Course on Decision Modeling
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Lecture 25
Module 5
Generation of Random Variates

Today we are going to discuss what is known as Generation of Random Variates. We have seen in our previous examples that the random numbers are coming from uniform distribution but most often for the purpose of simulation it is not sufficient that you have the uniformly distributed random variables because they particular thinks for example arrivals, or service, etc they are all coming from some other distribution. So how to convert those random numbers into the random variates or you know the random variables coming out from a given distribution that is what is required.

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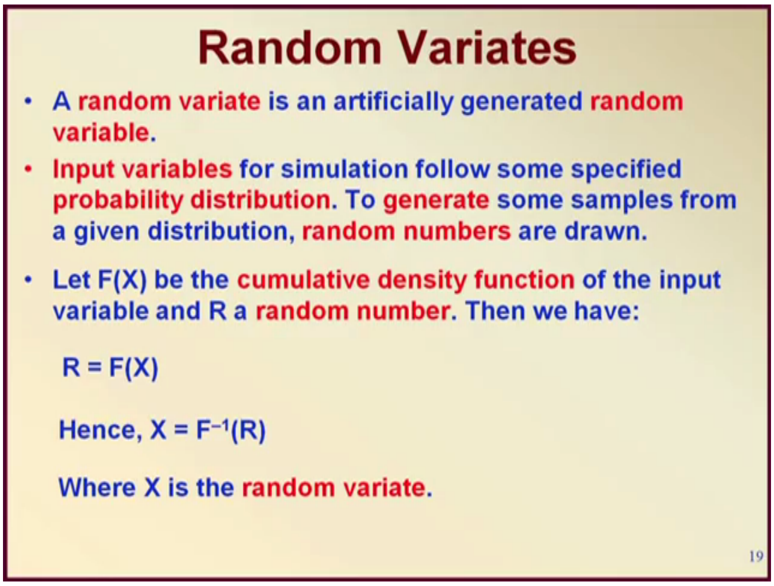

So a random variate is an artificially generated random variable, right. Input variables for simulation follows some specified probability distribution to generate some samples from a given distribution random numbers are drawn, right. So let F(x) be the cumulative density function of the input variable and R is a random number.

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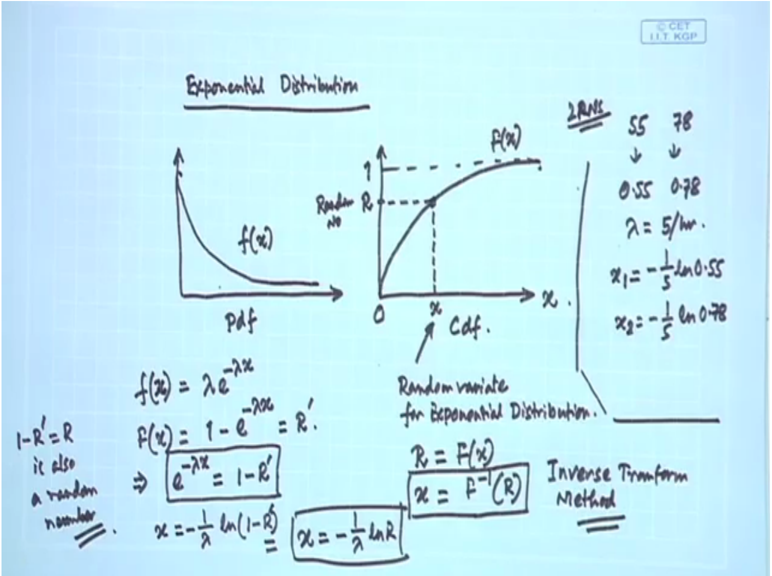

So exactly let us look at suppose we want a random variate for Exponential Distribution, we shall look at the mathematics little later but let us look at the Exponential Distribution how it looks you know this is usually the what is known as the F(x), right the probability density function. So what would be the right then that will be the cumulative distribution function, so this is Pdf and this is Cdf, probability distribution and this is Cdf.

Now fortunately for us this is 0 and this is 1, so if you really look at the Cdf and this side is x, right so what you can do if this number is between 0 and 1. Now see you can take a random number between 0 and 1. Let us say your random number is this. So a random number R which is between 0 and 1 is taken and this random number if you project it onto the particular distribution then we get an value of x.

So you see this is R is a random number and this x is a random variate and this is an for exponential distribution so this is a random variate for exponential distribution. Now mathematically speaking R equal to F(x), so x equal to F inverse R, right so this is called the Inverse Transform Method, right that is why it is called Inverse Transform Method, alright.

So we have the probability density function, we have the cumulative distribution function and after that if we take a random but this will be between 0 and 1, so if we take a random number we know a random number is also between 0 and 1 that random number can be projected on to the cumulative distribution function and that corresponding x value would be the random variate.

Essentially what are we doing? We are transforming the random number into a random variate from a given distribution that is what the Inverse Transform Method actually does, right. So if R equal to F(x), x equal to F inverse R that is the method. There are other methods but let us see this method first, so you see that is exactly what I told so far is here where R is a random number, capital F(x) is a cumulative distribution or density function and then x is the random variate.

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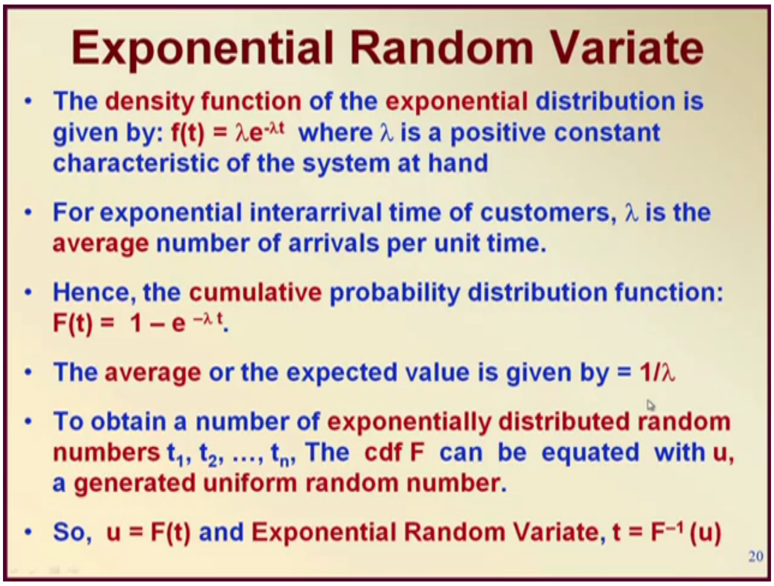

So what happens in exponential distribution in exponential distribution what is the value of f(t), right the value of f(t) equal to or even f(x) you can write because here we are writing x so let us write x also it is lambda e to the power minus lambda x. So what will be capital F(x)? The cumulative density function that will be 1 minus e to the power minus lambda x, right so that is our cumulative distribution function.

So if you equate this to R so what will happen then you can write down e to the power minus lambda x equal to 1 minus R, right 1 minus R so this is the equation that we can use we can see further how we can exploit this particular equation, right. So exactly that is written here the average is 1 by lambda to obtain the number of exponentially distribution random numbers the cdf F can be equated with u, or r as the case may be, so as generated uniform random number so Exponential Random Variate is F inverse u.

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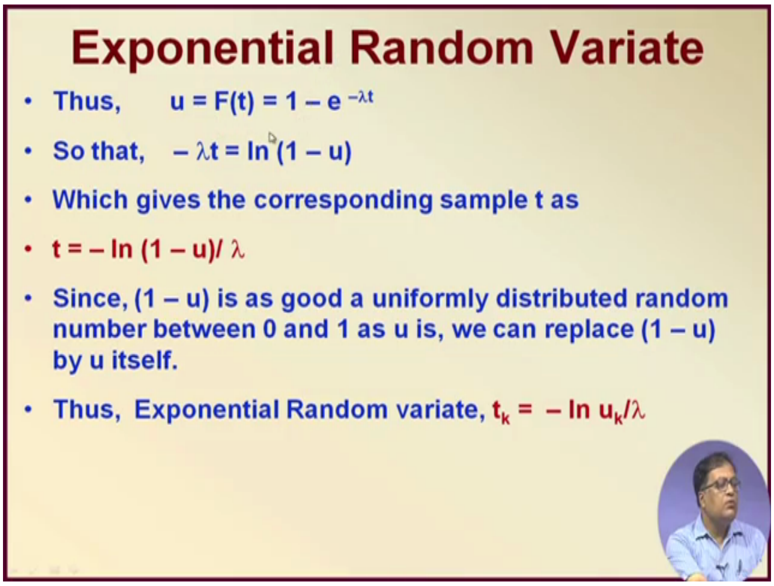

Now in this case since u equal to F(t) equal to 1 minus e to the power minus lambda t in this case. So minus lambda t is ln 1 minus u, right and therefore t will be minus ln 1 minus u by lambda. In our case you get what we can get it will be x equal to 1 by lambda ln 1 minus R, okay. Now there is a minus sign, right. Now one thing is look here that should we use 1 minus R, what is 1 minus R? Basically there is a random number R.

So a random number is between 0 to 1, so if R is the random number 1 minus R is also another random number we might call it R dash or you call the original one as R dash and 1 minus R as R, right 1 minus R dash equal to R is also a random number, right. So using that fact we can write random variate x equal to minus 1 by lambda lnR, okay. So that is how we can actually get a random variate from an exponential distribution, right.

So that is exactly if you look at that a particular one that is t equal to minus ln 1 minus u by lambda or t equal to minus ln u by lambda and thus exponential random variate tk equal to minus ln uk by lambda. So this is how we can actually compute and Exponential Random Variate, right.

So suppose let us go one step further, suppose we have two random numbers let us say 55 and 78, what we have to do? We have to convert them to these are two random numbers and suppose we have to generate an exponential inter-arrival time. So we have to convert them to random numbers between 0 to 1, 0.55 and 0.78 and suppose lambda equal to 5 per hour, alright. So two random variates will be for exponential distribution, inter-arrival times would be generated as x1 equal to minus 1 by 5 ln 0.55 remind you ln 0.55 is a negative number, so that negative and negative will make it positive and x2 will be minus 1 by 5 ln 0.78 is it alright. So this is preciously what we have to do to generate a random variate from an exponential distribution.

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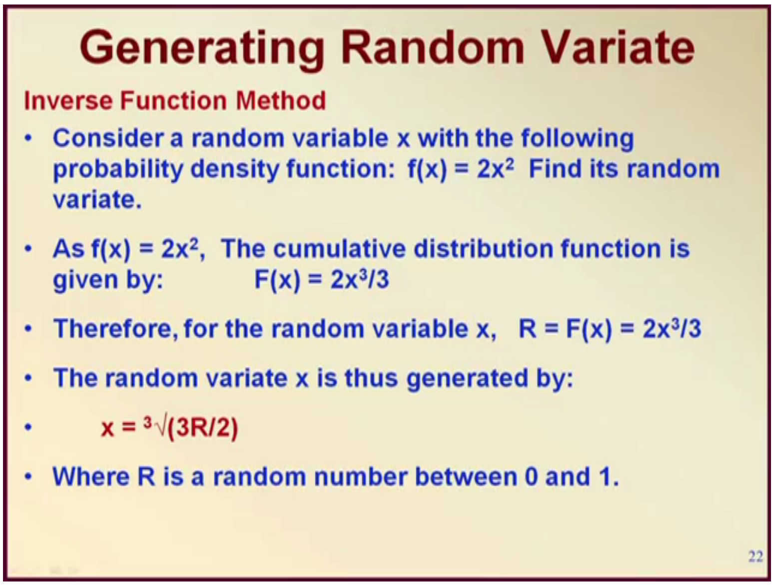

Sometimes we may not get an distribution as a something like exponential and all that suppose we have got a random variable x with the following probability density function f(x) equals to 2x square.

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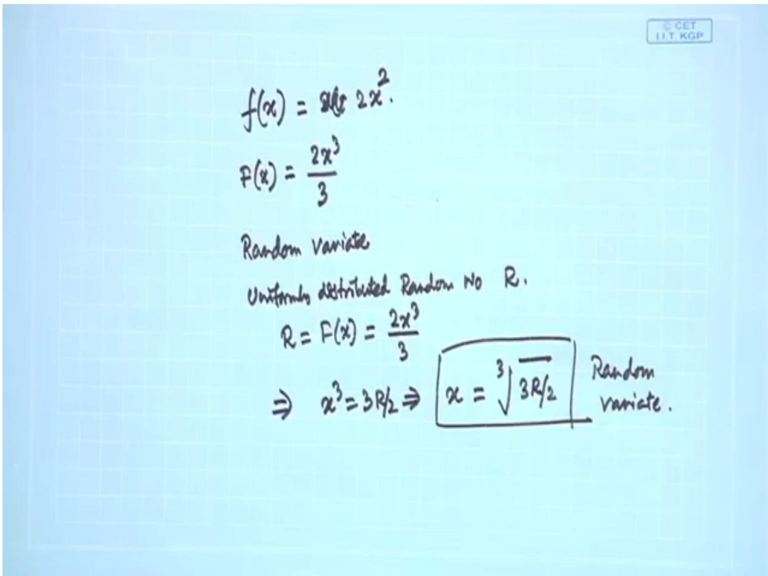

So what would be its random variate? You see interestingly what we have to do we have to compute if f(x) equal to 2x is 2x then we have to compute the cumulative density function

F(x) which would be how much, it would be (2) sorry 2f(x) is 2x square, it will be 2x cube by 3, right because that will be then integration so that integration would result 2x cube by 3. So if we want a random variate random variate and if we then equate random number so uniformly uniformly distributed random number R is taken so R equal to F(x) that is the inverse transform 2x cube by 3, right.

So from here we get x cube equal to 3R or x equal to that is what we shall get, right that 3R by 2, right so this is going to be our random variate for the given distribution, right. So you see if you have a function which is not something like an exponential distribution so we have to find out what is known as the cumulative distribution function cdf equate it to R capital R that is the random number and from there find out the given value of the random variate.

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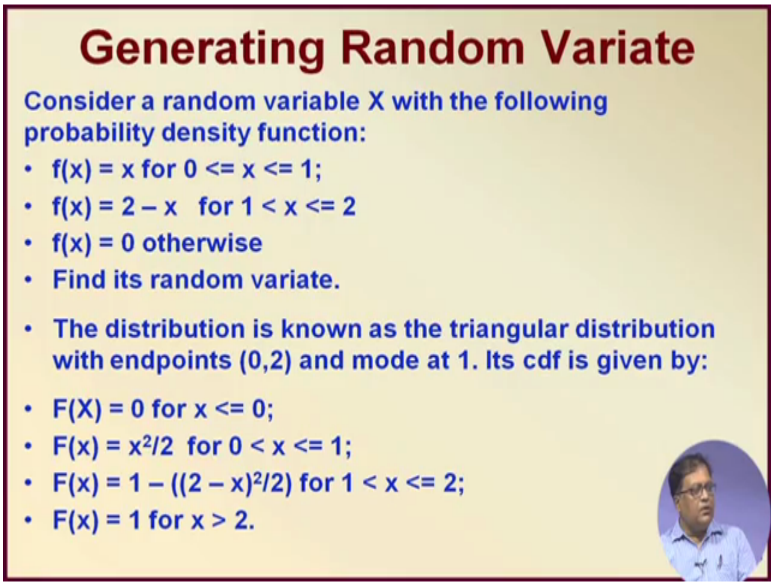

Now here is one more example consider a random variable X with the following probability density function that is f(x) is x for 0 that is 0 to x to 1, 2 minus x for between 1 to 2 and it is 0 otherwise. So find its random variate, right. So what we have to do? We have to obtain for each by an integration process what is the cdf, the cdf will be therefore f(x) equal to 0 for x less than equal to 0, f(x) equal to x square by 2, for 0 to x less than equal to 1 and capital F(x) equal to 1 minus 2 minus x whole square by 2 for 1 between x and 2, right and let us say capital F(x) equal to 1 for x greater than equal to 2.

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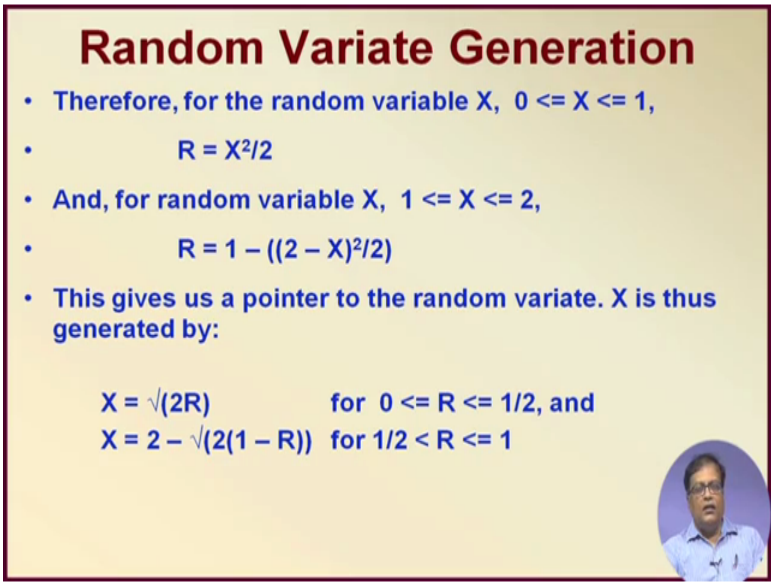

So when you have all these cumulative density functions and then if we equate then we find for the first case when it is between 0 to 1 then R equal to x square by 2 and when we get between 1 to 2, R will be 1 minus 2 minus x whole square by 2. So we get therefore you know random variate, X will be generated by root over 2R for between 0 to R to half but if the random number is between half to 1 then x will be 2 minus root over 2 into 1 minus R.

So you see if the distribution is different for different range we may have to use different formula depending on the kind of probability distribution function that we have at a given point of time, right. So the random variate generation therefore could be rather involved at different points of time.

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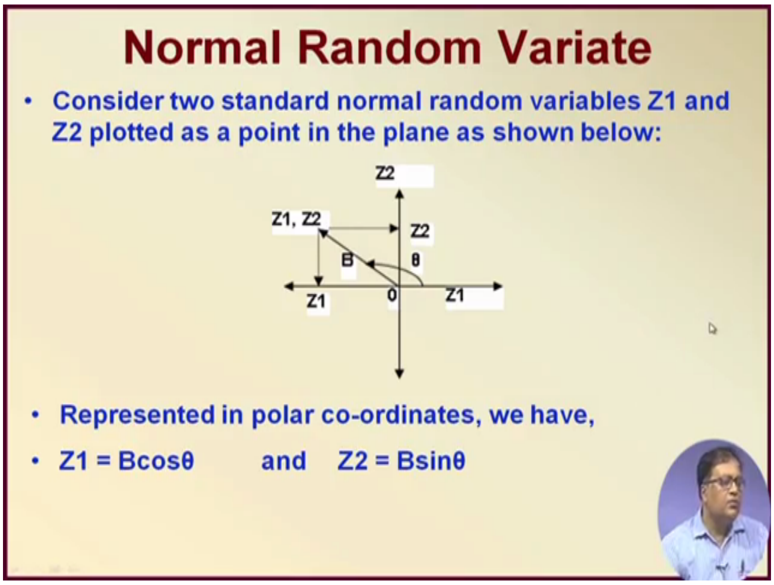

We shall now take slightly complicated case that is known as the Normal Random Variate. You see the normal distribution is some very important distribution and if something is normally distributed we actually may have to have a normal variate for our calculations. So in order to do that consider two standard normal variables Z1 and Z2 plotted as a point in the plane as shown below.

So what happens that two standard normal variate Z1 and Z2 usually what happens you know if they are orthogonal to each other then they will be in two different axis directions and you know they might have what is known as a particular intercept on various you now they are on the polar coordinates and in terms of polar coordinates they can be represented in terms of capital B and theta and we might show Z1 equal to B cos theta and Z2 equal to B sin theta.

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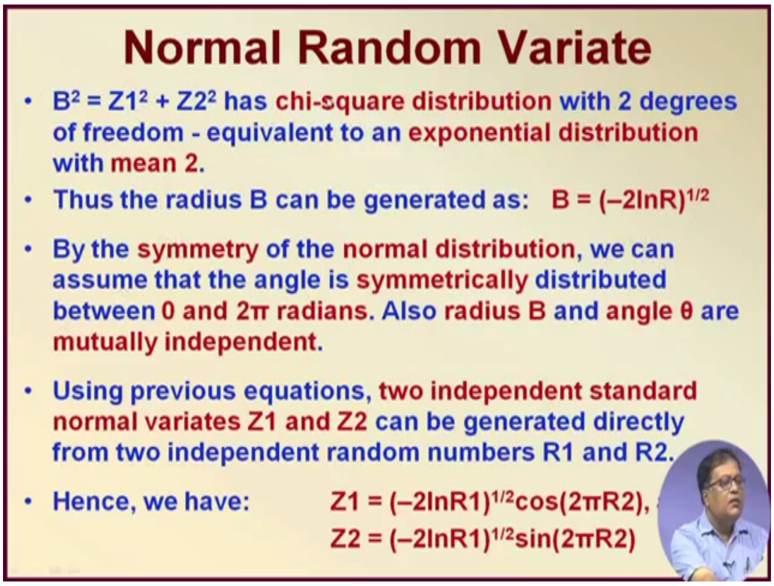

So once we have that you know we can therefore write the B can be generated as minus 2lnR to the power half, right. So because B square to Z1 square plus Z2 square, normally it can be seen that if they are you know standard normal variables then B tends to follow what is known as chi-square distribution. So by the symmetry of normal distribution we can assume that the angle is symmetrically distributed between 0 to 2pi radians and also radius B and angle theta are mutually independent.

So through that what we can do we can actually write Z1 and Z2 by you know Z1 equal to minus 2lnR1 to the power half cos 2pi R2 and Z2 equal to minus 2lnR1 to the power half sin 2pi R2. So you see forgetting the complications of the mathematics that is presented here these two relationships are very important because you know if we have two random numbers R1 and R2 by this transformation we can actually generate two normal variates, right.

So suppose it is said just for the sake of this let us assume that arrival and service are both normally distributed, what we have to do? We have to really obtain two random numbers one for arrival, one for service and we can actually generate you know two you know normal variates minus 2lnR1 to the power half cos 2pi R2 and minus 2lnR1 sin 2pi R2, so two normal variates can be obtained from this particular distribution, right.

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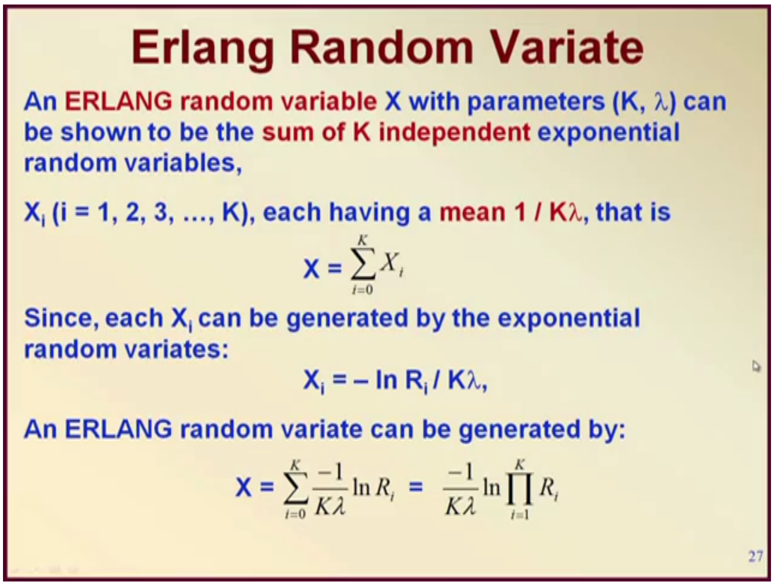

Now let us look at what is known as Erland Random Variate this Erland Random Variate we have not discussed while we discussed the queuing kind of thing but this is an very important distribution and particularly it happens you know when we have a K independent exponential random variates, right. Supposing there are large number of servers and all the servers are serving in an exponential manner, what you can do? You can do an Erland estimate estimation and replace all the servers by a single server and then make it simple, right. So that way you can aggregate the service process by the Erland assumption, right.

So if there are K independent exponential random variables then they some of all of them really is an Erland distribution. So how exactly it is possible to obtain an Erland distribution function that is what we have to do? We have to really come up with the individual you know Exponential distribution and then sum over them, right.

And the distribution X in this case would be the sum of all the distributions that means all the individual exponential distributions and from 0 to K because there are K independent exponential distribution which are summed here and then what we have to do? We have to find out the if they are all identical then each of them will have a random variate which will be govern by this 1 by lambda in this case K lambda because we are having each is having a mean of 1 by K lambda, so 1 by K lambda minus lnRi, is it alright.

And then when you sum them up you know sum them up all then you get an Erland Random Variate that is i equal to 0 to K minus 1 by K lambda lnRi. So that would be our Erland Random Variate and this Erland Random Variate can also be expressed in this form minus 1 by K lambda because it is a constant it can be taken out then it become sum of lnRi but we know that you know ln it is basically then lnR1 plus lnR2 plus R3, etc.

But then you know we know that sum of this ln is nothing but lnR1 into R2 into R3 into R4 so there is that multiplication symbol could be shown in these forms. So it finally becomes minus 1 K lambda ln R1 into R2 into R3 upto RK, right. So in this form it become simplest and that would be an Erland Variate. What is the advantage we get out of this? Suppose we know that there are K independent exponential random variables all of these things are happening at a simultaneously assume that people are arriving at a particular railway station.

Now while people are arriving at the railway stations all types of people are arriving, is it alright. Assume that there are some office goers who are coming back for train, there are some vendors they are coming from train, you know there are some what you call may be the students who are coming for train and if we know they are you know essentially they are all different independent exponential distribution then all of this can be summed up together and an Erland Random Variable can be used as an you know they are summed.

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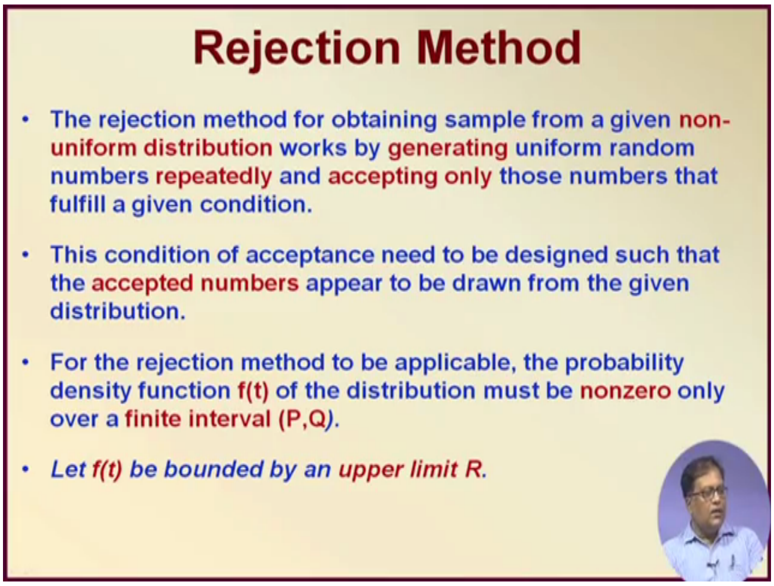

Now let us look at another method which is known as the Rejection Method. The rejection method is so basically there are 3 ways in fact there are 4 ways random variates can be obtained, what are those 4 ways random variates can be obtained?

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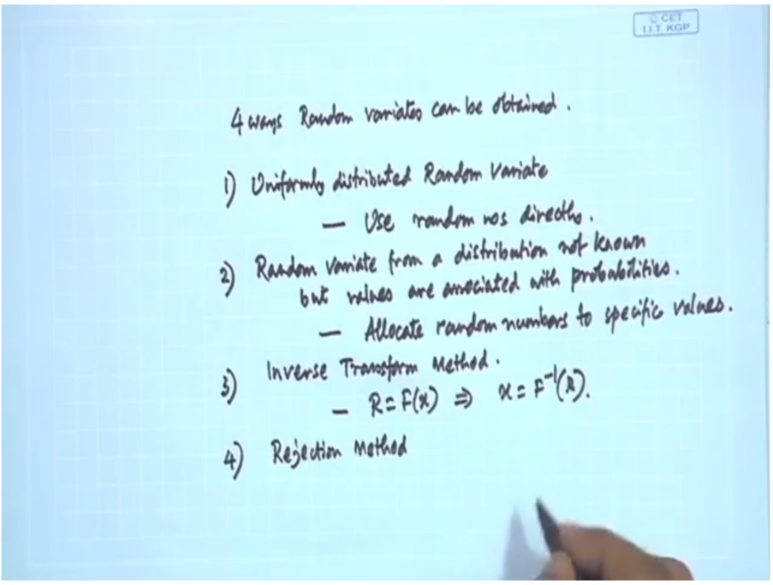

You know 4 ways can be obtained. If we want an uniformly distributed random variate, use random numbers directly, right use random numbers directly. Second random (number) random variate from a distribution not known but values are associated with probabilities. So like we have done the queuing problem example, right so random variate from a distribution not known but values are associated with probabilities. So what we can do? Allocate random numbers to specific values. So this we have seen in the queuing examples, right allocate random numbers to specific values.

The third is the Inverse Transform Method. The inverse transform method basically R equal to F(x) from here x equal to F inverse R, what happens the F(x) is known that means the cumulative distribution function is specifically known and that can be used before this the rejection method that is what we are discussing. Now basically here also the F(x) may be known not really only thing we have the values and we can actually draw the probability distribution function. But then it is not that all the random numbers that we have may not be possible to allocate.

So what happens we may have to reject some random numbers and we have to keep the remaining random numbers from a given distribution that is what is a rejection method. So the rejection method for obtaining sample works by generating uniformly distributed random numbers repeatedly and accepting only those numbers that fulfil a given condition. The condition of acceptance need to be designed such that the accepted numbers appears to be drawn from the given distribution.

For the rejection method to be applicable, the probability density function of the distribution must be non-zero only over a finite interval P, Q. Let f(t) be bounded by an upper limit R.

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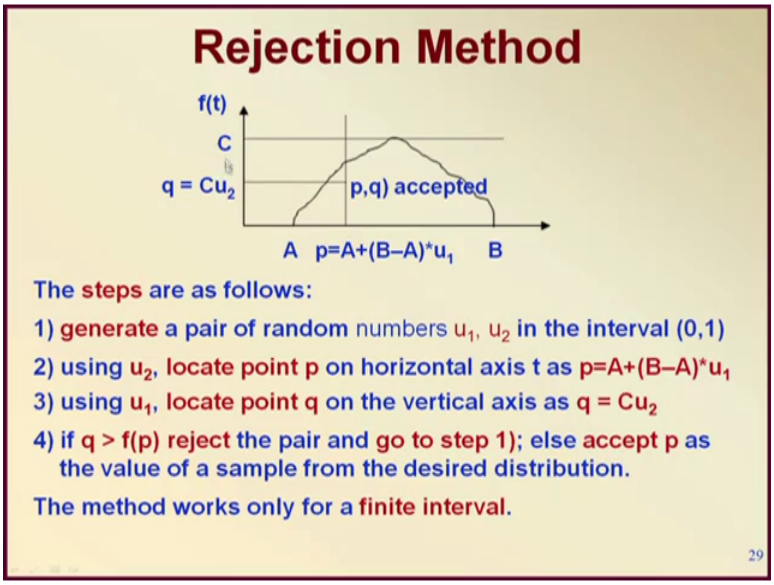

So here that is the you know the diagram, so what happens that this is the function and in this function what happens you know we have the function is bounded by one value A, another value B and let us say p is a number which is in between A and B and can be written as A plus

B minus A u1, where u1 is a random number, right. So let us take two random numbers u1 and u2 over the interval 0, 1, right.

So what you can always do? You can use u2 locate point p you know on the horizontal sorry using u1 this is wrongly written, so we have corrected it using u1 you know locate point p on the horizontal axis that is p equal to A plus B minus A into u1, right so this is the point p. Using u2 locate a point q on the vertical axis q equal to Cu2. So if we have those two points and if q is greater than f(p) then reject the pair and go to step 1, else accept p as the value of a sample from the desired distribution.

So see what is really happening then you know you are generating two points that is p and q and then you are seeing whether that point see these points suppose the q is here, right. If the q is here then we get this point here which is outside the function, this is the function between A and B this if we hatch this portion then this is where the function is but if the point is you known beyond this ft then reject the point otherwise accept the point, so that is the essential idea and this method works only for a finite interval.

So that is what is a rejection method and if the function is such that you know we really do not have any specific set of equations but we have a plot then we can also find random variate from this particular method, right. We stop here and in our next class we shall again we shall see some more examples and we shall then discuss some of the very important things about simulation like validation, verification and further how length of a simulation run can be obtained, how the input can be modelled and how the output we can have confidence on, right so thank you very much.