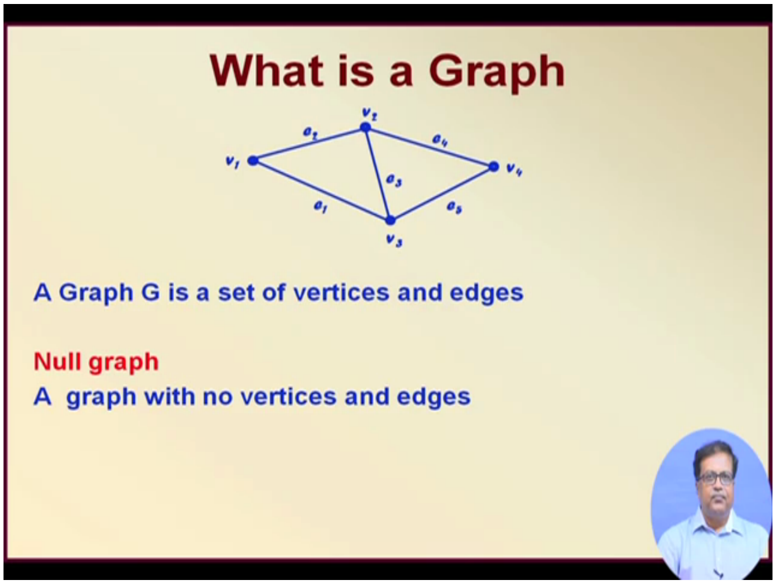

Now in an extreme case graph G4 I have drawn nothing, is it a graph? it's a very interesting question that it is also a graph in a sense and it is a null graph. So in the slide that is exactly what I was written, a null graph is a graph with no vertices and edges, that's called a null graph but what to do with a null graph? Really we have nothing to do with null graph, so therefore it is just a conceptual thing we will discuss about it.

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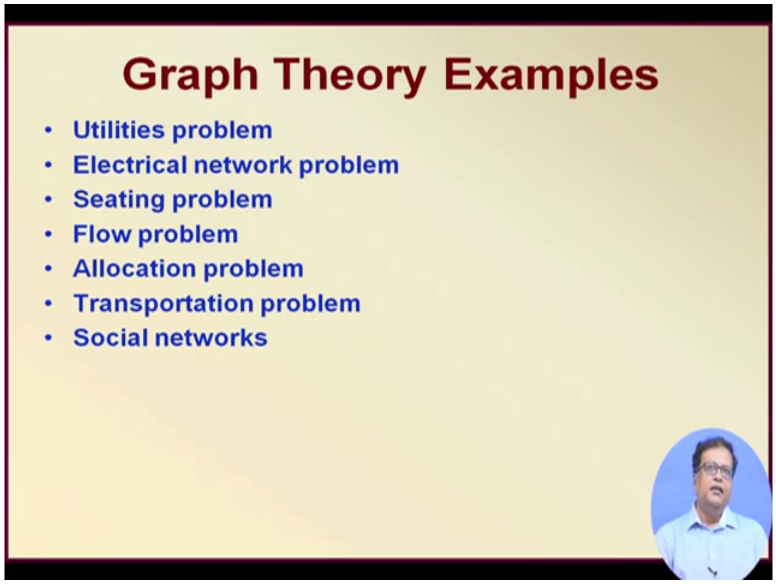

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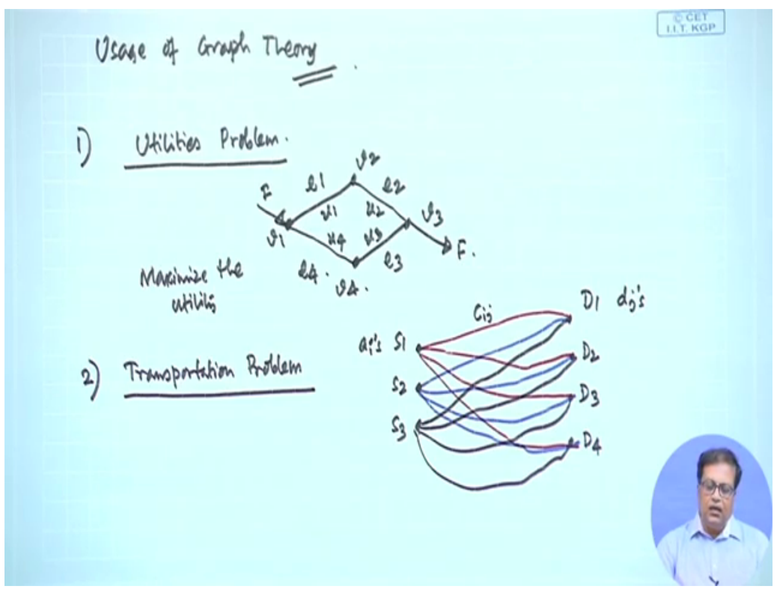

Now why graph theory? Why do we study graph theory? You know several problems, kind of problems that we can actually study by making use of graph theory. So let us look at those usage of graph theory, number1 say an utilities problem, see we have a first of all graph or sometimes a graph will be called a network, if you know we might think to have a flow between them, say these vertices are really representing, you know some junction points and these edges e1, e2, e3, e4 is some edges each in each edge. Suppose these may be a drainage line and in this drainage line you know on this several houses the draining system is connected, right?

So you have to maintain a certain amount of flow which comes here and say goes out from here. Now as the flow F goes the flow will be divided into this and it will pass through that, if it passes through this edge e1, suppose it has a utility u1, u2, u3 and u4. So a utility problem will really ask how I should assign the flow. So as to maximize my utility, as I you know put that flow that flow should actually increased by or maximize the utility.

The second example could be, let us say a transportation problem could be an example of a transportation problem. So you see there are several source, source 1, source 2, source 3 and let's say there are also several destinations, source 1 to several destinations and what we have to do? We have to send materials from source to the destinations and there are different parts that are available, right?

So I am drawing in different colors, so that it doesn't become to just oppose. So you know these are S1, S2 and then from S3 as well, right? So you see that from these 3 sources we have to send materials to these 4 destinations and each of these paths has got a certain amount of cost and at each of the sources there are some availability's and here there are some demands. So ai's and dj's. So each source has got certain available material and each destination has a demand.

So which paths are to be used to send this material and how much material should be flown through that? That is essential idea of a transportation problem. So you see perhaps you have solved those transportation problems through linear programming route but you can also have a network route to solve this problem. In fact you can also have you know a network-based linear programming method to solve this problem.

Apart from that there are several other kinds of problems like electrical network problem, the seating problem, the flow problem, the allocation problem. A very interesting problem is called the social network problem, right? A particular social network can be actually built with the help of maybe several entities, how are they related to each other? Suppose think of a share market.

In a share market there are several companies and they have based on the share market prices their values are changing with time. So if you calculate some kind of correlations between 2 companies how their prices are changing and if you find a certain amount of correlation then you draw a line means the behavior of these 2 companies are correlated and then we get a resulting graph, this graphic we can call the social network graph.

We can really analyze that social network graph and really look how dynamically it changes over time and we can predict a very interesting phenomenon about the share market. So like this there are several uses of graph theory, as I said our discussions will be basically based only on certain aspects to be more precise to the network flows and to the shortest path problem. Say I have not touched upon several other problems like by partition graphs, the coloring of graphs there are several very good applications that are available on the computer science literature these particular course network treatment that we are going to give is not meant for those kinds of students.

It is meant for those kinds of students who are interested in network flows and those were interested in shortest path algorithms specifically, you know in the linear programming or operation research literature, right? Having said that let us go into further discussions. Now once we have said that what is a graph? The next point that comes is you know what is a simple graph?

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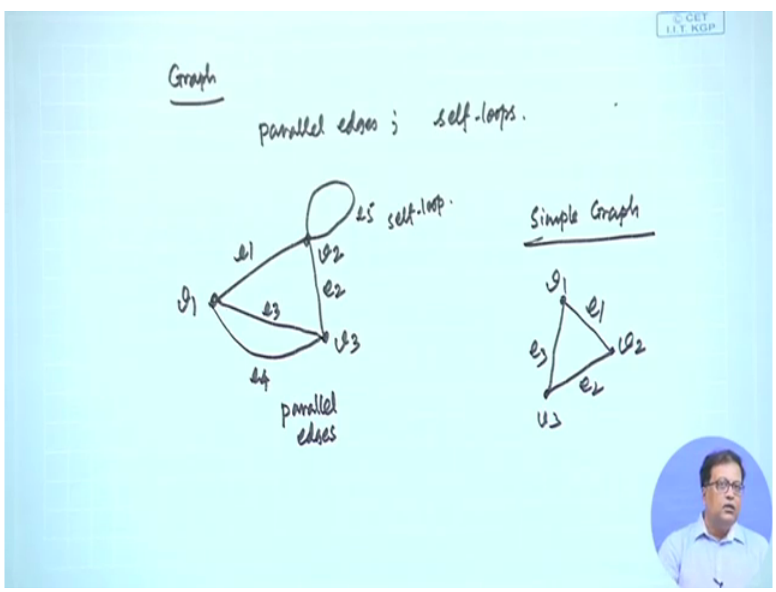

A simple graph is, see when we talk about the graphs then the graph is a vertices and edges that we have already seen but there could be additional things like that could be parallel edges and there could be self loops. So supposing we have these particular connections, so you see v1, v2, v3, e1, e2, e3, e4, e5. So look here, here is a graph this graph is having 5 edges and 3 vertices. Now can you see that there are some parallel edges, the edges are between same 2 vertices, so if the particular edge begins and ends, right?

Begins, these 2 edges begins at the same vertex and also ends at the same vertex, so they are called parallel edges. These particular edge e5 begins and ends at the same vertex, so this is called self loop. It is practically there are many situations where you can have parallel edges as well as self loops, right? So there are situations but there could be other situations where we have a graph that the not have parallel edges or self loops, those kinds of graphs are called simple graphs. So what is a simple graph? A simple graph is a graph that does not have parallel edges or self loops, so like this one is a simple graph, right? So this is a simple graph.

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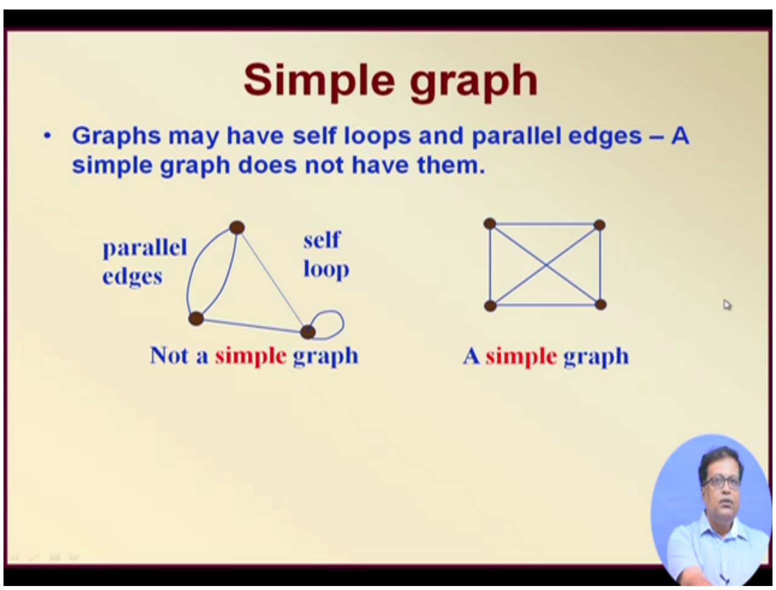

So here's slide also you can look at that this is a simple graph and this is not a simple graph because it has parallel edges and self loop, is it all right? That's a simple graph.

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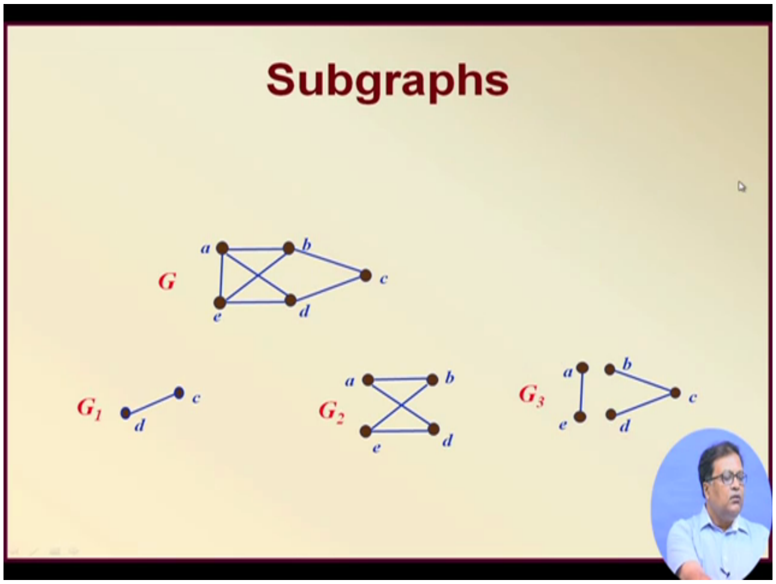

Now this is another example of subgraphs, right? So if you look at this particular graph G then you can see that you know it has got one component here which is G1 maybe this is another component G2 and this is also another subgraph because you know this and this. So there all of them are called subgraphs. So this is about subgraphs, that subgraph is a subset of the graph that means every vertex and edge of the subgraph has to be also available in a graph itself, if that happens then it is called subgraphs, right? So a subgraph is a subset of a graph.

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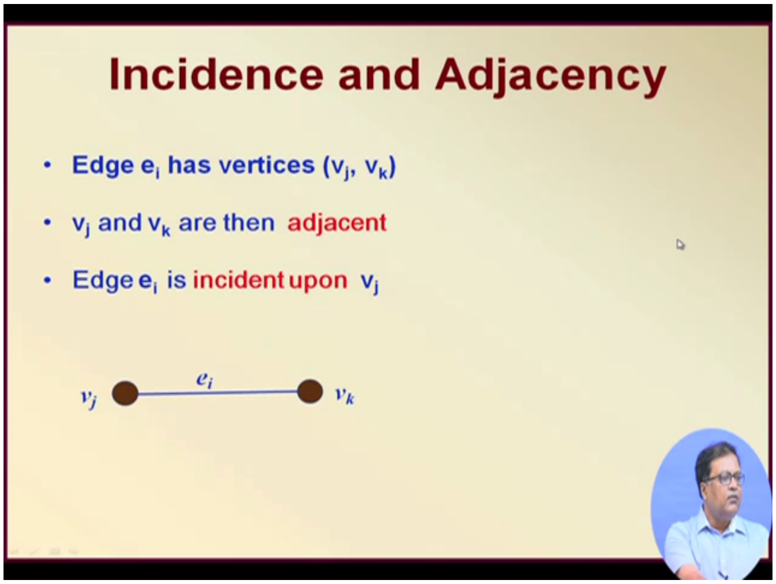

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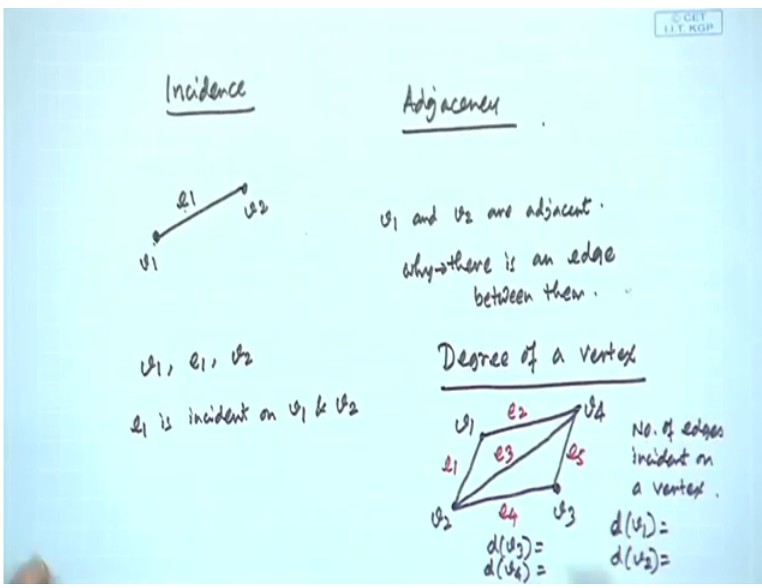

Now comes the next important point about the incidents and adjacency, so incidence, right? So what happens? We have a graph e1. So what happened, this edge e1 is an incident on the vertex v1 is it all right? And also these e1 and v2 are adjacent. So v1, e1 and v2 what is their relation? e1 is incident on v1 and v2, right? And otherwise we call that v1 and v2 are adjacent that's call adjacency, right?

So v1 and v2 are adjacent is, why? There is an edge between them, alright? So that is the incidence and adjacency. So e1 is incident on v1 and v2 then come a very important concept that is the degree of a vertex, right? So you see suppose we have a graph let us say this is a graph, so we have a graph of 4 vertices and I have not level the edges, let us let us level edges also, e1, e2, e3, e4 and e5.

So we have a graph of 4 vertices and 5 edges, so how many edges are incident on a given vertex? That is a measure of the degree of that particular vertex. So how many edges are incident on v1? You know number of edges incident on a vertex. So what are the degrees? So what is the degree of v1? What is the degree of v2? What is the degree of v3? And what is the degree of v4, right? So suppose we need this degree values to be obtained, how many edges are incident at V1? There are 2 edges, so this is 2 and v2 3 at v3 2 and at v4 3, so this is how we compute the degree of a vertex, right?

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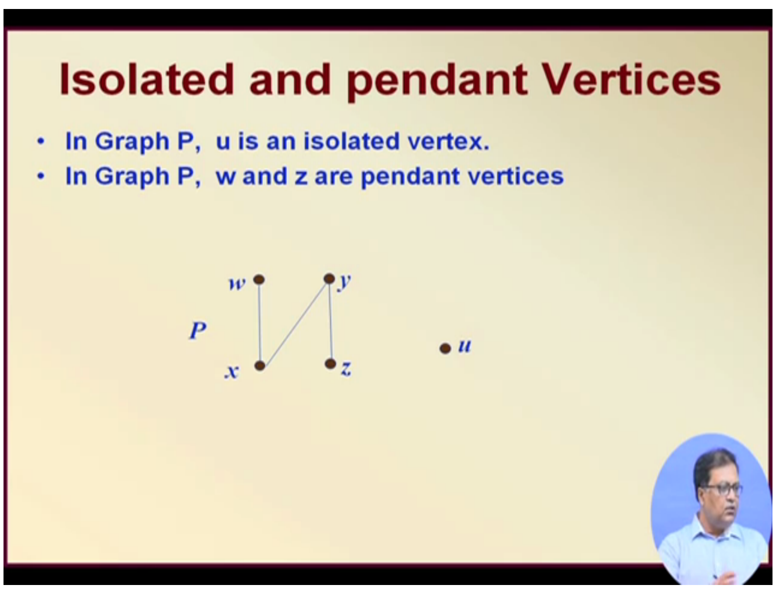

Now let's look at very important concept that is isolated and pendant vertices. If we look at this particular graph there are 2 graphs P I mean sorry one graph P you know in this P has got how many vertices? There are 5 vertices, what are they? U, w, x, y, z, is it all right? Now u is having no edge connected to that, so it's an isolated vertex and w and z their degree is equal to 1 that means they are extreme edges, right?

That means only one edge is connected, so they are called pendant vertices, right? So this point might be required at some point of time.

Now a very important concept that of isomorphism, you see the essential idea of isomorphic graphs you see we called to graphs as isomorphic if they have one to one incidence between their vertices and edges, so these particular example look at very carefully w, x, y , z, right? They are connected in a certain manner and here also a, b, c, d they are connected in the same manner although they don't look exactly t same if you just look at them.

In simple words w and x is connected, a and b are connected. So w corresponds to a and x corresponds to b, is it all right? Similarly x and y and b and c, x is connected to y and b is also connected to c. Similarly y is connected to z and c is connected to d. So these 2 graphs are isomorphous, right? That means although they look different but they have the same one-to-one relationship between their vertices and edges. So that means 2 isomorphous graphs not only they should have the same number of edges and vertices but the way they are connected will be exactly same and degree definitely the degree of the vertices should also be exactly same corresponding degrees.

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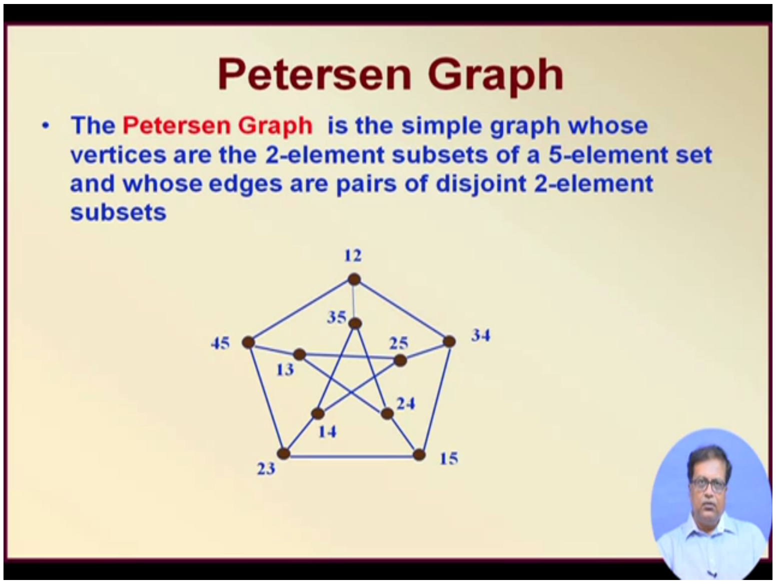

A particular type of graph is called the so-called Petersen graph. So you see a very interesting graph is this particular 1 the Petersen graph, what is really happening here in the Petersen graph if you really look at that, if you start from any vertex, so this is called a Petersen graph you know if you start from any vertex you want to reach any other vertex the distance of the path will be exactly 2 the path length will be exactly 2.

So supposing you are in this vertex you want to reach this vertex, just see you have simply to go like this. Suppose you have this vertex and this vertex, you have to go like this. You have this vertex and you want to reach this vertex you have to go like this. So you can see that each vertex is connected to every other vertex in a 2 connected manner, so that is the beauty of a Petersen's graph and you know with 10 you know with 10 vertices and how many edges?

Just count the edges 1, 2, 3, 4, 5 then 6, 7, 8, 9, 10, 11, 12, 13, 14 and 15. So with 10 vertices and 15 edges, it's a very interesting arrangement you know and if you can connect a graph with 10 vertices and 15 edges in this manner you can really find that everything is to connected with everyone else and is at least 2 connected. Obviously there are different vertices they may be just connected one connected you can say, is it all right? So that's the good thing about the Petersen's graph.

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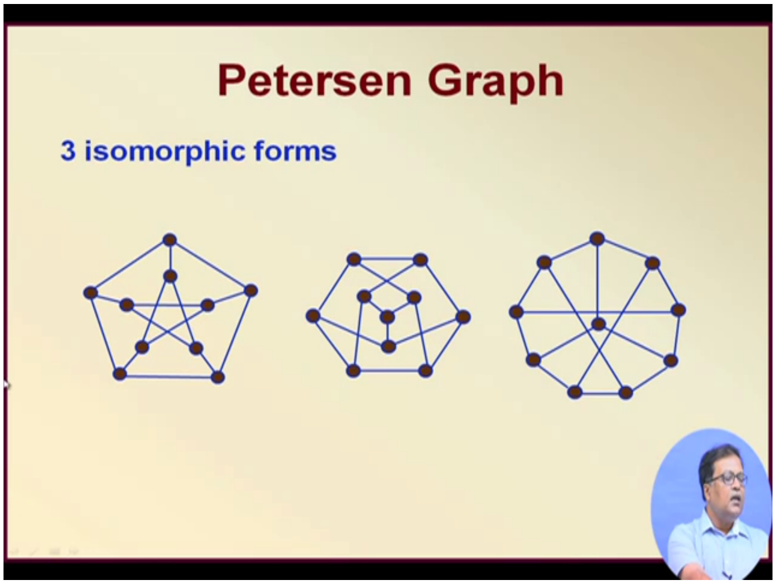

Petersen's graph also has 3 isomorphic forms, right? You can you can look at this slide again and again and you can really try to put those 10 vertices numbers the similar numbers like if you call it a, B, C, D, E then E, F, G, H, I try to put the same A, B, C, D, E, F, G, H, I, J, I mean sorry A, B, C, D, E, F, G H, I, J, so both the same numbers you know in these graphs also and see that they have the same incidence. In other words these 2 are also Petersen's graphs, right? They are isomorphic forms, so these isomorphic forms are sometimes good to really understand and identify, so that it helps us in analyzing systems in a simple manner.

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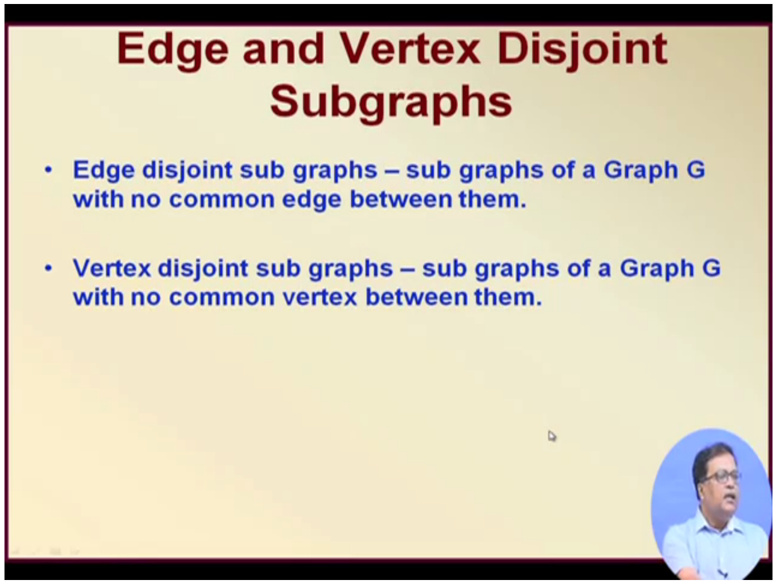

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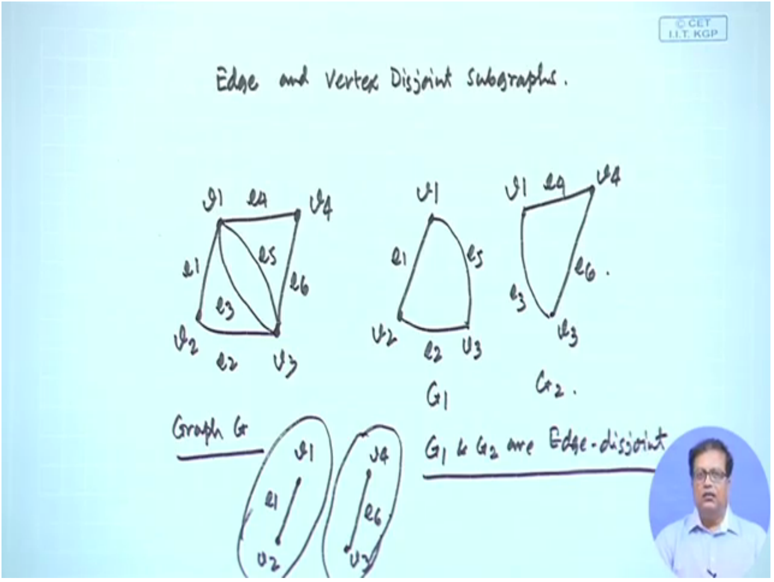

Now edge and vertex disjoints subgraphs, you see sometimes when we have a particular graph we may be able to find out edge and vertex disjoint subgraphs, to really understand edge and vertex disjoint subgraphs to make better understanding of a problem, alright? So you see suppose I have a graph here, alright? So this is v1, this is v2, this is v3, this is v4 and here is e1, e2, e3, e4, e5 and e6.

So if I draw like this, right? So look at very carefully this is graph G, the graph G is partitioned into 2 subgraphs G1 and G2 and they are edge disjoint. What is the advantage? If you can really make these kinds of subgraphs we can partition a problem, right? And if this represents a total problem these 2 are representing 2 partitions where there are no common edges, its vertices are common but you can still make very interesting observations out of such situations, right? Similarly we can also have vertex disjoint subgraphs. So maybe we can make say v1, v2 this is one connection and v3 and v4. So you see here we are not able to partition the whole thing but at least what we have done, we have got 2 vertex disjoint subgraphs we have obtained out of that graph, right?

So we will stop here now and we shall continue with our discussions in our next class, so thank you very much.