Decision Modelling.
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Lecture-12.
Poisson and Exponential Distribution

So today we are going to discuss the Poisson and exponential distribution and which is very important put of them are very important distributions particularly in context of the queuing system or waiting line. In our previous class we have seen that the most usual queuing system has this kind of assumption, the Poisson distribution for a number of arrivals and exponential distribution for service time. But since both distributions are basically what is called one is the compliment of the other that means the number of arrivals are Poisson distributed then inter-arrival time is exponentially distributed right, so these are their connection, now how are they related?

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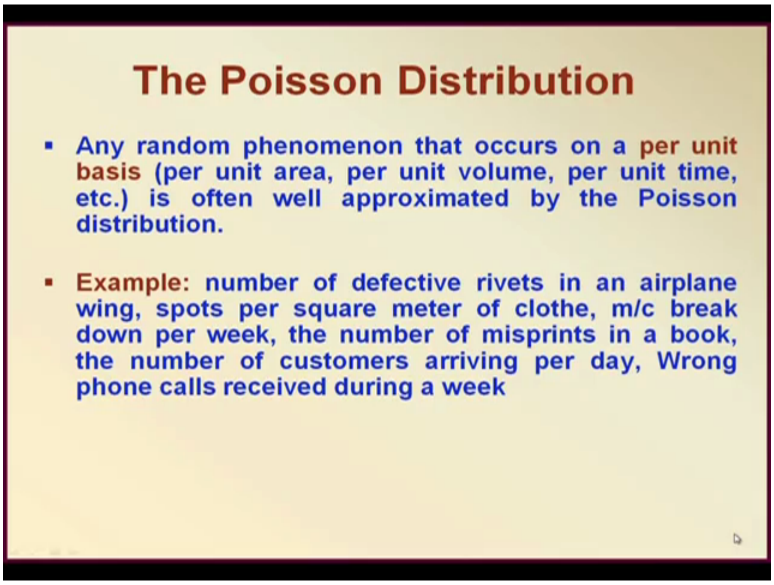

First of all, the Poisson distribution is occurring for any random phenomena that occurs on a per-unit basis like per unit area, per unit volume, per unit time, you know they are well approximated by the Poisson distribution example, number of defective rivets in an aeroplane wing, spots per square meter of clothe, machine breakdown per week, number of misprints in a book, number of customers arriving per day, wrong phone calls received during a week, so they are all different distributions, different examples of the Poisson process.

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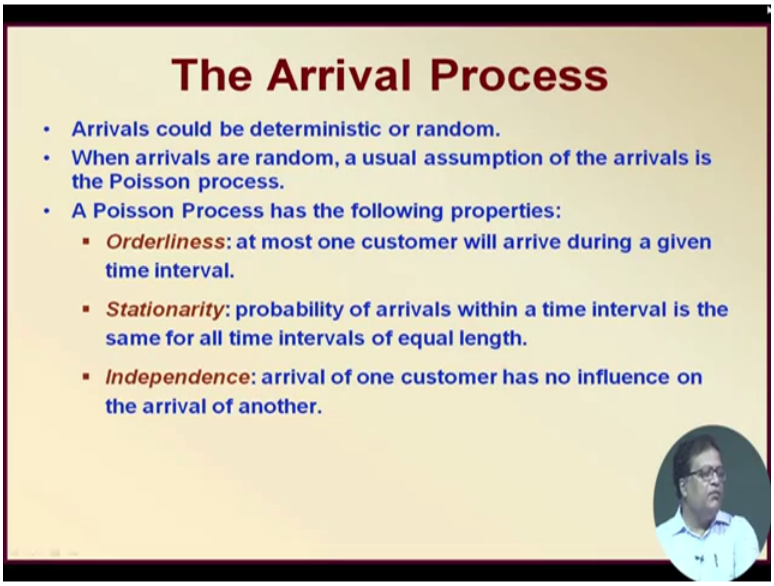

Now if you look at the details of the Poisson arrival, the Poisson arrival could be deterministic or random. First of all when arrivals are random and usual assumption of the arrivals is the Poisson process right, a Poisson process has the following properties, what are they? First one orderliness; utmost one customer will arrive during a given very small time interval. Suppose we talk about a small time interval Delta t then you know we expect only one customer at most will arrive during that very small Delta t time as you note that is called orderliness. Second one is called stationarity, probability that of arrivals within a time interval is the same for all time intervals of equal length.

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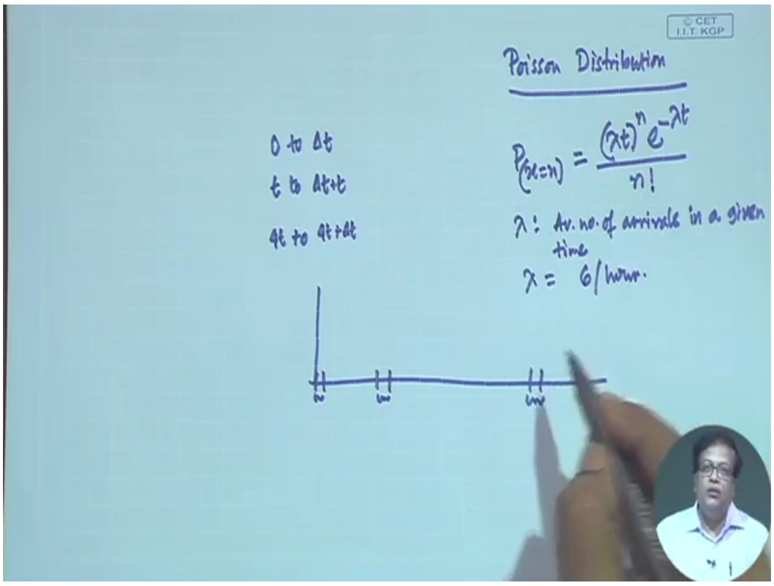

That is very important that is that if we take two different time intervals, one is say 0 to Delta t and another is t to Delta t right so Delta t + t obviously or maybe another you know 4 t to 4 t + Delta t, you see all of them 0 to Delta t, t to t+ Delta t and 4 t to 4 t + Delta t, all of them are different time intervals of equal length for example, this could be 1, this could be one or this could be right, so all of these in this time interval the number of arrivals you know that may happen will be similar distribution right, so the probability of arrivals will be same in all 3 cases that is called stationarity, so that means is we have one, we can know the other as simple as that.

Finally independence, arrival of one customer has no influence on the arrival of another right, so that means you know sometimes we think o achcha within 5 customers have come in the last time period so this time period may not get more, not like that. This time period also 5 may come, the 5 is equally likely like 0s and this is you know the most important property that is why they are called Markovian they have what is known as the memory less property, what is memory less et cetera we shall look a little later.

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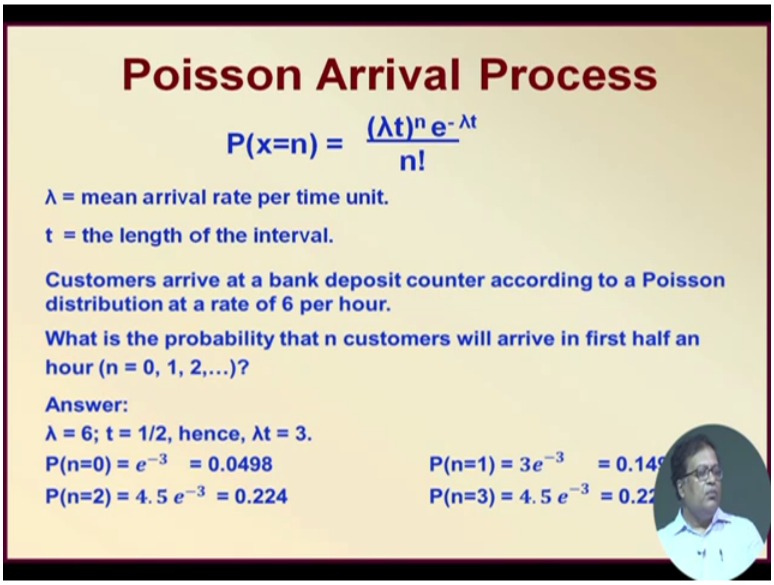

So this is the Poisson distribution, or by ability of x = n you know this is Poisson distribution Poisson distribution, probability of x = n equal to lambda t to the power n e to the power – lambda t by factorial n, so that is the you know the Poisson distribution expression. Now let us say lambda is the average number of arrivals average number of arrivals in a given time, let us say Lambda = 6 per hour right. And let us say that this Poisson process has been established for a time period between 8 o’clock to 9 o’clock and we are talking about half an hour time between 8 to 8:30, so what is t? t is basically half hour right, for this period t we are doing all this calculation.

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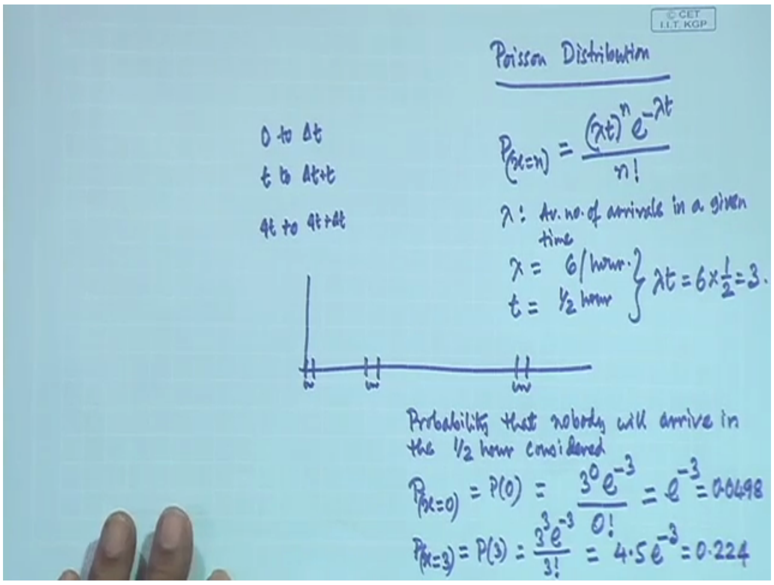

So you know therefore lambda t will be 6 into half = 3, hour is our unit. So we got the value of lambda t, so what is the probability that nobody will arrive during this half an hour time, probability that nobody will arrive in the half hour considered, so probability of x = 0, sometimes it is simply written P 0, so P 0 lambda t is 3 will be 3 to the power 0, e to the power – 3 by factorial 0. So you see 3 to the power 0 and they are all one so it all comes to e to the power – 3, and e to the power – 3 that value I got here that is 0.0498. So what will be P x = 3, if x = 3 sometimes it is called P 3 now it should be then how much? Lambda t is 3, 3 to the power 3 multiplied by e to the power – 3 by factorial 3. How much is 3 to the power 3? It is 27. How much is factorial 3? 3 into 2 into 1 = 6 so 27 by 6.

How much is 27 by 6 if you cancel 3, 3 then 9 by 2 or in other words 4.5, 4.5 E to the power – 3 I have the figure here that is 0.224, so look at the slide once again. Customers arrive at a bank counter according to a Poisson distribution at a rate of 6 per hour, what is the probability that n customers will arrive in first half an hour right, so lambda = 6, t = half so lambda t = 3 and P n = 0 is e to the power – 3, P n = 1 is 3 e to the power – 3 and similarly we can do all these calculations. So basically we can actually find the probabilities of the number of customers arriving at a particular service station with regard to the Poisson arrival process very easily right, so this calculation can be very easily done.

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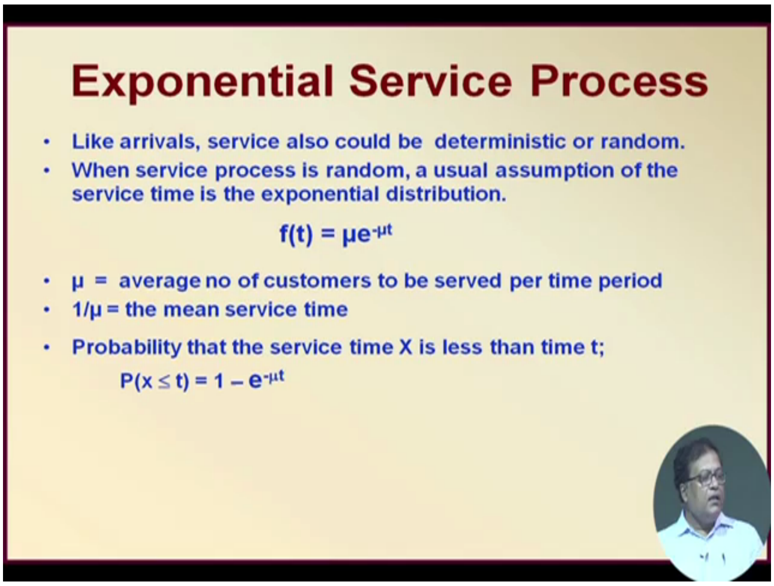

On the other side you know like arrivals, service also could be deterministic or random. When service process is random and usual assumption of the service time is the exponential distribution. So corresponding to the Poisson distribution we can also think of an exponential distribution which is given by f t = Mu e to the power – Mu t, so Mu is the average number of customers to be served per time period and 1 by Mu is the mean service time right. So the probability that the service time x is less than the time t is given by 1 – e to the power – Mu t, so probability that let us write it here.

Probability that the service time x will be less than time t that is probability of x less than equal to t is given by 1 – e to the power – Mu t right, so this particular thing is given by what is known as exponential distribution exponential distribution.

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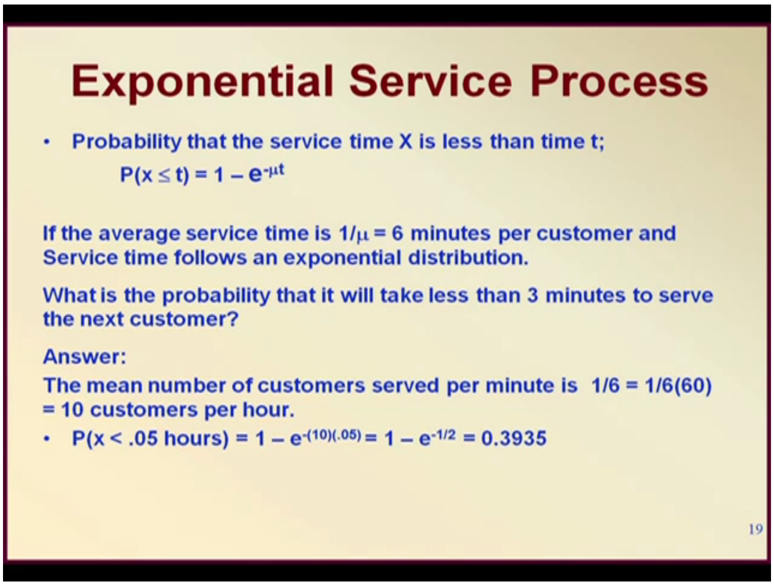

Now looking further, probability that so once again the same thing we have repeated that service time x is less than time t is 1 – e to the power – Mu t. So if the average service time is 6 minutes per customer and service time follows an exponential distribution, what is the probability that it will take less than 3 minutes to serve the next customer right, so in this case what is it that we are going to take?

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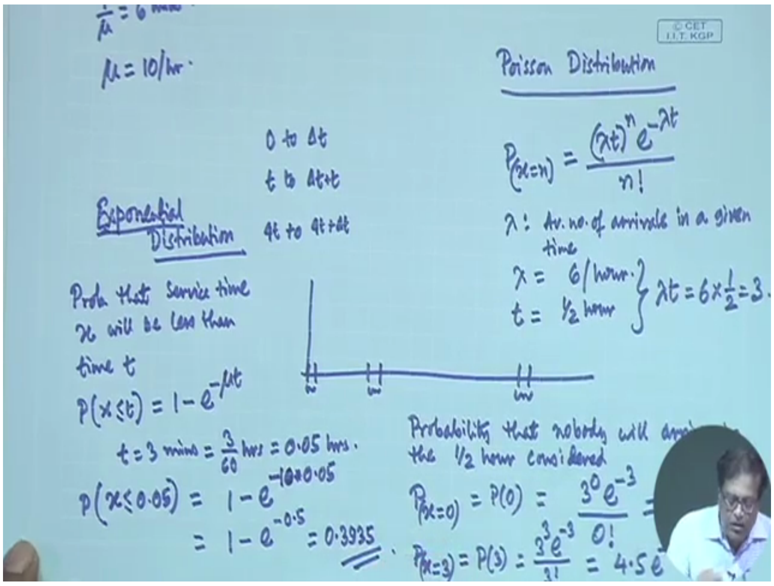

So we take here t = 3 minutes, so 3 minutes is 3 by 60 hours = 0.05 hours so what is the probability of x less than or equal to 0.05? It will be given by 1 – e to the power – 6 into 0.05 right, so how much is that in this case no 6 was the you know the service time was 6 so basically Mu was basically 1 by Mu us 6 minutes, so Mu = 10 per hour right, so in this case not 6 it will be 10, so this is equal to 1 – e to the power – 0.5 and this value comes out to be 0.3935 right.

So basically this is how the calculation goes, once again let us look at the slide, so if the average service times 6 minutes per customer then Mu will be 10 per hour that is the average service per hour right, average number of customers serviced per hour that is called Mu which will be equal to 10 customers per hour. So the probability of service time less than 3 minutes or probability of x less than 0.05 hours will be given by 1 – e to the power – 10 into 0.05 is 1 – e to the power – half which comes out to be 0.3935 alright. So you can do that kind of very simple calculations of Poisson distribution and exponential service process very easily.

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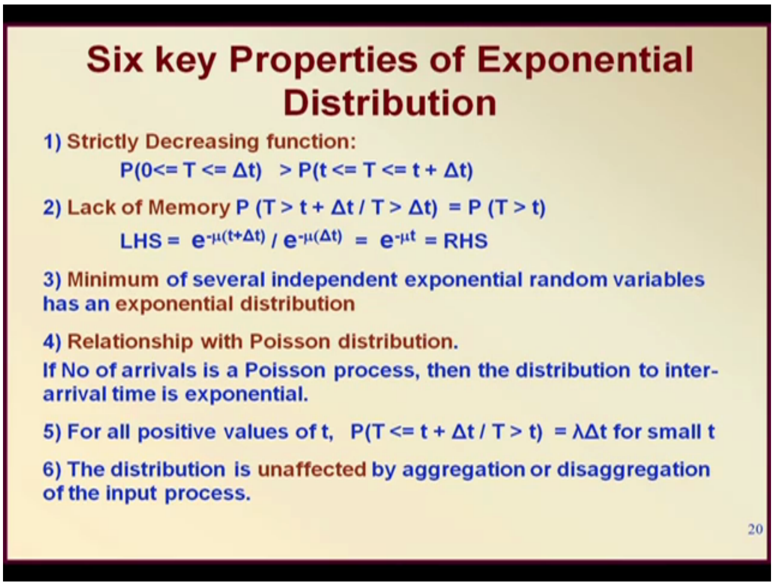

Now look here, this is little complicated but very important slide that means there are 6 key properties of exponential distribution, what are they? It is a strictly decreasing function, second one there is a lack of memory or what is known as the memory less property that is why it is called a Markovian process. Since exponential distribution and Poisson distribution they are very closely related therefore if one is having Markovian property, the other will also have the Markovian property. The minimum of several independent exponential random variables has an exponential distribution right. Then there is a relationship with Poisson distribution, if number of arrivals is a Poisson process then the distribution to inter-arrival time is exponential.

And then for all positive values of t probability of t between the you know t should be less than t + Delta t given that the random variable t is greater than t is lambda Delta t for small t and then finally the distribution is unaffected by aggregation or desegregation of the input process. Anyhow, let us look at it once more time so first of all the first function that is called the exponential distribution that is the there are strictly decreasing functions. Supposing there are 2 equal intervals right, in the 2 equal intervals the P equal to you see what is happening in an exponential distribution, you see this is the kind of plot that you get.

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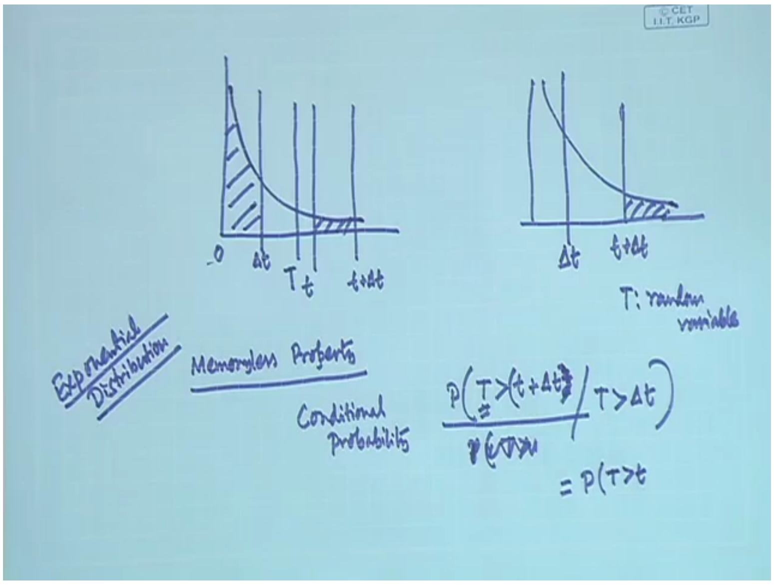

So the probability of 0, suppose this is 0 and this is Delta t, this side is time so between this you know in this area, in this area this is the kind of probability distribution and if you take another same amount of area between t and t + Delta t then this area is less right, so this is what is we are calling as a strictly decreasing, so look here that is decreasing that is the probability of the service time between this and the probability of this you know this is how the comparison goes right that here it is high and here it is low that is what it is called a strictly decreasing function.

The second one is lack of memory, the lack of memory essentially tells what? Let us look at the memory less property that is a very important one. So here we are talking about a conditional probability what is that conditional probability, that probability of T greater than t + Delta t given that probability of T I mean not like this, given T is greater than Delta t right so bracket closes here actually. So say, what is the probability of T greater than t + Delta t given that T is greater than Delta t. The thing is that memory less property therefore says that does it depend on the period between 0 to Delta t because you see what we are telling, suppose T is here, now probability that this T is greater than t + Delta t right, this is sorry let us draw the diagram once again.

So this is our exponential distribution, this is Delta t, this is t + Delta t right, then what is the probability that this random variable T, T is a random variable, what is the probability that the random variable T is greater than that means this portion, is it dependent on what has happened before right? Given that it is already known that T is greater than Delta t that means this period has already elapsed, once this period has already lapsed, you know essentially the good thing is that when we try to find out this probability that probability of T greater than t + Delta t given T is greater than Delta t is nothing but P T greater than t right P T greater than t. So in other words it is not necessary that we have to really think of this

So let us take an example, suppose nobody has arrived nobody has arrived in let us say first 10 minutes right, so what is the probability that somebody will arrive in the next minute? So should we take the fact that nobody has arrived in the first 10 minutes right, this fact is not required that is what is the memory less property is all about. That we have to only take that last minute that this between 10 to 11 whatever has happened that is the only consideration here right. Now the therefore you know the proof is also here that you can see that left-hand side that probability is T greater than t + Delta t is given by e to the power – Mu t + Delta t, and the other one is e to the power – Mu Delta t, so when you will calculate then you will find e to the power – Mu t right so that is RHS so that is the memory less property is all about.

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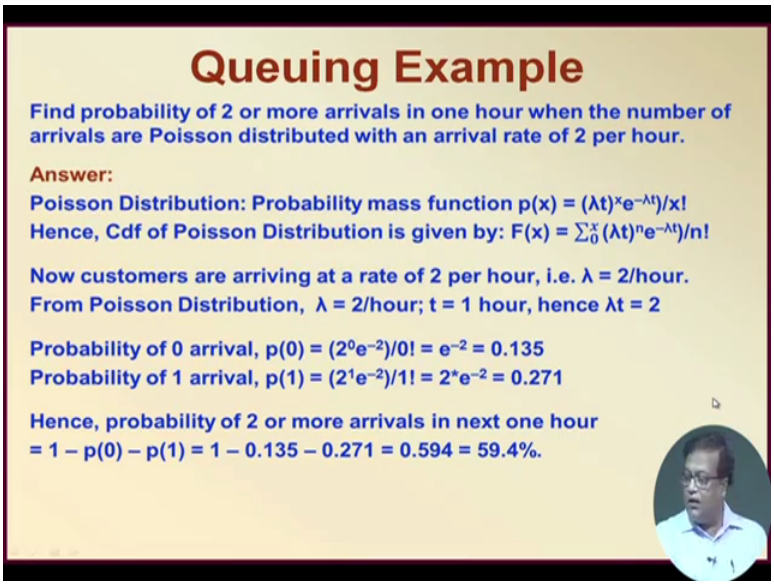

Let us look at the particular example right, the example is find probability of 2 or more arrivals in one hour when the number of arrivals are Poisson distribution with an arrival rate of 2 per hour. So here you know you can really calculate this from a simple addition that probability mass function here is lambda t to the power x e to the power – lambda t by x factorial, so that is the Poisson distribution. So what will be the CDF or cumulative distribution function? It is sum over 0 to x lambda t to the power n, e to the power – lambda t by factorial n. So what is the arrival rate? The customers arriving at a rate of 2 per hour that is lambda = 2 per hour. From the Poisson distribution we get lambda = 2 per hour, t = 1 per hour and therefore, lambda t = 2.

So what will be the probability of 0 arrival? P 0 that will be given by you know this sum 2 to the power 0 because that is a lambda t, then e to the power – 2 because that is what is the value of lambda t and then we can calculate by factorial 0 that is e to the power – 2 is coming to 0.135. So what is the probability of 1 arrival, p = 1 that is given by again lambda t to the power n e to the power – lambda t by factorial n 0.271. Hence the probability of 2 or more arrivals in next one hour that will be given by 1 – P 0 – P 1 = 1 – 0.135 – 0.271 = 0.594 = 59.4 %, let us look at it in a slightly more detail.

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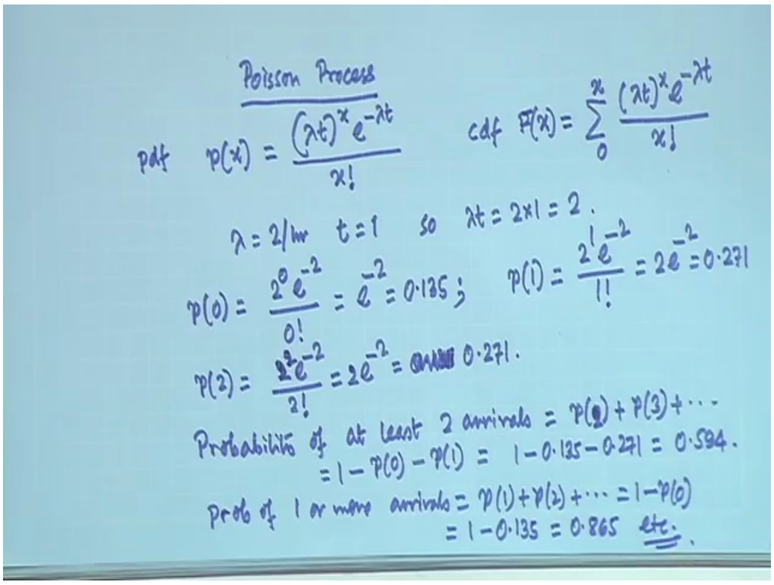

See, what we have got here that we have a Poisson process, in a Poisson process we have the P x = lambda t to the power x e to the power – lambda t by factorial x so this is the Poisson process. Now what is the CDF, CDF will be, this is the PDF, CDF will be that F x equal to because this is a discrete process so F x will be some over 0 to x this. Now here we know lambda = 2 per hour and t = 1 so Lambda t = 2 into 1 = 2, so already we know lambda t. So what is P 0? P 0 will be this one, 2 to the power 0 e to the power – 2 by factorial 0, so it comes to e to the power – 2 right this value is already known 0.135. What is P 1? P 1 is you know 2 to the power 1 e to the power – 2 by factorial 1 = 2 e to the power – 2, this value comes to 0.271 actually because of rounding errors.

Similarly P 2 will be P 2 2 e the power – 2 by factorial 2 equal to you know it will be again 2 to the power 1 yeah again 2 to the power 1 by 2 so it also will be same 2 by 2 that is e to the power – 2 = .135 right. So you can see that sorry 2 to the power 2 so this also will be 2, 2 to the power to so 4 by 2 so 2 e to the power 2 this also will be 0.271 right. So one interesting thing that is to be noted here that number of 0 arrivals is 0.135, single arrival is 0.271, probability of 2 arrival is 0.271. So what is the probability of at least 2 arrivals ?

Because there are 3 things that probability of 0 arrival, probability of single arrival, probability of 2 arrivals right, so it could be equal to P 0 no, it will be P 2 + P 3 + dot dot dot that is equal to 1 – P 0 – P 1 = 1 – 0.135 – 0.271 = 0.594 right. So similarly you can do all these calculation, probability of at least 1 arrival right or 1 arrival or more, so this is let us make it last one, probability of 1 or more arrivals = P 1 + P 2 + dot dot dot = 1 – P 0 = 1 – 0.135 right so = 0.865 etc. So like this what we can do, we can you not calculate such kind of things from Poisson process or exponential process and we can make some what you call conclusions out of them right, thank you very much.