Advanced Financial Instruments for Sustainable Business and Decentralized Markets

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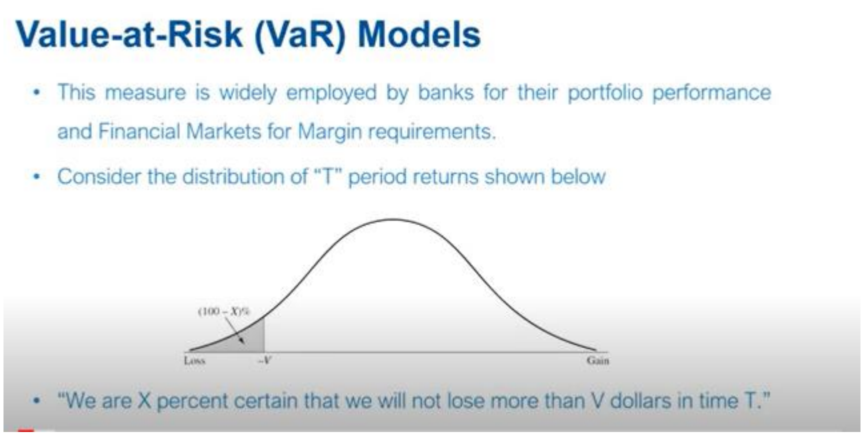
Lecture – 25
Week 8

In this lesson, we will discuss the Tail Risk Measures Namely Value At risk, VAR and expected shortfall or ES or Conditional Value At risk models. First, we discuss the theoretical underpinnings behind the VaR measure, then we provide the mathematical formulation behind the same. Then we conclude the discussion on VAR with a few important numerical examples. Next, we provide the shortcomings of VAR that offer the motivation to explore the CVaR or Conditional VAR measure. We introduce the CVaR measure and its useful properties. Then we provide the mathematical formulation to CVaR measure. Subsequently, we concretize our understanding of the C-VAR measure with a few simple examples. We close the discussion with a summary and concluding the marks.

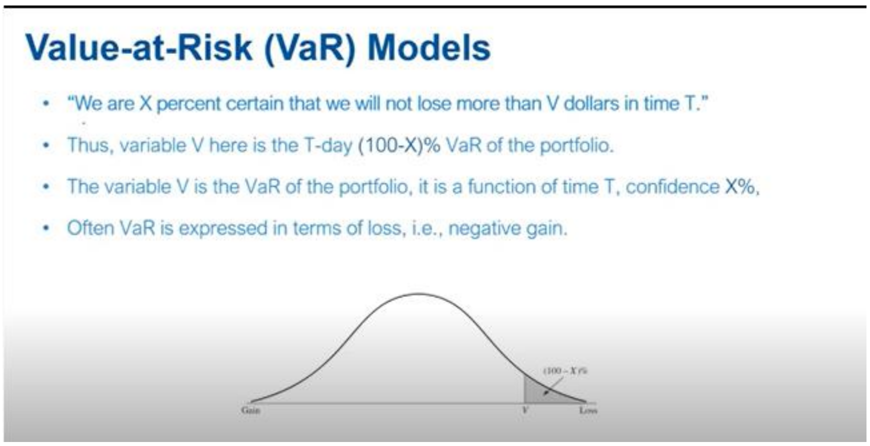
In the series of the next two videos, we will discuss a very important Tail Risk Measure that is Value at Risk Measure or VAR Measure. The measure is often employed in risk management by financial institutions.



For example, employed by banks in portfolio performance and risk management, employed by financial market exchanges for their margin requirements. Consider the distribution of T-period returns. It appears to be a normal bell-shaped curve like this. On the positive side, generally you have positive returns or what we call gains on the right side and on the left side you generally have losses or negative returns. Now let's say we want to make a critical statement about our position that we are X percent certain that we will not lose more than V dollars in time T.

So two important parameters of this VAR measure is time T, the period over which we are examining our data and our confidence or certainty level at X percent. Let's say this X percent is our confidence level, then in that case, if this is a gain distribution that means on the right side, we have gains and on the left side we have losses which are generally negative returns, then we are looking at 100 minus X percentile value. Which means if we want to be X percent certain, then we are looking at on this distribution 100 minus X percentile. For example, let's say if the value is 95 percentile, X takes the value of 95 percentile, then what we are looking at is 100 minus 95 which is 5 percentile of the distribution which should be somewhat here, 100 minus X and this would be generally a loss value, negative return or negative value associated with this. So we will say that this value V is the loss that we are X percent certain, in this case 95 percent certain that will not lose more than this amount, will not lose more than this amount 95 percent times in time T.

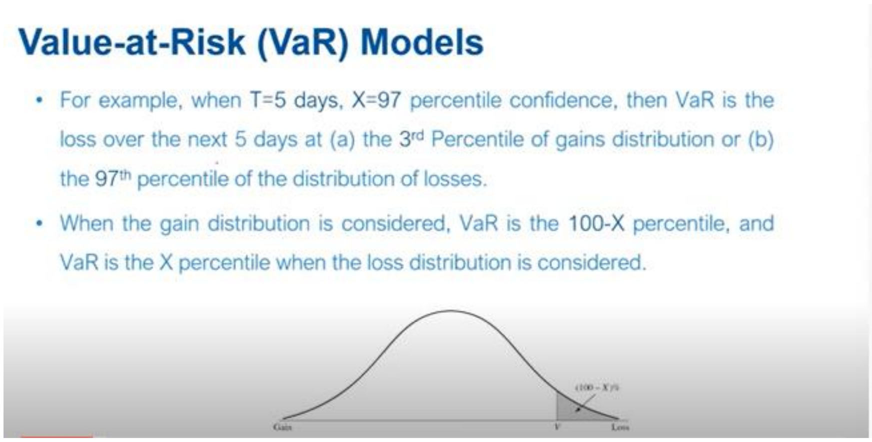
T is the time period that is under consideration which also means basically this is the area under the curve which is 95 percent. So, 95 percent chance is there that will not lose more than this value and what is this V? This is 100 minus X percentile on this distribution.



So, we are essentially making a critical statement that we are X percent certain that will not lose more than V dollars in time T, a period of T. It can be daily, weekly or any time period under consideration. So, the Variable V here which is the value that we will not lose is the T Day 100 minus X percentile VaR of the portfolio.

I repeat V here is the VaR on Tth day 100 minus X percentile VaR of the portfolio that is 1 percent VaR of the portfolio. This VAR Variable V which is the VaR of the portfolio it is a function of time, time period which is under consideration and confidence level X. Often this VAR is also expressed in terms of loss function which is opposite of gain sort of negative gain. In the loss function your gains will be on this side on the left side and your losses will be on this side. So if you are talking about loss function and we are associating a 90 let us say 95 percent confidence then we are looking at 95 percent percentile itself.

We are looking at this side right side of cut off which is 95 percent of the area sort of this area cumulative area of 95 percent and then in that case VAR is expressed in terms of losses or negative gain and we say that on the gain for negative gain or loss function this is 95 percent VAR on the negative gain or loss function 95 percent VAR. So, the cutoff point is 95 percent VAR, and we make again the same statement that we are 95 percent certain that we will not lose more than V dollars which is this cut off point V dollars in time T.

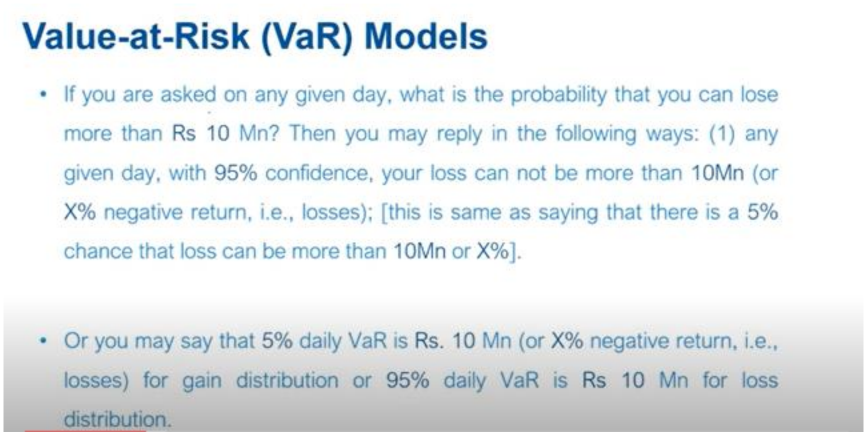


Let us put some values here for example, when T equals to 5-day horizon when we are looking at possible losses over 5-day horizon and the confidence level of 97 percent. So that means you want to be 97 percent confident then VaR measure value at risk measure is the loss over next 5 days either on gains distribution it is the third percentile value if you are looking at gains distribution then this is the third percentile value that means if it is a gains distribution here you have gain and here you have losses then in that case third percentile or if you are looking at loss distribution that means losses on this side and gain on this side then you are looking at 97 percentile this side 97 percentile of distribution of losses. So again to just repeat that if you are considering the loss distribution you have losses on this side and you want to be let us say 97 percent confident then this cut off value is 97 percentile that means the area under the curve area and this probability distribution this area is 97 cumulative area is 97 percentile or essentially this point is cut off point is 97.

In case you are looking at the gains distribution where gains are on this side positive returns and losses on this side then you are looking for V on the left side on the left side of distribution the losses are here at around 100 minus 97 which is 3 percentile. So this cut off value would be 3 percentile and both of the values will be same because there is a symmetry in the distribution so both the values will be same and whatever value you find V as VaR you would make a statement that you are 97 percent certain that over a 5 day horizon your loss will be no more than V dollars. To summarize, in this video we discussed and introduced the VAR value at risk measure. We noted that VAR on a given distribution of returns gains and losses it is sort of maximum loss that would occur over a given horizon or a given period let us say t days maybe 5 day or a week or month with a certain confidence level x that is maybe 95 percent or 97 percent. The interpretation changes depending upon whether we are looking at the loss distribution or gains distribution.

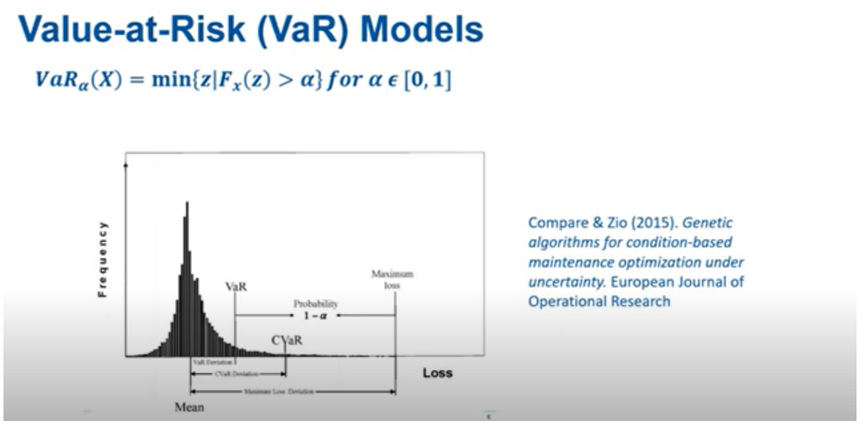
We said that on a loss distribution we look at the right side and if you want to be let us say x percent certain then we look at the cutoff point as x percentile which gives us the cumulative probability here on the left side and this x percentile becomes the cut off to estimate the VaR V on the right side. While when we are looking at the gains distribution where gain side is the right side and loss side is the left side then if you want to be experts in certain then we look at x 100 minus x percentile on the left side which is the loss side. So for example if you are looking at 95 percent confidence then you look at the 5 percent cut off value on this side which is the V or rather it will be negative some kind of loss value which is your potential losses maximum potential losses over a given horizon t with the confidence of 95 percent.

In this video we will continue our discussion of VAR and we will provide a more formal mathematical definition of the theoretical concept that we have discussed till now.



Recall how we expressed VaR we said that if you are asked on any given day let us say we are talking about daily periods and we say that what is the probability that you can lose more than 10 million.

So you may reply by saying in the following manner you may say that on any given day if daily is my horizon I am looking at daily periods then with 95 percent confidence I cannot lose more than 10 million on my portfolio or it can also be expressed in terms of rupee or dollar value or also the return negative return amount which is essentially losses. So, I can express my loss maximum loss for this 95 percent confidence as some value in rupee or dollar amount as 10 million or something or negative return of x percent. This is also if you think of it this is also same way of saying that though there is also a chance of 5 percent that my losses can exceed I repeat there is also a chance or there is a way of saying that there is a 5 percent chance that my losses can be more than this amount of 10 million or x percent and therefore we may express our VaR value in two ways. We can say that when we are looking at gains distribution then we are looking at left tail for losses then we may say that 5 percent daily VaR is some loss amount like 10 million or x percent negative return that is losses or if you are looking at loss distribution then you may say that I am looking at right tail and then we can say that 95 percent daily VaR is 10 million or the amount that I think that I would lose the maximum amount that I would lose with 95 percent confidence.

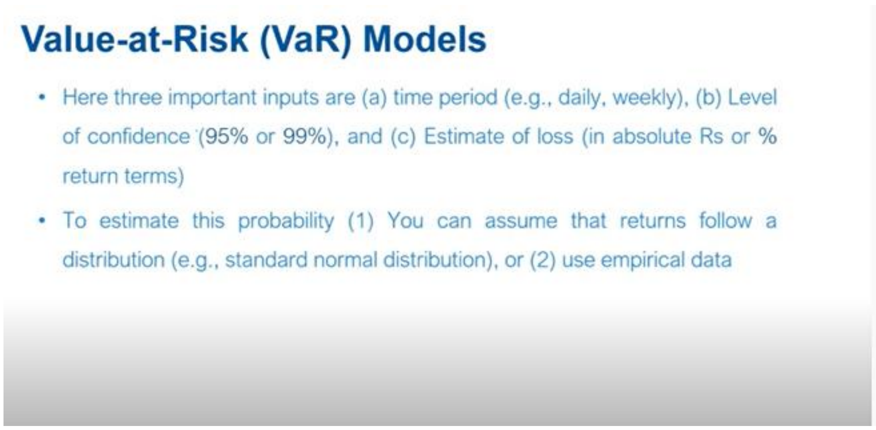


Think of a return distribution rather loss distribution appearing like this for now we will ignore this CVAR and the extreme tail we are talking about VaR only so we are looking at normal scenarios with some 95 or 99 percent confidence.

In that case on y axis we have frequencies or probability distributions probability densities rather and on the x axis you would have losses on the right tail and gains on the left side. We can express losses in rupee dollar amount or also in percentage return terms but we generally tend to convert them into what we call as standard normal distribution or z values or z distribution or z distribution which is z equal to x minus mean upon standard deviation where x can be for example returns or rupee dollar gain losses mu is mean and sigma standard deviation so you convert to make it more easy to interpret you convert into z values or standardization this is called standardization and that is why when you have z values or z values on x axis this is called standard normal or some kind of standardized distribution. Now when we are talking in terms of standardized distribution let's say you are looking at 95 percent VaR because we are talking about loss distribution so we are saying 95 percent VaR that means essentially we are looking for a loss value on this side or corresponding z value to a loss where the area the cumulative area on the left side is 95 percent why 95 percent because we want to cover a 95 percent chance that we are well covered we want to attach a 95 percent confidence so we are looking at the cumulative probability on this side at 95 percent. Now we are looking at a z value that gives us this confidence or this cumulative probability on the left side of 95 percent or rather alpha equal to 95. Once we have this z value we can convert it into the actual rupee dollar loss rupee dollar loss or loss in the form of return percentage x percentage but we are defining or providing the formal definition of VaR in this z terms.

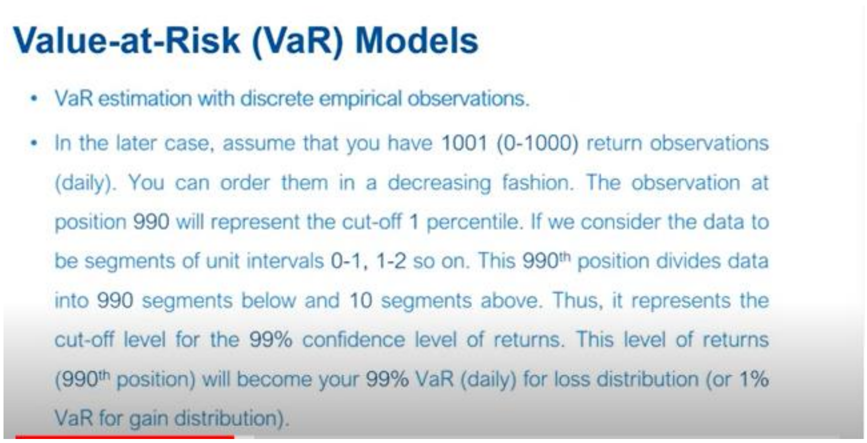
So this is nothing but the definition of percentile itself which is to say that this is the minimum value of z this is the minimum value of z for which the cumulative probability almost exceeds this 95 you can express it as greater than equal to or greater than which will almost because of the continuous distribution it would not change much. So, this is an amount which is greater than equal to alpha which was 0.95 or 95 percent. So this is nothing but the definition of percentiles only and we are looking for a cutoff point cut off point z which gives us the cumulative value of 0.95 here on the left side of it which on the loss function becomes our percentile.

If you are looking at the gain function we would have looked for the 5 percentile but we are looking at loss function so we look at 95 percentile and that becomes our VaR value of x. Later we will convert this z value into corresponding x loss value in terms of rupee dollar or percentage terms but this is the formal definition of our VAR measure that is the minimum value of z which has a cumulative probability on the left side equal to that cut off which is nothing but the definition of percentile itself. To summarize this video, we provided a more formal mathematical definition of VaR based on previous discussions that we had previously we discussed in a more qualitative manner but now we have provided a more formal mathematical definition which is same as the definition of percentiles or quantiles. In this video we will conclude our discussion about theoretical underpinnings of value at risk or VaR models. We will also discuss how to compute VaR from empirical dataset.



To conclude our VaR discussion, we said there are three important inputs to define VaR or value at risk for a position or portfolio. First the time period the daily or weekly time period over which we are going to estimate the losses. Second the level of confidence which leads us to probability values may be 95% confidence or 99% confidence. Last and a very important measure which is the estimate of loss which can be in absolute rupee dollar terms or percentage return terms. Now the missing piece here is how to attach this confidence level to the estimate of loss and that requires us the distribution of returns or losses may be a standard normal distribution or we can have empirical data.

In case if we have standard normal distribution of returns daily or whatever horizon we are interested in that periodic return distribution we have we have already seen the mathematical formal definition of VaR to estimate the loss using this standard normal distribution. The percentile definition we have already seen applying which if we have the standard normal distribution for daily weekly or any kind of periodic returns, we can estimate this loss by modeling the returns through the distribution. But what if we have the empirical data then what to do.



So let's discuss how to estimate VaR when you have discrete empirical observations rather than although more desirable is to have some kind of distributional properties like standard normal distribution but in the absence of that how to estimate with the empirical discrete observations and for a case for explanatory purposes I'll use a rather simple case that I have 1001 observations which includes starting from observation number 0 to 1000. Here it is interesting that I am noting the first observation as 0 you could have also noted it as first but then the last observation is 1001, you'll see why I'm doing this.

Now let's say you are interested in daily horizon, so these are daily return observations. The first thing you can do is you can order them in a decreasing fashion so the maximum return is let's say one observation number 1000 and the lowest return is observation number 0 or 1 to 1001 that also you can pick. In that case the observation number which is 990 is sort of a very important point here. Let's say I'm looking at 99% cut off or 99 percentile value then the observation number 990 you can think of the entire set as 1000 periods so the 99% period will end sort of will cut off the entire segment into two parts so this 990th observation will essentially cut into 99% and 1% two segments starting from this maximum value this 990th value will be the 99 percentile. Now think of this data into segments and this 990th observation will divide the upper right side into 1 percentile segment and therefore it represents that cut off level or that 99% confidence level that we are interested in that means if I choose this cut off value than 99% of the intervals will fall on the left and 1% on the right this 1% is the extreme sort of observations and therefore this cut off point becomes my 99% daily bar because it distributes the two or segments the entire horizon into two segments of 99% versus 1% so this becomes and obviously because we have started the maximum value here and minimum value here so this is our loss side where negative returns or negative losses are there so this cut off value will become the point at which we can be 99% confident that my maximum loss will be this value.

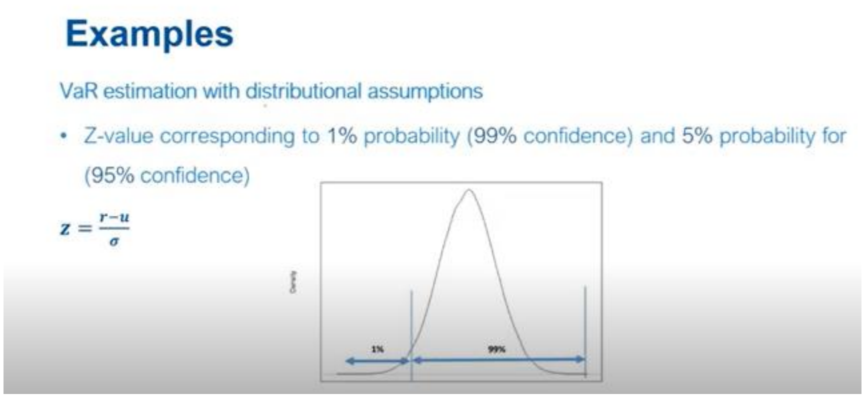
Similarly you can think of 95% so then I can use the observation number 950 as a 95% cut off because the observation because the 950th this 950 observation starting from this 0 to 1000 will distribute this segment into 95% length here and 5% here so this becomes my 95% cut off that means 95% of the observations on the left and these are the 5% extreme loss of the sort of observations so I can say with 95% confidence that this will be my maximum loss. It is customary to have this side there is nothing theoretically that stops this side also having positive returns or not exactly losses but lower returns but generally in a practical sense you will tend to have negative numbers of negative return numbers here. To summarize this video, we discussed and concluded our understanding and theoretical discussion of VaR. We noted that while it is desirable to apply some kind of distribution like standard normal distribution to model the returns and obtain VaR measures but also often you have discrete empirical observations and how to model our returns with these discrete empirical observations and find certain VaR values like 99% VAR and 95% VaR that we saw in this video. In the next two videos we will try to reinforce and concretize our understanding of VaR value at risk through two simple numerical examples.

Let's start with this first example where a sort of hypothetical gamble is designed that from a loss of 50 million to a gain of 50 million all outcomes are likely over a period of year. So, this is a sort of example of a uniform distribution that means the outcomes are distributed on probability distribution table where on y axis we have probability densities, probability PDEs, probability densities on x axis we have possible outcomes. They start from a value of minus 50 and they end up with a final value of plus 50 million. You can think of mid value or mean value as zero, but the point here is that this height is same for all the outcome that means all the outcomes are equally likely. Also please recall that this is a version of gain distribution because here on the right side we have gains and left side we have losses.

The other way round would have been starting from 50 million gains here and losses here then this would have been the loss distribution. So now if you are looking at either 1% on the gain distribution which is on this side 1 percentile cut off or sort of 99 percentile here. To simplify things on either of these graphs you can think of 100, 1 million intervals of 100 pieces where for example this is minus 50 to 49 and 49 to 48 and so on starting here 49 to 50. So, you can think of 100 such intervals of 1 million dollar each or INR 1 million rupees of each. You can construct such 100 intervals.

Now if I am interested in 1% work then means on the gains distribution either I am looking at this cut off which is at minus 49 million or 49 million loss or if I am looking at loss distribution either I am looking at this 99 percentile point cut off which is nothing but again minus 49 million rupees loss. So, either I look at gain side or loss side if I want to have that 99% confidence then I am looking at 1% on the gain side which is this or 99% on the loss side which is this and both values are same as 49 million losses. Similarly, if we are looking at 95% confidence then let us start with the gain function. This is minus 50, this is plus 50 and we are looking at 95% confidence, which means on the gain distribution we are looking at 5% cut off which is 100 minus 95 which is this 5% cut off. Now it is too easy to know that this will be 5 intervals of 1 million rupee each which will end up at what point 45 million.

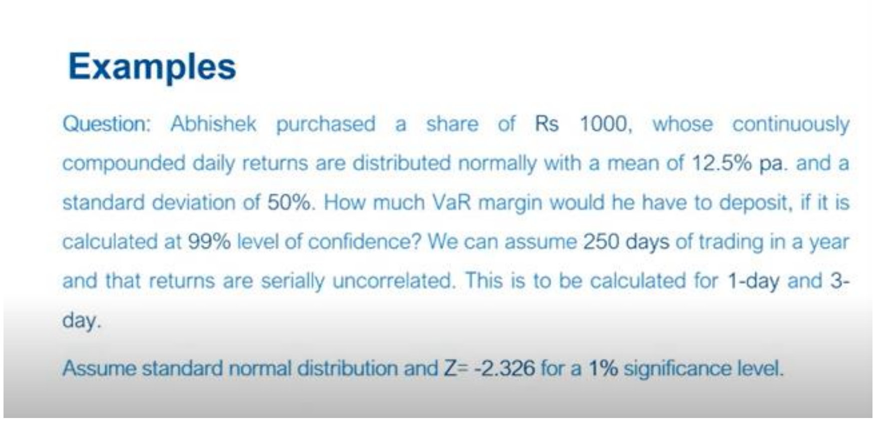
So this would be the 45 million loss point or minus 45 million. Similarly, if I am looking at the loss function then essentially it would be 50 to minus 50 and in that case also, we are looking at on the right side 95% cut off point which would again be a loss of 45 million. So, because this was uniform distribution things were rather simple for us to understand and estimate the cutoff points at 95% and 99% confidence levels. To summarize this video here with a simple example numerical example where the approach employed uniform distribution, we saw how to compute VaR for 99% and 95% confidence levels. In this video we will discuss a more systematic approach to VaR estimation with the help of probability distributions.



Before we start with our numerical example, please recall that when you are making an assumption about probability distributions you need to connect that distribution probability distribution with your return data. For example, take a case in point with normal distribution. Generally, when we are talking about normal distribution, normal probability density distribution we tend to use standard normal distribution which are defined by z values. So, on x axis you have z values, so you have to convert let us say you have a return data in percentage returns or continuously compounded returns then you have to convert this return data into z values or z distributions. How to do that? You subtract the return data from its mean and divide it by standard deviation it is a pretty simple procedure that is R return data minus its mean mu which is the average of return divided by standard deviation sigma so this is how you get the z values.

Now instead of return values on x axis you have z values and as we will see shortly it has lot of beneficial properties. Now let us say you are interested in sort of 99% confidence interval and assuming that is regular gain distribution where on positive side we have on the right hand side we have positive returns sort of gains and on left side we have negative returns or losses then in that case if you are looking at 99% confidence then essentially your 1% area will be this. This will be your 1% area where you are interested in. So, what you will do is use the benefit or sort of properties of normal distribution to find this cut off point. Cut off point in the sense that right hand side probability cumulative probability on the RHS is 99% and therefore this becomes your cut off value at 1%.

Similarly if you want to find the cut off value or sort of 95% confidence level the procedure remains identical. All you need to do is find a point to which right side cumulative probability is 95% and therefore this left side tail area becomes 5% which is the cutoff point in z terms you will obtain. Once you have the z value corresponding to this cut off point then you can convert the z value into return form and subsequently compute the losses in actual rupee or dollar terms. This was about gains distribution so now if you are thinking in terms of loss distribution you have to follow a similar procedure only that now on the right side you will look at 95% or 99% cut off depending upon your confidence level. So now let's start with our numerical example.

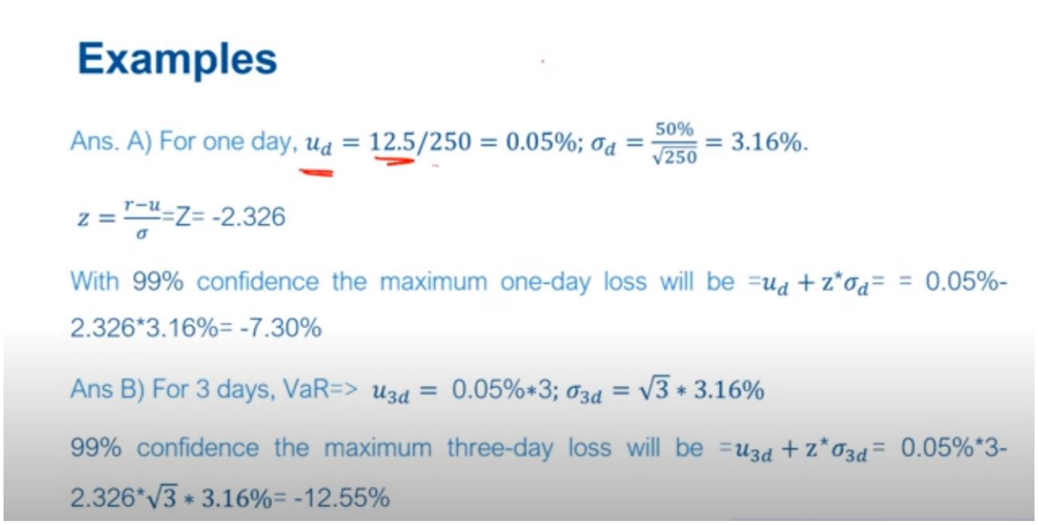


The example is as follows. Abhishek purchased a share of Rs. 1000 where continuously compounded daily returns are distributed with a mean of 12.5% per annum. So, this 12.5% is per annum so we need to convert it back to daily level and a standard deviation also of 50% per annum.

So we will convert this back as well at daily level. The question is how much value at risk or VaR margin needs to be deposited if you want to have 99% confidence. Also, it is given that there are 250 days of trading in a year and returns are serially correlated. This is very important assumption before modeling or employing any kind of distribution that returns are serially uncorrelated. Particularly when we are translating volatility across different periods from annual to daily and so on then this assumption is very important.

Now we are supposed to compute the VaR margin for 1 day and 3-day period and it is given that z value corresponding to 1% significant level is minus 2.326. The application of this z value goes like this. So if we recall our previous discussion this 1% point is corresponding to minus 2.326 here z value and the right side probability would be 99.

Similarly if we were looking at the loss distribution then on this side z value corresponding to 99% is positive 2.326 you can employ them either way either looking at left side or right side. The value is symmetrical only that here it is negative and on ARCH it is positive. So let us start with our numerical. First, we will compute the returns on daily because our horizon is on a daily basis.



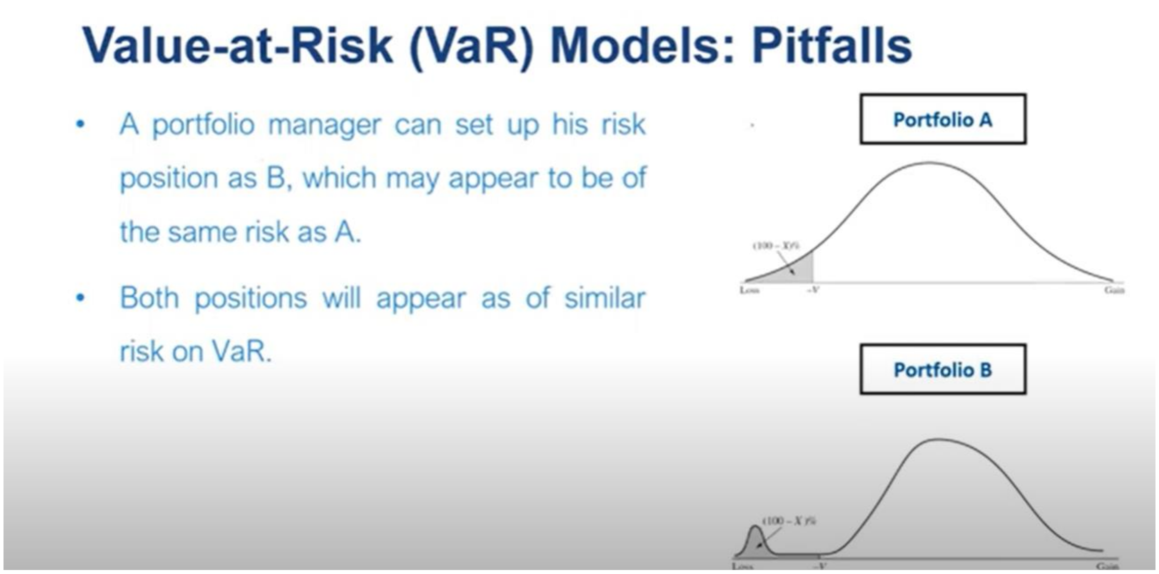
If you recall the period was one important input what period, we are looking at that was one important input in our analysis. So, 12.5% is given at the annual level. So, let's compute it at daily level and it turns out to be 0.05% which is we got after dividing by 250. Why did we divide by 250 because there were 250 dividing days in the year, so we divided 12.5% by 250 to get 0.05%. The computation of standard deviation from annual to daily is not as simple. The formula for any periodicity is sigma t is equal to under root t into sigma for one period.

$$\sigma_t = \sqrt{T} * \sigma_1$$

So here the period is total 250 days so I need to divide it in order to compute it for daily I need to divide it by 250 that is sigma t upon square root t equal to sigma 1 this is what we needed to do. So, we have 50% total volatility annual basis, so we divided by total trading days sigma root 250 to get this 3.16%. Now the last missing piece is that Z value which is given to us minus 2.326 so that means we are looking at the sort of gain distribution, so we are looking at the left side of it and Z value is -2.326. All we need to do is connect this Z value and find a corresponding return value on the normal distribution. Already this Z value gives us that 1% sort of VaR number or maximum loss with 99% confidence. So how do we compute it the formula is simple mu mean mu D plus Z into sigma D mean?

$\mu_d + z * \sigma_d$ is 0.05% Z value being negative at minus 2.326 into sigma which is 3.16% which we got here so our return value is minus 7.3%. Now let's say if your portfolio value your investment in this portfolio was 1000 rupees then in that case the losses that you can expect on any given day the maximum loss with 99% confidence is 73. I will repeat this a very important statement that I am making from vast perspective that one day VaR one day VaR with 99% confidence or 1% significance here was 7.3% in return terms or in value terms loss of 73 rupees with 99% confidence this is my maximum possible potential loss over a daily horizon with 99% confidence.

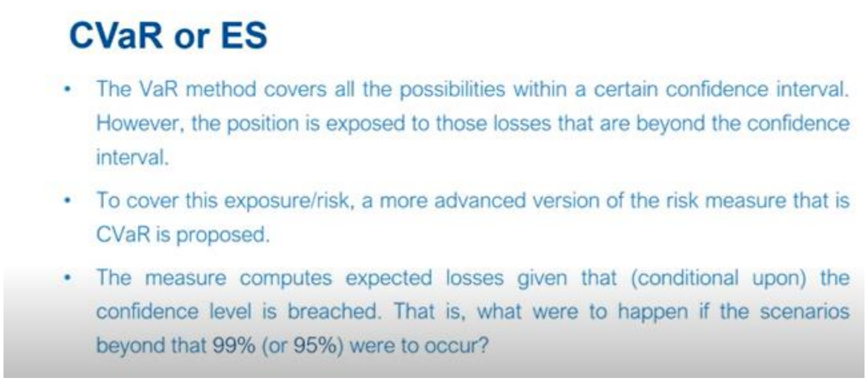
Let's see how to work this number for 3 days. So first we will convert our return to 3-day period by multiplying the daily return with 3 so we get this number and second we also need to multiply the daily volatility sigma one day which is 3.16% with square root 3 so this is nothing but under root D multiplication to one period standard deviation. So now that we have our return and standard deviation values it's very easy to compute the potential maximum 3 day loss with 99% confidence it's exactly identical procedure we have mean return plus Z into sigma 3D mean we have already computed as 0.05% into 3 minus 2.326 into square root 3 into 3.16% which works out to slightly higher than minus 7.3 at 12.55% which is to suggest that if my portfolio is of 1000 rupees then the maximum 3 day loss the maximum potential 3 day loss with 99% confidence would be 125 rupees 50 paise that means 125.5 in loss terms. So to put it in simple terms what we are saying that your loss value for a 3 day period with 99% confidence is minus 12.55% in negative return terms or loss value or let's say if you are holding 1000 rupees worth of portfolio then your loss with over a 3 day period maximum loss over a 3 day period with 99% confidence is 125.5 rupees. To summarize in this video, we saw how to apply a normal distribution to compute value at risk measure in a more systematic manner. In this video we will conclude our discussion about value at risk models with their shortcomings and therefore the motivation to study expected shortfall or what we call conditional value at risk or CVAR models.



Let us consider the profile of two portfolio managers or fund managers A and B while up till a certain point this point the value remains same however portfolio B differs in the sense that it has taken certain position in certain securities where in the extreme tail, extreme tail maybe 5% or extreme tail of 1% the loss probabilities are extremely high.

So we can see extremely large possibilities but these are hidden these probabilities are in tail manner so to a portfolio manager who is specifically focused on the value at risk that is the 95% scenarios may not be able to look at these loss measures because he is solely focused on this region which is identical for both the positions but the main difference lies here in this loss which is an extreme loss which is not for portfolio A but for portfolio B. The question you may ask why portfolio manager of portfolio B manager would be interested in such a position maybe because in this position on the positive side also he may have some kind of upside he may witness some kind of upside on the positive side and if the positive side were to appear his bonus would be very high but his sort of negative side his performance is judged based on this bar number and his performance is not evaluated based on this loss that may happen if extreme scenarios were to take place because this is hidden as per our model so if he is judged or evaluated on bar model he will be penalized only for this value V not for this extreme risk that he has taken because this risk appears in the extreme tail and not captured by the bar so to somebody who is looking only at a VaR as a measure both the positions portfolio A and portfolio B will appear as of similar risk on the VaR. Like I said his motivation would be probably there is some on the positive side some excess gains that he is expecting because of this extreme risk taking and this motivates us to look at something called conditional VaR or expected shortfall models that precisely look at what were to happen if indeed some extreme events were to take place. So, if this extreme events were to take place then what would be his losses, that is what we examine in conditional value at risk or CVaR models or what we call it expected shortfall models that are the topic of next set of videos. To summarize this video we discuss the shortcoming of value at risk models that is they are not able to identify if there is some kind of extreme risk position on negative tail and this motivates us to study what we call as expected shortfall or conditional value at risk models.

Starting from this video and in next few videos we will introduce and discuss a very important measure of tail risk that is conditional VaR or expected shortfall.



Previously in the VaR or value at risk measure we said how bad things can get but now here we are saying if things do get bad what is our average expectation or our expected losses. So, the VaR method covers all the possibilities with a certain confidence level. If that confidence level were to be held then what is my estimate of maximum loss. However, there are chances that this position is exposed to those losses that they are beyond confidence level. So if you recall the area that we ignored in the VaR is the tail region.

So whatever we said we stopped at this cut off point which we called as VaR but here we also look at that tail region. So we are more interested in the tail region, that is if these extreme scenarios were to materialize what would be our losses. And to cover this kind of exposure the extreme tail losses a more advanced version of risk measure that is CVaR or conditional VaR or expected shortfall is proposed. This measure computes expected losses given that or conditional upon the fact that confidence level is breached in mathematical terms that means your losses exceed your

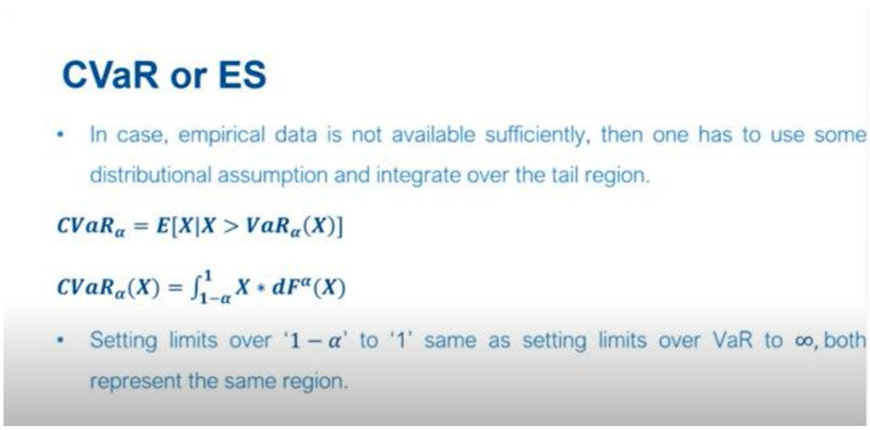
VaR level. So VaR level tries to compute certain level of maximum losses this VaR level if this VaR level were to breach then what were to happen if this VaR level was breached then what were to happen that is the scenarios beyond a certain confidence like 99 or 95 percent have been breached.

So in simple terms essentially with CVaR or ES what we are saying that earlier we had a VaR measure which was this which gave us idea of how bad things can get but if things do materialize the bad things do materialize that means this VaR level is breached and conditional to the fact that our losses are more than this VaR that means we are in this region then what is my expectation of losses how much losses I can expect.

Let's put some numbers here so for conditional VaR CVaR or expected shortfall also we need to compute the VaR measure again and then only we can proceed with the CVaR. So, let's say we have for my X value of 99 percent confidence for over 10 days period T equal to 10 VaR is given as 10 million that means with 99 percent confidence we can say that over a 10 day horizon our maximum loss would be 10 million. So, this is the maximum loss with this much confidence in this period. Now what C VaR or expected shortfall says is that if my VaR is more than 10 million assuming that this 10 million is breached that means my losses are more than 10 million that means we are beyond this 99 percent level so if there was this 99 percent cut off which was my VaR at 10 million given that this 10 million level is breached and now we are talking in this level what is my expected level of loss for a 10 day period given that this loss is greater than 10 million or this 10 million VaR level is breached.

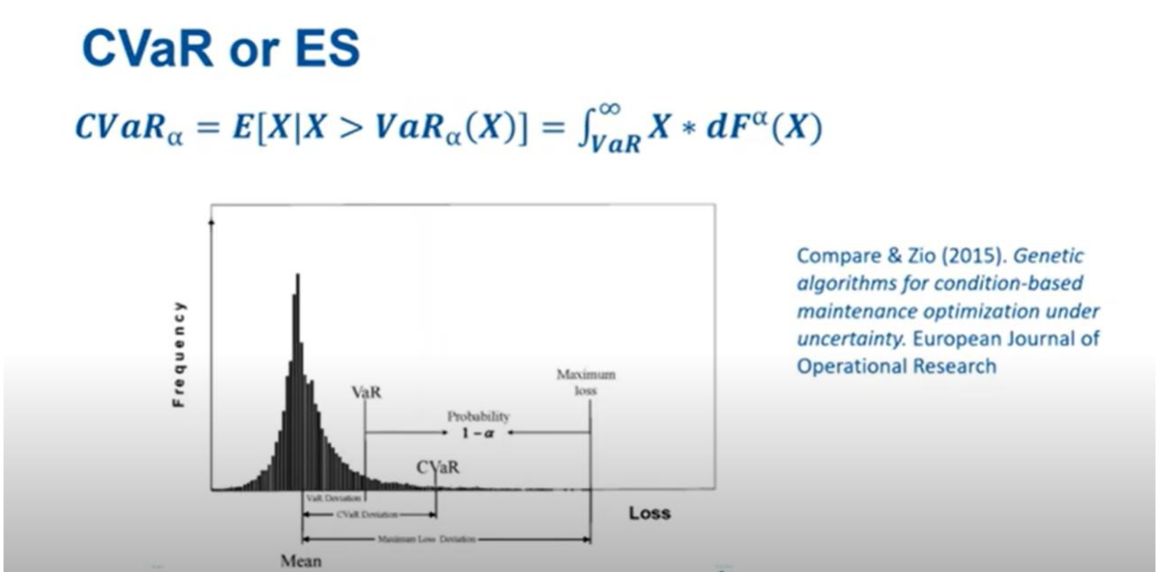
So in simple terms what we are saying that we have already assumed that this this level is breached we are already beyond this redemption level of this much and we are already dealing with this level and in that case what is my expected losses. Now the intuition that I want us to understand is that in this particular case if this is your risk measure recall our one of the two portfolios portfolio A and portfolio B where we said that with VaR level there is a motivation to set up this kind of portfolio where there is an extreme possibility in the tail this extreme tail possibility of loss with this kind of C VaR or expected shortfall measure the tendency of fund managers to set up this kind of position where there is a in the tail there is extreme negative loss possibility that will be mitigated. So the motivation for portfolio manager B to set up this extreme position where there is an extreme loss possibility in the tail which probably may not be identified by the VaR measure but identified by the CVaR measure. So if the management or holders of this fund are also looking at CVaR measure they would easily identify this kind of position and therefore the motivation and probably the fund manager will be penalized and therefore the motivation to set up this kind of position is less. To summarize in this video we introduced our CVaR or conditional VaR method or what we are calling as expected shortfall we saw how it improves upon the conventional VaR method and how it demotivates the fund managers it demotivates the fund managers from taking positions that may entail a very sizable risk in extreme tails which probably would not have been identified by the simple VaR models.

In the previous videos we have discussed the theoretical underpinnings and provided the basic understanding of CVaR or ES measure. Now in this video we will concretize and reinforce those understandings with mathematical formulation and providing the mathematical underpinnings or a way to measure this conditional VaR or ES measure.



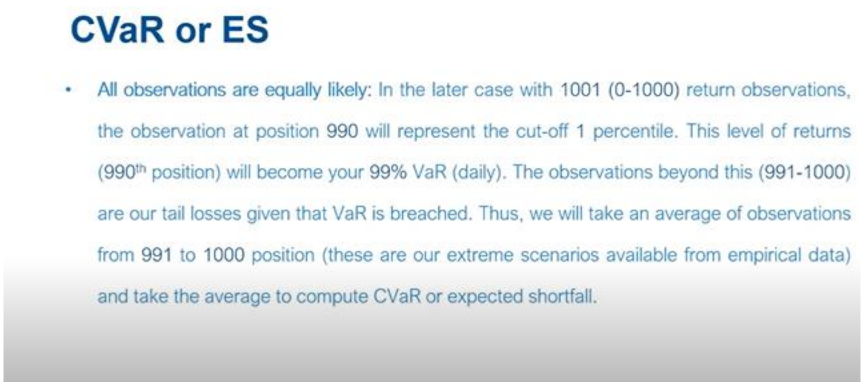
So often there are two ways like VaR there are two ways employed to compute ES either you can use the empirical data if empirical data is sufficiently rich data is available but if not then when you can use some distributional assumption and integrate over the tail region. What it means is that by basic definition the basic definition of CVaR is nothing but the fact that if this X is my loss let's say I'm looking at the loss Variable X given that this loss Variable X is greater than the VaR level so already we are assuming that VaR level is breached so conditional to the fact that this loss is greater than VaR of X then what is the expected value of that loss X. For a continuous normal distribution kind of probability distribution let's say we are looking at the loss distribution that means this side we have losses and this side we have gains.

So let's say this is my VaR level at one minus alpha probably we are looking at 95 percent confidence level which means this is my 95 percent level one minus alpha and this is my VaR so I would like to integrate I would like to integrate the loss into probability density let's call it DF so I would like to integrate this value from this one minus alpha this is the cumulative probability one minus alpha for this cut off point up till one so one is the final last point so one minus alpha to one I would like to integrate this X DF where we know that this F is a probability density function in this case it is a normal distribution so we sort of integrating over normal distribution from one minus alpha to one. Now if you recall the cutoff point the value of X corresponding to this one minus alpha is nothing but the VaR value and on the right side it goes till infinity so theoretically it can be in finite level so integrating this X DF from one minus alpha to one is almost same as integrating X with values from VaR so one minus alpha has a corresponding value of X as VaR and one as a corresponding value of infinity so the same can be transferred in terms of X as integrating VaR to infinity which have the same interpretation.

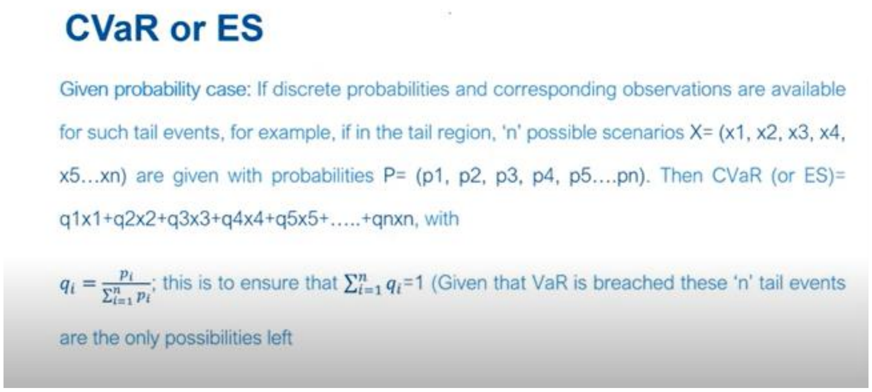


So essentially what we mean here is that if we are looking at this loss function this is our cut off VaR point which corresponds to a cumulative probability of one minus alpha maybe 95 percent or 99 percent and starting from this point one minus alpha till one so the point the maximum loss point which theoretically is an infinite value has a cumulative probability on the left side as equal to one so sort of integrating from one minus alpha to one which is integrating from VaR value to infinite value and therefore the original formula where we said that given that X is greater than VaR at certain level of significance maybe five percent or one percent if it is five percent then one minus alpha is 95 percent if it is one percent then one minus alpha is 99 percent depending upon this number given that X is greater than this VaR what is the expected value of X which is to suggest that integrate X starting from VaR so we integrate at lower limit VaR up till infinite the maximum possible theoretically infinite but in real life you may have based on your empirical data you may have some expectations of the maximum possible loss integrated Df X where F is the probability density function so generally we assume normal distribution to describe returns so this Df would be the probability density function for a normal distribution. To summarize this video, here we provided the mathematical intuition and also the formulation of equations to describe the C-VaR or expected shortfall measure. In the previous video we saw how to approach the problem of C-VaR or ES estimation when probability density distributions are given sort of continuous probability density distributions are given.

In this video we will see how to compute conditional VaR or expected shortfall using empirical data.



Recall our computation of VaR with the empirical approach we said that if 1000 observation 1001 observations are given we can segregate them let us say we are looking at 99 percent VaR then we can find the 990th position we will order the data starting from maximum to minimum and we will find the 990th observation the idea was that this 990th observation which sort of segregates the entire observations into two length segments one is having 99 percent observations there is 1 percent observation similarly for 95 percent we will find the value at 950th and this 950th will segregate 95 percent and 5 percent length segments. Now with the C-VaR or ES the idea is that this value this corresponding value of loss is already breached that means we are already on the right side of it on the negative loss side and therefore the way to approach this problem is to find all the value let us say we are looking at 99 percent expected shortfall or C-VaR then the idea is that to pick up all the values that are higher than this 990th position on the right side of it all the values that are on the right side and take the average of them that average would be the expected shortfall or expected value of loss given that 99 percent VaR is breached. Similarly think of 95 percent VaR so this is the 95 percent cut off then you look at all the values on the right side of it because we have ranked them from max to minimum all the values on the right side of this 950th would be part of those 5 percent sort of extreme loss observations that is scenario where maximum loss is materialized and therefore once you put this cut off of 950 observation as the VaR value as a cut off minimum cut off and take all the values that are a higher loss in terms of magnitude they are higher that means indicating loss on this side you take the average of all those x1 plus x2 plus and so on and take their average if there are no observations this will be your expected shortfall or average loss given that extreme scenario of 5 or 1 percent were to materialize.

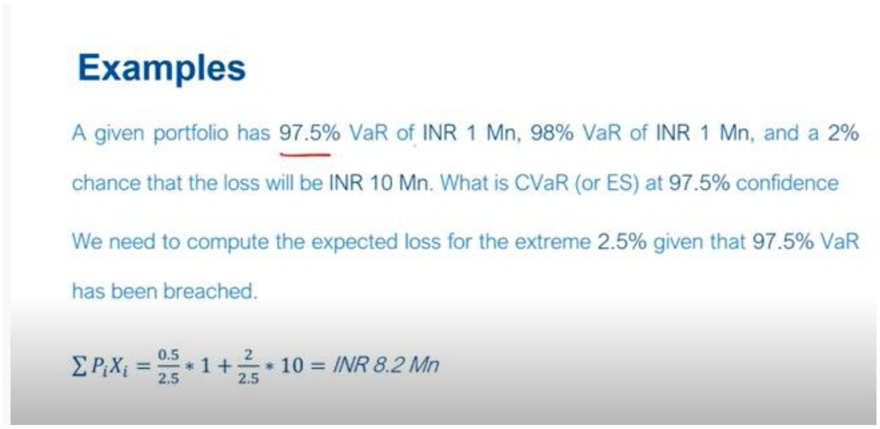


Now let us generalize this understanding suppose in a given data you have been given probability mass function that means let us say there are n observations in the tail in the empirical data and tail observations are already given to you and for those n tail observations the observation loss observations are x1, x2, x3, x4 up to xn these are the losses possible scenarios losses in possible scenarios in the tail region negative losses and the corresponding probabilities as p1, p2, p3 and up till p1.

So in that case when the probabilities are also given as a probability mass function rather so the probabilities are given and the values are also given then your conditional VaR or s measure is simply nothing but the expected value like this Pi into Xi summation i equal to 1 ton which is nothing but this one which is q1, x1, q2, x2, q3, x3 and so on. Now here this q1 is different from p1 how? Please note once you decide once you once you have decided that your VaR level has breached once you have decided that your VaR level has breached then only these are the possible scenarios. So for example in the actual universe of population let us say this p1 was 5 percent in the original scenario but now in the revised scenario conditional we are looking at conditional probabilities conditional that these are the only possible events so there may be some other events which are on the left tail which are on this side as well but that area is now negated so we are only looking at these possibilities so what it means is that now we have to look at the conditional probability q1 that conditional probability is p1 upon summation of p1 to n. $q_i = \frac{p_i}{\sum_{i=1}^{n} p_i}$

Why we are doing this because we already know that this region is breached so these possibilities are ignored and only these are the remaining possibilities therefore their summation has to be 1. But in original scheme of things this was not 1 probably this is 5 percent or 10 percent or 1 percent depending upon our criteria for VaR and therefore if this is the case then in order to standardize or sort of obtain the conditional probability qi we compute qi as pi upon summation i equal to 1 ton pi.

Now how it is done we will see with the help of a numerical example in the next set of videos. To summarize in this video, we saw how to compute CVaR or expected shortfall from a given set of discrete probabilities or probability mass function if empirical data is employed. In the next two videos we will try to concretize our understanding of CVaR and Es with the help of two simple numerical examples.



Let us go through this example here it is given that 97.5 percent VaR is 1 million 98 percent VaR is 1 million and then a 2 percent chance that loss will be 10 million and we are supposed to compute CVaR or expected shortfall at 97.5 percent confidence. To simplify this problem because these observations are discrete we will assume some kind of uniform distribution in the tail that means at 97.5 percent VaR it is given to us that this is 1 million level we will assume it to continue till 98 and beyond 98 it is given that there is a 2 percent probability that loss can be 10 million. So there are two particular discrete regions available to us A and B and it is given while computing CVaR we assume that this 97.5 percent is breached and the only two possibilities are A and B.

So the corresponding matching probability of 0.5 percent here and 2 percent here needs to be standardized to 1 that means now probability of A plus probability of B has to be 1 conditional because we are dealing with a conditional event conditional to that VaR is breached and therefore we are in this region only. How to achieve this? So the idea here is to make this summation as 1 and therefore we need to standardize these probabilities on a scale of PA plus PB because now this new P(A) plus P(B) supposed to be 1 will standardize the probability of 0.5 by dividing it with P(A) plus P(B) which is 2.5 originally and will standardize the other probability which is 2 percent also by standardizing it with 2.5. So there now the summation P of A plus P of b this becomes 1. So, in that case our new expected loss would be P(A) which is 0.5 upon 2.5 into 1 million plus 2 upon 2.5 into 10 million which is our expected loss which is equal to 8.2. What is this 8.2 million? This 8.2 is our expected loss given that our VaR of 97.5 percent is breached that means given that this level was to breach at 1 million what is the possible scenario, and that value is 8.2 million is our expectation of loss. In this video we will conclude our understanding of conditional VaR or expected shortfall with the help of simple numerical example.

Examples

A portfolio has 95% VaR of INR 1 Mn, there is a 3% probability that a loss of INR 2 Mn, 1% Probability that a loss of INR 5 Mn, 0.75% probability that a loss of INR 10 Mn, and 0.25% probability that a loss of INR 20 Mn may occur. What is 95% ES

$$\sum P_i X_i = \frac{3}{5} * 2 + \frac{1}{5} * 5 + \frac{0.75}{5} * 10 + \frac{0.25}{5} * 20 = INR\ 4.7\ Mn$$

Let's see how this numerical example works out. So we have been given that a portfolio position has a 95 percent VaR at a 1 million level rupees 1 million then there is a 3 percent probability that a loss of 2 million may happen another 1 percent probability that a loss of 5 million may happen 0.75 percent probability that a loss of 10 million and a 2.0.25 percent probability that a loss of 2 million may occur. Now we are supposed to compute 95 percent ES expected shortfall or C VaR which means it is given to us that what were to happen what is the expected shortfall or expected loss if this 95 percent VaR of 1 million is breached.

Essentially what they are saying is let me help you visualize this what they are saying that given this 97.5 percent VaR is breached this level is breached what were to happen next. Now it is given that let me plot a level of 2 million here with a probability of 3 percent. So, this is 95 percent sorry. So this is 3 percent that means 95 to 98 there is a 2 million loss we are assuming a uniform kind of distribution because these are discrete observations.

Next from 98 to 99 there is let me plot a slightly less thick bar 98 to 99 1 percent probability this is 1 percent previously it was 3 percent so now it is 1 percent and the height of this is 5 million it is given to us as 5 million. Next another 0.75 percent and I will draw rather thin bar to indicate that this is 0.75 percent in this region first is A this is B this is C and height is 10 million.

Another very thin bar with a very remote possibility of 0.25 the region D which has an area of 0.25 20 million. So, these are the four regions. The interesting thing here is that in the previous case if I sum up all the probabilities corresponding to event A which is this A this is B this is C 0.75 and this is D if I sum this up P(A) to P(B) plus P(C) plus P(D) it would add up to 5 percent but now the world has changed it is given to us that this 95 percent level has breached and these 5 percent are the only four possibilities with us that means conditional to the fact that this 95 percent level VaR has breached these probabilities are the only probabilities and therefore they should sum up to 1 that means probability of A given that VaR has breached plus probability of B given that VaR has breached plus probability of C given VaR has breached plus probability of D given that VaR has breached should be equal to 1 which means we need to standardize these probabilities conditional to this fact that this VaR has breached.

How to do this? Let us see that. So in order to standardize these probabilities all I need to do is I need to divide them by their summation that means summation is 5 percent so let us say 3 percent I divide it by 5 percent so I get probability of A conditional to VaR breached similarly I divide 1 by 5 0.25 by 5 and so on to get all the conditional probability given that VaR has breached now that I have revised probability that is PA dash PB dash that are conditional to the fact that the VaR has breached I can compute the expected shortfall with the same method of expectation that is 3 by 5 into 2 plus 1 this is PA into RA plus PB into RB plus P rather PA dash PB dash PC dash into RC plus PD dash into RD which is nothing but an expression like PI dash dash I am putting for the fact that conditional to this fact that VaR has breached into RI where I equal to 1 to 4 there are four possible events and we get expected value of 4.7 million let us see how to interpret this 4.7 million to conclude and summarize this video this 4.7 million is my expected or sort of average loss what kind of average probability weighted average loss given that VaR is breached and these are the only four possibilities the extreme tail these extreme tails are the only four possibilities available to me and this VaR has been breached given that this VaR has been breached these are the only four possibilities and their expected value given these probabilities 3% 1% and so on this is that expected value of 4.7 million.

Conventional risk models such as ARCH, GARCH do not provide sufficient emphasis on the negative tail of the written distribution this inefficiency with the conventional risk models is overcome with tail risk measures such as value at risk VaR and conditional value at risk C VaR measures to begin with value at risk models these models estimate the maximum loss value expected over a given horizon t with a certain confidence x percent while this method tells you how bad things can get but in case things do get bad what may happen to our investment portfolio is not known fund managers may set up positions with a lower maximum loss value with a certain confidence level however these positions may carry extreme tail losses outside the confidence band such inefficiency in the portfolio position design will not be identified by the VaR models to account for this inefficiency of the VaR model we go to conditional VaR or expected short form models these models examine those losses that exceed VaR or to put it another way what is the expected loss given the losses exceed VaR .