Course on Decision Modeling
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Module 03
Lecture No. 13
Birth and Death Process

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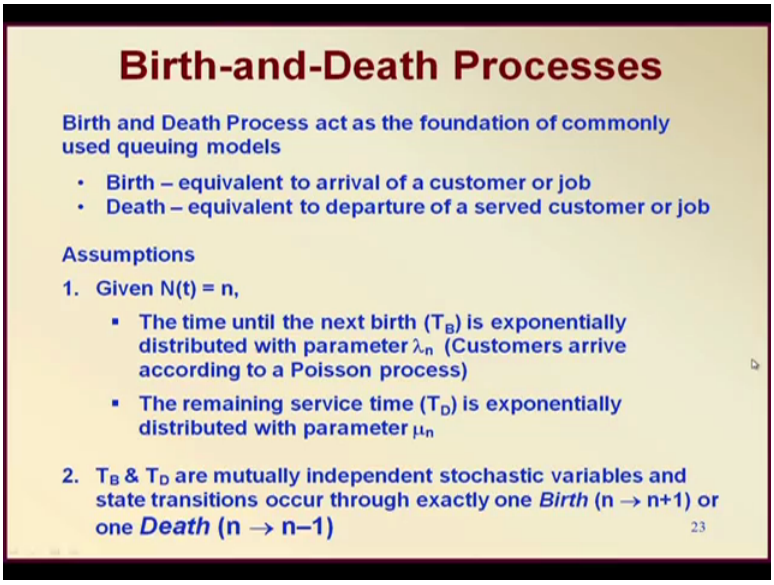

Right, now we shall start what is known as a new process that is called the birth and death process, right. The birth and death process is an integral portion of any queuing theory and for any queuing theory analysis we need to look at this birth and death process. So birth and death process act as a foundation of commonly used queuing models, here the birth can be as an equivalent to arrival of a customer or job and death could be equivalent to the departure of a served customer or job.

So assumptions given Nt equal to n, the time until the next birth that is TB is exponentially distributed with parameters lambda n. What is lambda n? Customer arrival according to a Poisson process. Why lambda n? Because in a general way we are making our assumption that the arrival is dependent on the state. So suppose n is equal to 0, then lambda 0 is different from n equal to 1, that is lambda 1.

What is lambda 1? If the state is 1, that means there is 1 customer in the system then number of arrivals will be determined by lambda 1. Then the remaining service time TD is exponentially distributed you know with parameters Mu n, similarly Mu that is a service rate is also dependent on the what is called the current number of people in the system that is called n.

Now TB and TD are mutually independent stochastic variables and state transition occurs through exactly 1 birth or 1 death. Because you know last class you have seen that the exponential and so called the process and the Poisson process they have the unique thing that in a short period of time there could be only exactly either 1 birth or there could be exactly only 1 death, so that is about the birth in the process.

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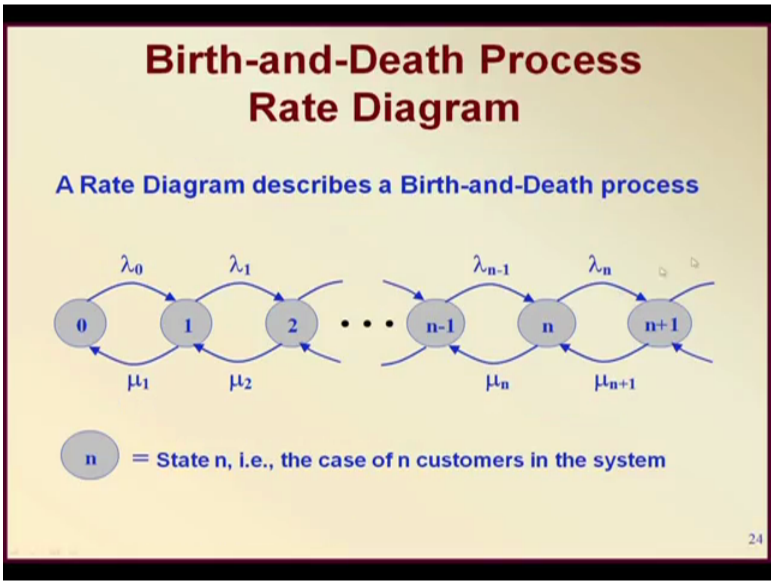

Now you see we have already told that the usually whenever a queuing system begins in the very first you know what is known a transient period. During the transient period the queue system does not stabilize, till the queue system does not stabilize we cannot really think about equilibrium condition and we cannot really think of giving process to begin. The kind of analysis that we do in a queuing process we cannot do until and unless the queuing system comes to equilibrium. So that the moment when system comes to equilibrium that time we can analyze very easily by simply drawing a rate diagram, right.

So what is the rate diagram? Look at this rate diagram very carefully this describes the birth and death process. What are these 0, 1, 2, n minus 1, n, n plus 1? They actually determine the system state. Say system state is 2, what is the meaning of this? That means there are 2 customers in the system, right there are 2 customers in the system. Now how system can have 2 customers? If there are let us say 3 customers, suppose n equal to really 4, then if there are 3 customers then 1 customer can get service. When 1 customer get service at a particular rate what will be the rate? That rate will be dependent on the state, so we will call it Mu 3 in Mu 3 rate the person will be serviced and it comes to too.

Similarly, how does a system state is 1, how can it go to system state 2. There is another way that if 1 customer arrives and joins the queuing system. So 1 customer arrives and joins the queuing system then the system states become 2, so that is what a rate diagram basically determined. So you can see very simple, you have to simply draw the states and you have to just draw these arrows in such a manner that from 0 customers arrives it becomes 1, and the customer arrives it becomes 2, another customer arrives it becomes 3, like this so on and so forth.

Now how it is general? Because you know assume a multiple server, if the multiple server is there then you know then it is not necessary really that you have these values of lambdas and Mu's will change, but although a value of lambda Mu's will change, since the transmission can occur only 1 at a time the rate diagram will still be valid, right. So this is about the essential idea that how birth and death process can actually be model with the help of a rate diagram.

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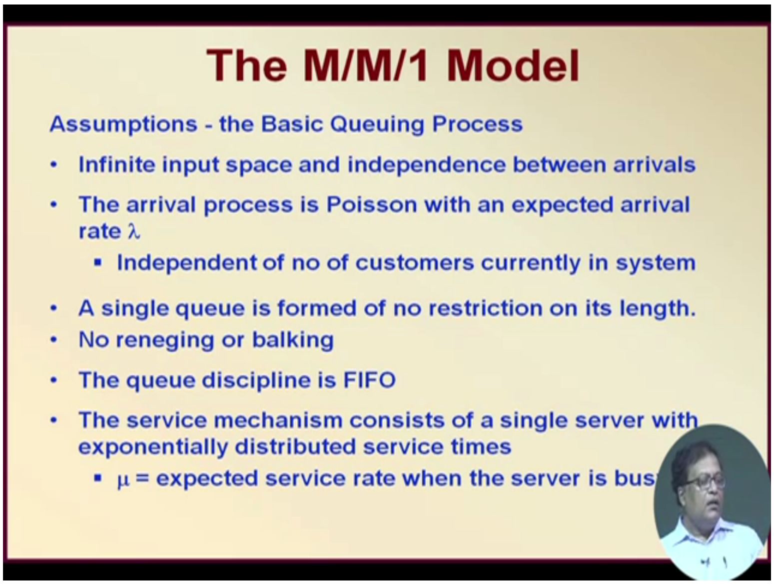

Now let us make some very basic assumptions and consider the simplest model that is so called MM1 model. In a MM1 model what happens we have these assumptions. What are they? First of all we assume an infinite input space and independence between arrivals, right. That means archival of 1 person is independent of another person arriving. The arrival process is Poisson with an expected arrival rate lambda, independent of number of customers currently in the system. A single queue is formed of no restriction on its length also assume no reneging or balking or such special cases and let us simply assume the queue discipline to be first in first out. The service like as consist of a single server is exponentially distributed service times where Mu is the expected server rate when the server is busy.

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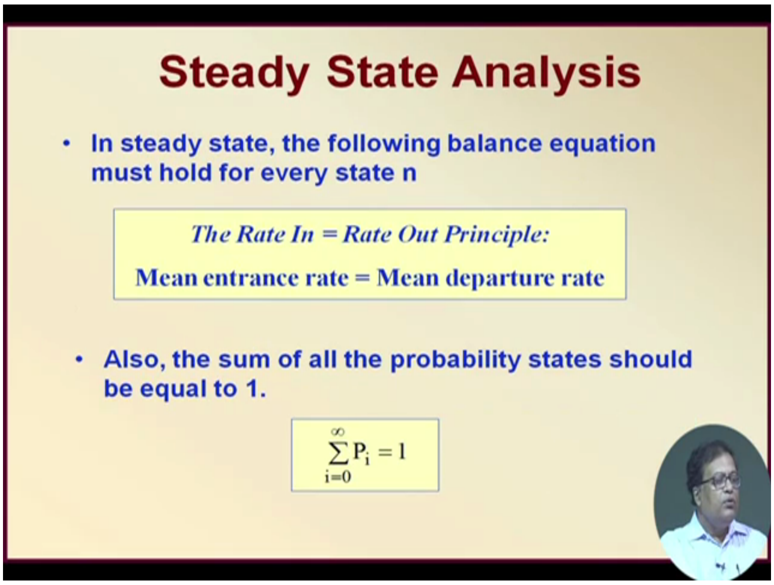

So all of these put together this is the steady state analyses, and the steady-state analyses the following balance equation must hold for every state n, the rate in equal to rate out, that means the mean entrance rate equal to mean departure rate and also the sum of all the probability state should be equal to 1. So Sigma I equal to 0 to infinity, PI is equal to 1.

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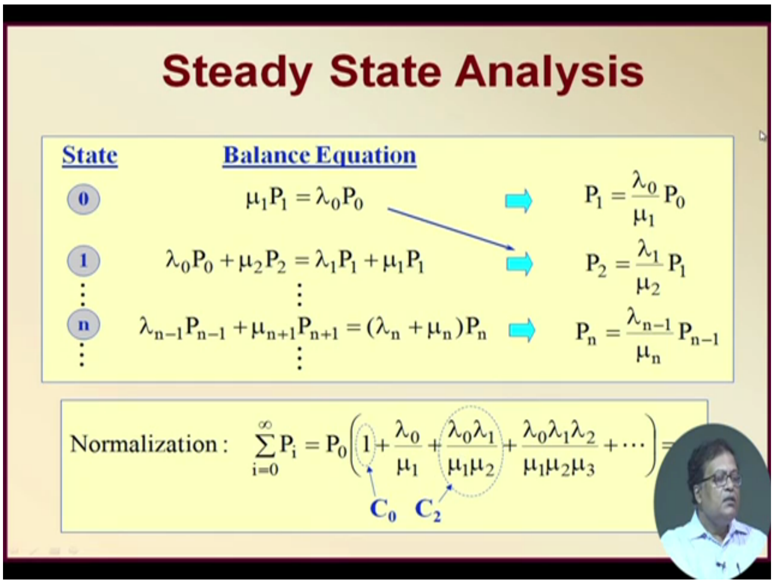

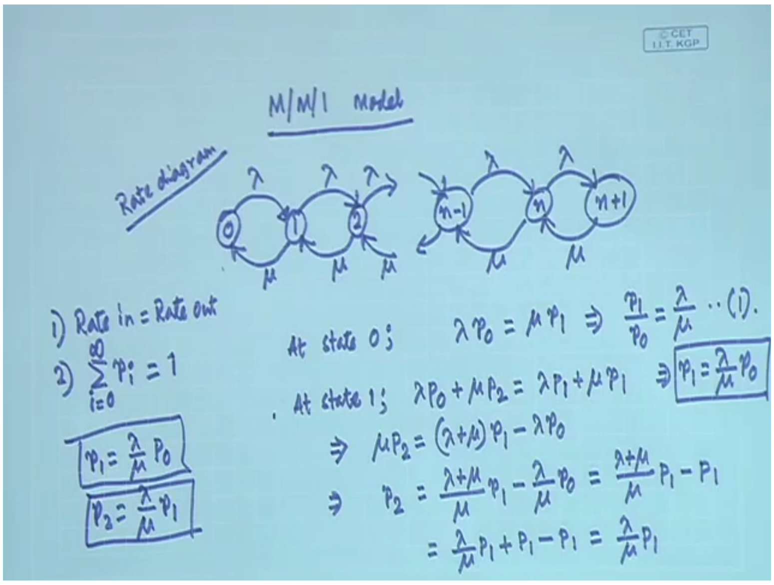

Then let us look at the balance equations, in fact these balance equation can be obtained from the rate diagram these rate diagram, how it is obtained? Let us see for an MM1 model let us draw it over here so then it will be understood easily, let us say we have an MM1 model. Now for an MM 1 model these are the system state 0, 1, 2 then we can draw n minus 1, n, n plus 1. Now 0 to 1 if there is an arrival similarly here, similarly here, and on the other side and since it is simple MM1 model, so all these are Mu’s and all these are lambdas, so this is our rate diagram.

Now look here what is the balance equation says, let us put those equations before us, number 1 rate in equal to rate out, number 2 Sigma Pi, i equal to 0 to infinity equal to 1 only 2 they are very simple only 2 simple principle and we have nothing else. So if that is so then how we make the balance? You see at state 0, what would be the balance? You see what is going out and what is coming in and going out is lambda and that value is P0, so lambda P0 equal to Mu P1.

So very simply we can write, how do you get? This 1 is the probability at here at P0 so what will be the rate lambda times P0 and here the state is 1 probability of the system having 1 and Mu is coming out so rate in equal to rate out. We simply can right lambda P0 equal to Mu P1, so if we do that then from here we can write P1 by P0 equal to Lambda by Mu, all right, so very simply we can make this balance. Let us make 1 more balanced that is at state 1, what is the balance at rate 1? What is coming in and what is going out. So rate in, what is rate in? Rate in is lambda P0 this is 1 rate in, what is another rate in? Is Mu P2 and what is going out? Going out is lambda P1 plus Mu P1, is it alright? So these are the things that are going out.

So from here what can we make out? We can make out is Mu P2 equal to Lambda plus Mu P1 minus lambda P0, is it alright? So you can further simplify this gets P2 equal to Lambda plus Mu by Mu P1 minus Lambda by Mu P0, is it alright? But look at the first relation this is the first relation from the first question we can say P1 equal to Lambda by Mu P0, is it not? This can be derived from the first relation.

Now here we have a Lambda by Mu P0 so we can write therefore Lambda plus Mu by Mu P1 minus simply P1 all right, so basically it is minus 1. So here it is lambda by Mu P1 plus 1, you know you can simplify it is lambda by Mu P1 plus P1 minus P1 equal to Lambda by Mu P1. So basically we got 2 very interesting relations, we have got P1 equal to Lambda by Mu P0, P2 equal to Lambda Mu P1. So you know very interestingly we have been able to really obtain 2 of the very simple but fundamental relations.

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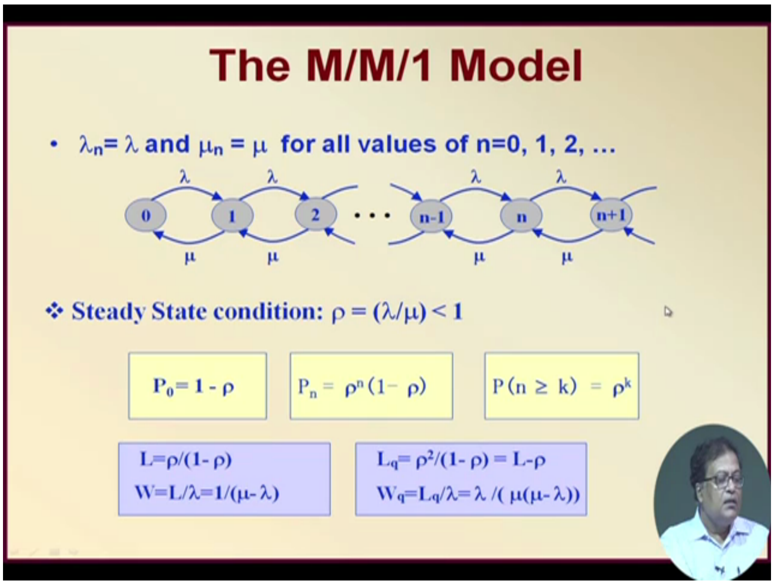

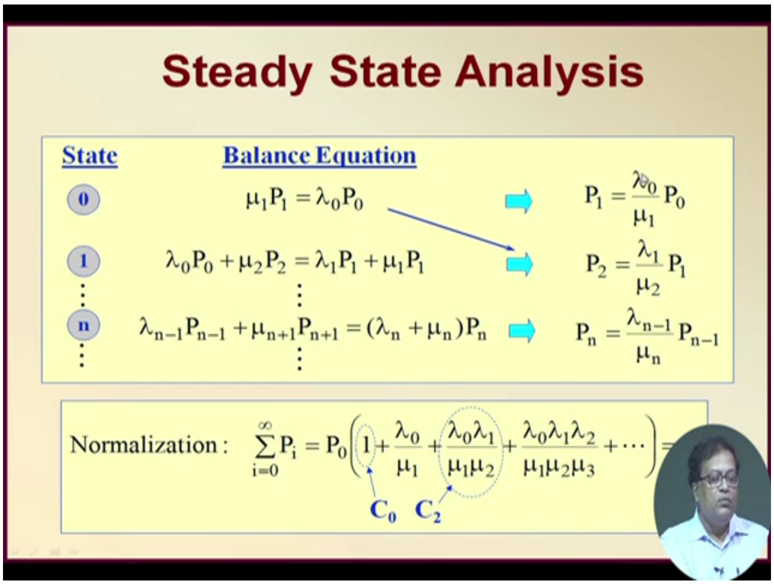

So look here you know this is how we have been able to now generalize, how do we generalize that P1 is equal to you know this is not here okay. The advantage that we can get is we can actually you know put this in that format, what is that format? That P1 equal to Lambda by Mu, because all lambdas and Mu's are same, so P1 equal to Lambda by Mu P0, P2 is equal to Lambda by Mu P1, and Pn is again Lambda by Mu P n minus 1.

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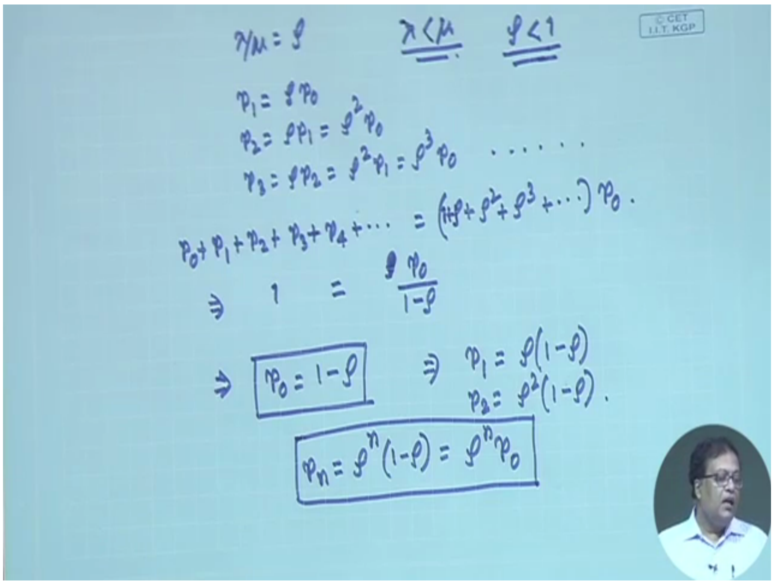

So when you add them all right so you can see before you can write that in another way since this is happening that let us see we have got say let us put lambda by Mu equal to rho. So we have got P1 equal to rho P0, we also know that P2 equal to rho P1 equal to rho square P0, P3 equal to rho P2 equal to rho square P1 equal to rho cube P0, , so like it goes. Supposing we put them all together if you put them altogether then P1 plus P2 plus P3 plus P4 plus dot dot

dot equal to 1 plus sorry rho plus rho square plus rho cube plus dot dot dot P0. So all these relations they can be obtained in this format.

But what are these this 1 is you know if you really compute then you can add P0 this side also, so you can add P0 both sides you can add P0 also here, movement you at P0 then it will be plus P0 which basically means 1 will coincide, so it will be 1 plus rho plus rho square plus rho cube plus etcetera. Now all of this comes what is it? It is nothing but 1, is not it? So all of this adds to 1, right? So the side is 1 and this side is this term, since rho is less than 1, because you see we have already told that Lambda should be less than Mu Lambda is less than Mu. If Lambda is less than Mu then rho should be less than 1, because rho is Lambda by Mu because a steady-state in the separation it should be higher than the arrival rate.

So if that so then rho should be less than 1, if rho is less than 1 then what is the sum of this quantity 1 plus rho plus rho square plus rho cube plus etcetera, this is nothing but 1 by rho, so this becomes P0 by 1 minus rho. So in simple terms from here we can make this P0 equal to 1 minus rho, okay. So now we have got P0, once you get the P0 we can compute P1 in equal to rho times 1 minus rho, P2 equal to rho square 1 minus rho, in simple terms Pn equal to rho to the power n 1 minus rho or rho to the power n P0, all right.

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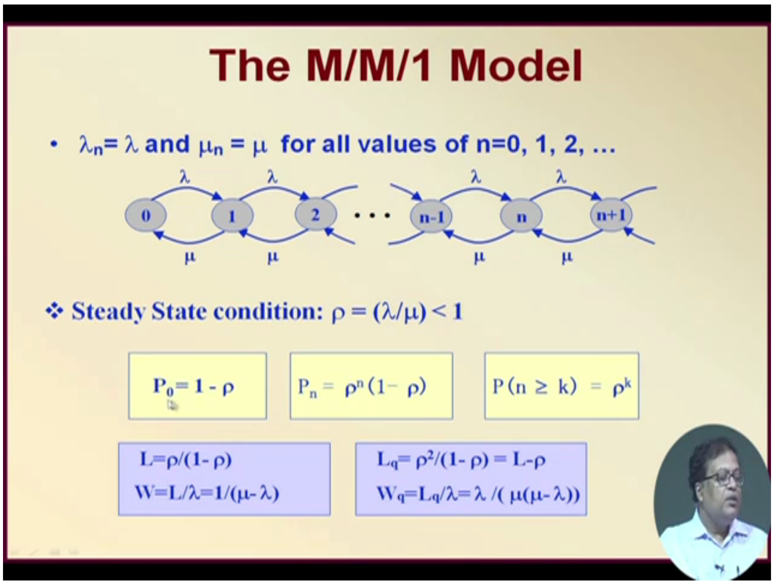

So these are the essential terms that we can actually compute and we can actually therefore see all these relations that this is P0 equal to 1 minus rho, Pn equal to rho to the power n 1 minus rho, P n rate than equal to k is out of the power K and similarly you know this formula can also be obtained that L equal to rho by 1 minus rho. So I am not showing it but let us assume but we can show it in detail that is L equal to rho by 1 minus rho.

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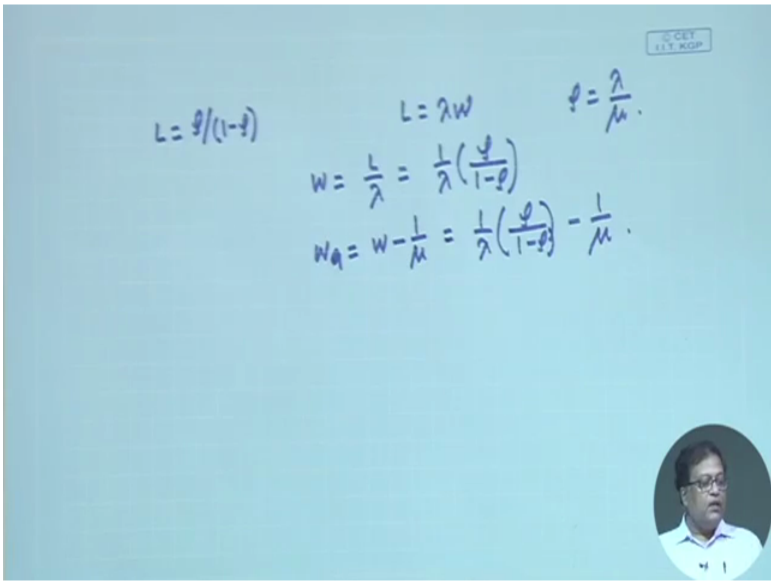

Now you can use little's formula, what is that? Since we know L is equal to Lambda W then and what will be W then? The W will be then L by Lambda, so it becomes 1 by Lambda rho by 1 minus rho, what is row? Rho is Lambda by Mu, okay, so this is how you can also obtain what is known as a W. Then you can also obtain LQ and WQ, so WQ will be W minus1 by Mu, because W equal to WQ plus 1 by Mu, so it will be 1 by Lambda rho by 1 minus rho minus 1 by MU, so all these calculations can be definitely done and simplified forms can also retained.

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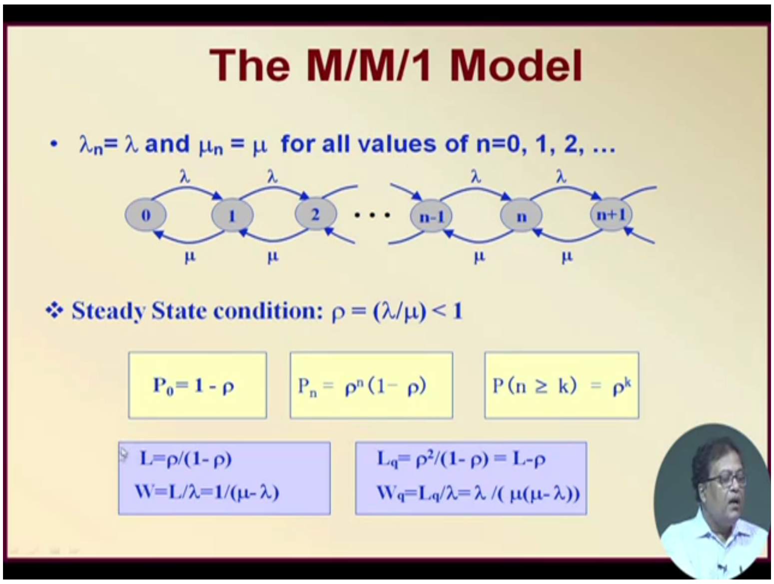

So look at the slide here you can see L equal to by 1 minus rho, LQ equal to coming out to rho square by 1 minus rho or in simple terms L minus rho. And W equal to L by Lambda that is 1 by Mu minus Lambda and WQ equal to LQ by Lambda that is Lambda by Mu into Mu minus Lambda. But any how all these are not required we just do some simple calculation is on the basis was the formula is known.

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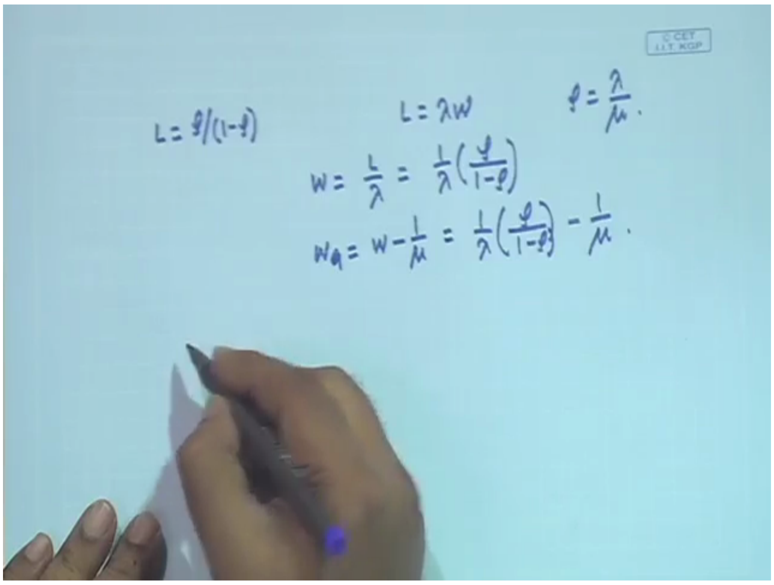

1 very important interpretation is required here, what is that interpretation? So we have already seen that P0 equal to 1 minus rho and what is rho? Rho is Lambda by Mu. You see P0 means probability of nobody in the system. So P0 is probability of nobody in the system which is 1 minus rho then what is 1 minus P0, 1 minus P0 is probability of somebody in the

system, so it is other side of the Picture, 1 side of the Picture is probability of nobody in the system, so all these are MM1 right.

So probability of nobody in the system is P0 1 minus rho, so probability of somebody in the system is 1 minus P0 and what is the value? 1 minus P0 if P0 equal to 1 minus rho then 1 minus P0 equal to rho, so that is the beauty of the thing rho equal to Lambda by Mu equal to 1 minus P0 which is nothing but probability of somebody in the system and this is called the utilization nothing but the utilization of the system.

So actually rho or Lambda buy Mu really determine the utilization of the system and in particular case for MM1 it also signifies 1 minus P0 that is the probability of somebody in the system, is it all right? So you know what we basically can therefore look at is that for an MM1 queue we can see the system parameters and always system parameters they have something very unique and that unique ways that we can have the on the system parameters are they are based on some simple calculation on rho and 1 minus rho that is 1 minus P0 that gives you the system utilization.

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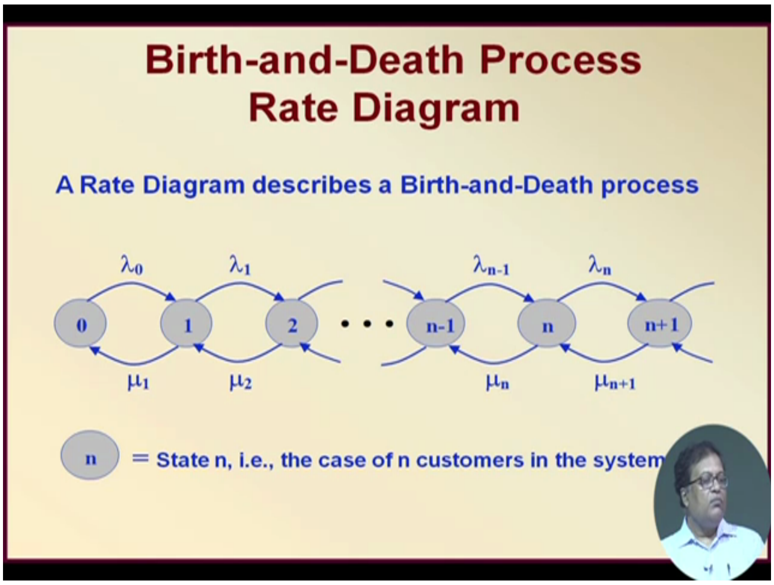

So let us quickly review of what is it that we have d1 in this particular in discussion on birth and death process. Once again if you look at the bullpen then process very quickly because it is so important topic and this rate diagram we shall keep using again and again. Let me again read the rate diagram of birth and death process which is shown here is only useful when the system is in a steady-state, if the steady-state is not reached that this rate diagram cannot used.

Rate diagram can only be used when system comes into the steady-state, but does the system comes in a steady-state right in the very beginning? First of all in most situations where arrival rate is not lower than the service rate the steady-state does not happen, there is no steady-state. Supposed you go to a queue where there is only 1 service process and that too servicing at a very low rate and people are coming in a continuous manner, then what is happening? There cannot be a steady-state. The arrival rate is much higher than the service rate, there is no steady-state no such rate diagram of a birth and death process can be constructed.

So simple advice queuing system is not applicable here only thing that you can do is you can try to achieve service rate which is higher than the arrival rate till you achieve that kind of service rate there is no point talking about queuing systems queuing system analyzing, it is definitely a queuing system, but it is in a translate state the steady-state has not reached a precondition for reaching steady-state is that Lambda should be less than Mu or in other words the utilization rho which is should be Lambda by Mu that should be less than 1, is it all right? That is our first requirement.

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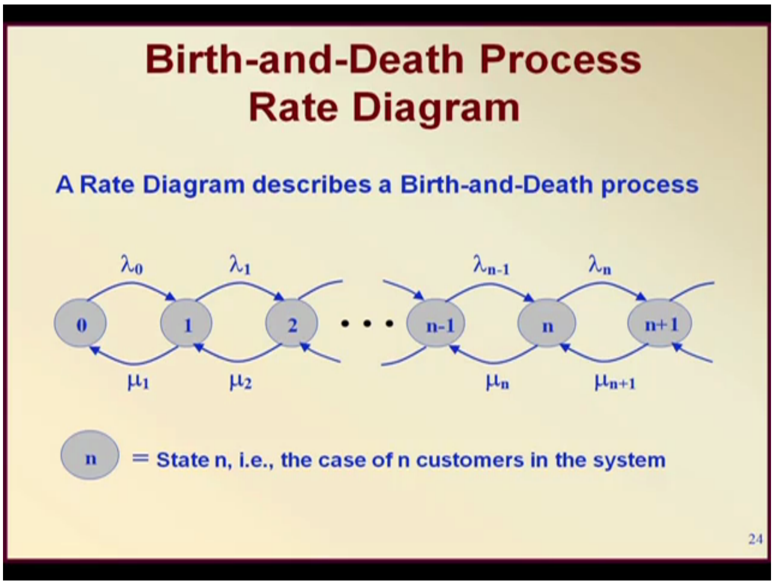

Once that is possible only then the birth and death process the rate diagram can be constructed. Now once again diagram is constructed you can see that the rate diagram shows the state of the system that is zero, 1, 2, etcetera-etcetera and they are state dependence and arrival happens at state zero you know because you are also making an assumption that the there is a single exactly 1 trusted transmission occurs through exactly 1 birth and 1 death, so this actually happens and these are the rates.

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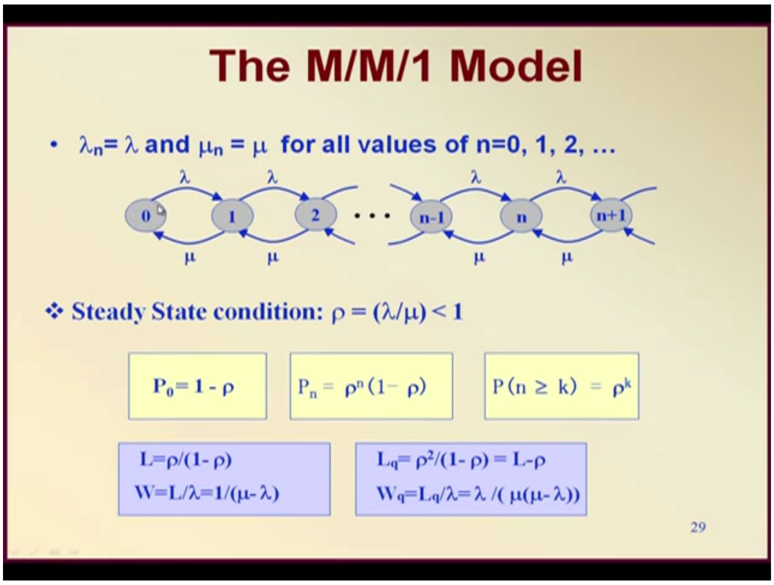

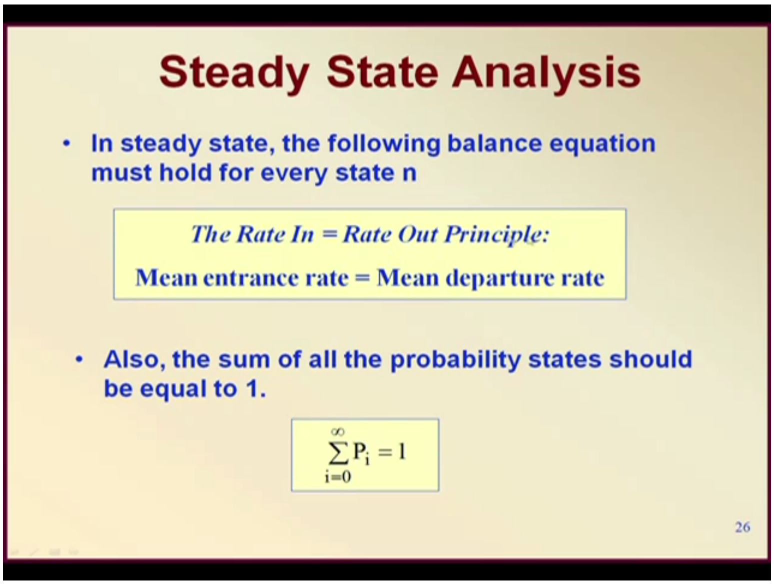

Once we know that in MM1 queuing system you know shown by queuing diagram then you know you can see that from the state 0 there is an arrival through these Lambda process then it can come to state 1, then if there is a service it comes back to 0. Then what we did? We did what is known as the steady-state balance equations. The balance equation is rate in equal to rate out, main entrance rate equal to main departure and also the sum of all probabilities should be 1.

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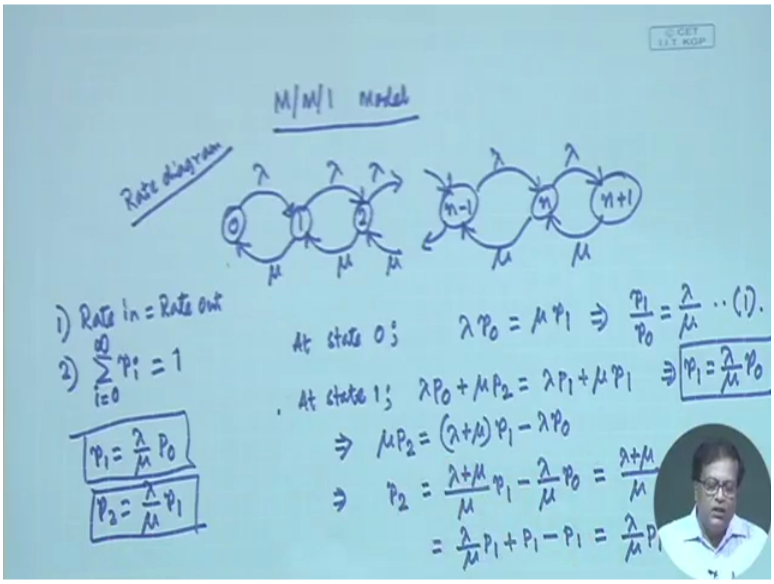

So then what we did? We have made these computations, these computations let me show once again. You know look at these rate diagrams, then in this rate diagram what we did at state zero report Lambda into P0 because that is where is the probability and equal to Mu P1 and from here we got P1 is equal to Lambda by Mu and therefore P1 is equal to Lambda by Mu P0 and 2 we have written Mu P2 is equal to Lambda plus meal P1 because that is what is happening that Mu P2 Lambda P0 Lambda P0 and state 1 Lambda is P0 plus Mu P2 that is sorry Lambda is P0 and Mu P2 they are arriving and Lambda P1 and Mu P1 they are living . If you put that you can compute Mu P2 equal to Lambda plus Mu minus P1 minus Lambda P0. From there if you simplify then you find that you get P2 equal to Lambda by Mu P1.

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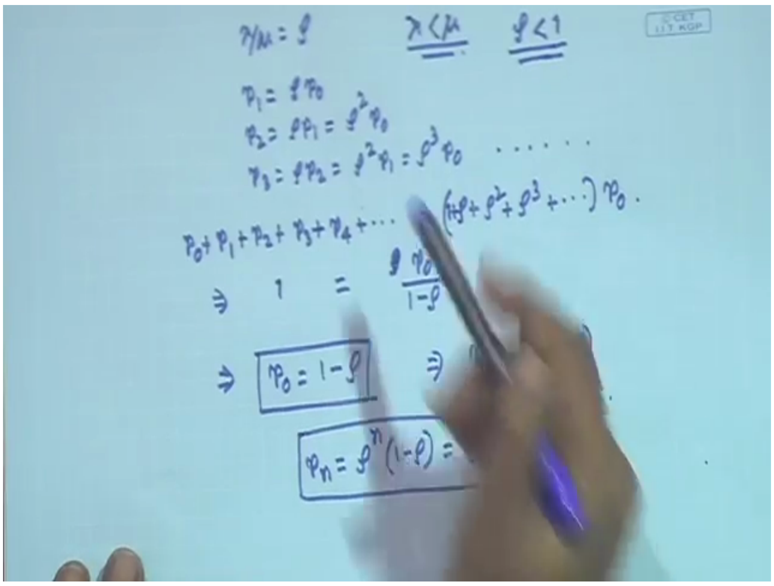

So those things are written here rate in equal to rate out, Sigma Pi equal to 1, P1 equal to Lambda by Mu P0, P2 equal to Lambda by Mu P1. So all these calculations further if you know what you can do if you can see that since Lambda by new equal to rho then you can write P1 equal rho P0, P2 equal to rho P1, P3 equal to rho P2 this will follow and then you know you can add both sides and you can get a simple calculation that this side becomes 1 because some of all probabilities is 1 and this side is that sum is equal to 1 minus rho, so P0 by 1 minus rho, from here you can compute P0 which is 1 minus rho.

So then from you can also compute Pn equal to rho to the power n 1 minus P0, all right. So from here the different parameters like L, L cube, W, WQ can also be computed, right. In our next class shall again see all these parameters you can solve some problems and carry on from there. So thank you very much.