Advanced Financial Instruments for Sustainable Business and Decentralized Markets
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Lecture – 10
Week 3

In this lesson, we will revisit the problem of portfolio construction, computation of expected returns, risk, correlation and covariance. We will also summarize our learnings of portfolio optimization with two-security case and multi-security case. We will construct portfolio possibility curve and examine the feasibility region where all the possible combination of securities in the form of portfolios are available to investors.

We will examine various properties of the feasible region with and without short sale and minimum variance portfolio. We will try to find the most efficient portfolio on these available set of portfolios. Subsequently, we will introduce risk free lending and borrowing and examine the implication of the efficient set of portfolios. Lastly, we will introduce the concept of market risk and beta.

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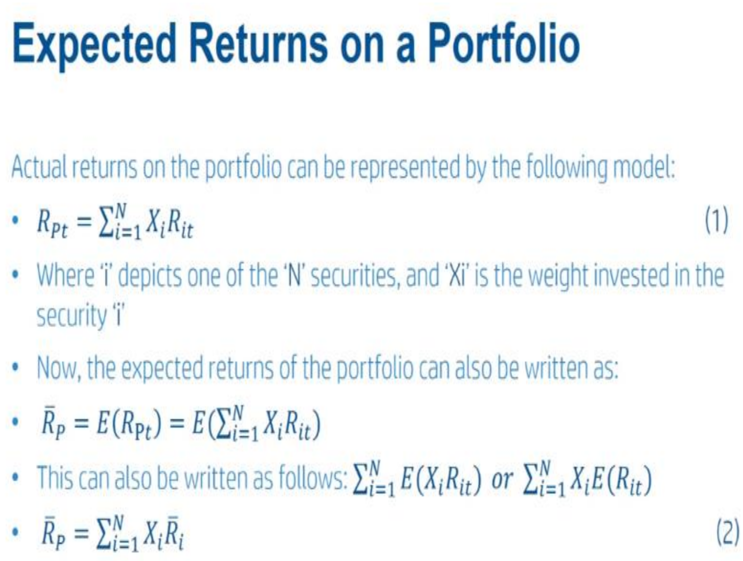

Portfolio construction recap one: In this video, we will discuss the expected return risk and correlation structure for multi-security portfolio. The actual return on a portfolio of assets for any given period is the weighted average of the actual return on the individual securities for that given period. The weights are the fraction of the portfolio invested in securities. Now, we will try and compute the expected returns of the portfolio.

But what is the need to compute the expected returns? Because it is the expected returns that are matched with the risk profile in various portfolio applications, not the actual returns. Now, keeping this rule in mind, we know that expected returns on any security for any period are the simple average if all the return observations are equally likely of the returns sampled from the past periods. What does it mean and what are its implications? This simply means the following.

If you have three observations for example 10 percent, 20 percent and 30 percent and also that the returns on a security are sampled in an unbiased manner or equally spaced periods like daily, weekly and so on, then the expected returns for any given period are the simple averages, for example 10 + 20 + 30 percent divided by 3 which is equal to 20 percent. Mathematically, this can be very easily written as return R i bar which is the expected returns equal to 1 upon T summation T = 1 to t Rit.

Now consider the actual return on the portfolio for period t as shown here. $R_{pt} = \sum_{i=1}^{N} X_i R_{it}$ Where i depicts one of the N securities and Xi is the weight invested in the security i. Now, the expected returns of the portfolio can be easily written as $\bar{R}_P$ = E (R_{pt}) = $\sum_{i=1}^{N} X_i * R_{it}$this can be also written in the following manner.

$$\sum_{i=1}^{N} E(X_i R_{it}) \sum_{i=1}^{N} X_i E(R_{it})$$

Summation of i = 1 to N expectations of X_i R_{it} or summation of i = 1 to N X_i times expectation of R_{it} because X_i constant and this can be effectively written as $\bar{R}_P = \sum_{i=1}^{N} X_i \bar{R}_i$. Even if you are not comfortable with the derivation above, keep the last result in mind. In simple English, this result suggests that the expected returns on the portfolio are simply the weighted average of the expected returns from the individual securities.

Here the weights are the proportion of the portfolio invested in the securities. Please note that while equation 2 looks similar to equation 1, there is a fundamental difference. Equation 1 describes the relationship between the actual return on portfolios and individual securities in that portfolio, while equation 2 describes the expected returns.

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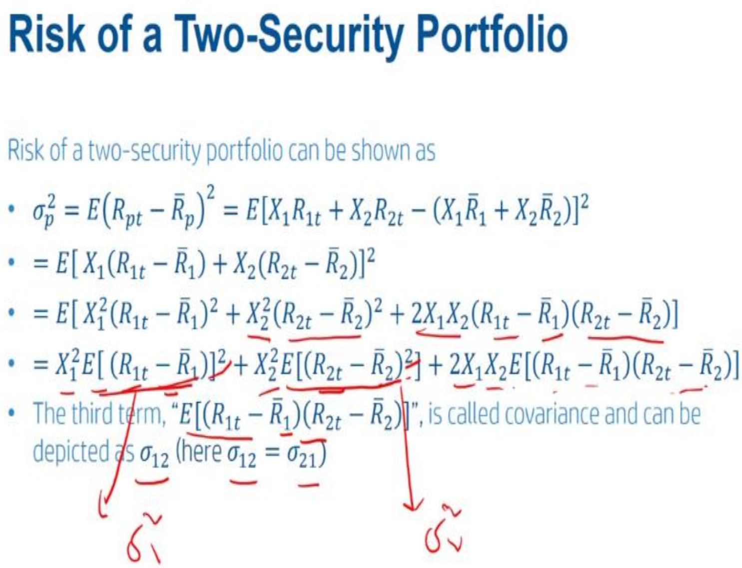

Now, let us talk about computing the risk or standard deviation for a portfolio of assets. Now that we have understood the return part, we will move to the risk part. However, it is not so simple for a multi-security case, so let us start with the two securities case with returns being R 1, R 2. Therefore, σ_p^2 = E $\left(R_{pt} - \bar{R}_P\right)^2 = E[X_1R_{1t} + X_2R_{2t} - (X_1\bar{R}_1 + X_2\bar{R}_2)]^2$

Which is also equal to $E[X_1(R_{1t} - \bar{R}_1) + X_2(R_{2t} - \bar{R}_2)]^2$. This expression can be further expanded to σ p square times $E[X_1^2(R_{1t} - \bar{R}_1)^2 + X_2^2(R_{2t} - \bar{R}_2)^2 + 2X_1X_2\,(R_{1t} - \bar{R}_1)(R_{2t} - \bar{R}_2)]$, which can be further simplified like this $[X_1^2E[(R_{1t} - \bar{R}_1)]^2 + X_2^2E[(R_{2t} - \bar{R}_2)]^2 + 2X_1X_2\,E(R_{1t} - \bar{R}_1)(R_{2t} - \bar{R}_2)]]$

This third term here which is denoted as $E[(R_{1t} - \bar{R}_1)\,(R_{2t} - \bar{R}_2)$ is also called the covariance between securities 1 and 2 and it can be easily depicted as σ_{12} which is the covariance between security 1 and 2 and please note a very fundamental property that $\sigma_{12} = \sigma_{21}$. Also, we must notice that this expectation of $R_{1t} - \bar{R}_1$ raised to the power 2 is nothing but σ_1 square, this is by definition σ1 square. Similarly, expectation of $R_{2t} - \bar{R}_2$ raised to power 2 is nothing but σ_2 raised to the power 2.

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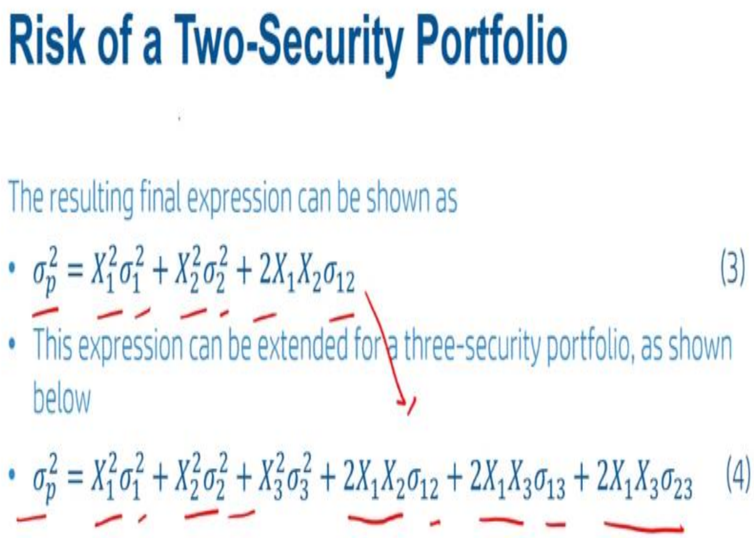

Therefore, a resulting expression can be written here. This is our resulting expression of risk for a two-security portfolio that is $\sigma_p^2 = X_1^2 * \sigma_1^2 + X_2^2 * \sigma_2^2 + 2 * (X_1 * X_2 * \sigma_{12})$. Now, this expression can be further extended for a three-security portfolio and here it is $\sigma_p^2 = X_1^2 * \sigma_1^2 + X_2^2 * \sigma_2^2 + X_3^2 * \sigma_3^2 + 2 * (X_1 * X_2 * \sigma_{12}) + 2 * (X_1 * X_3 * \sigma_{13}) + 2 * (X_2 * X_3 * \sigma_{23})$ which is very simply an extension of three-security case.

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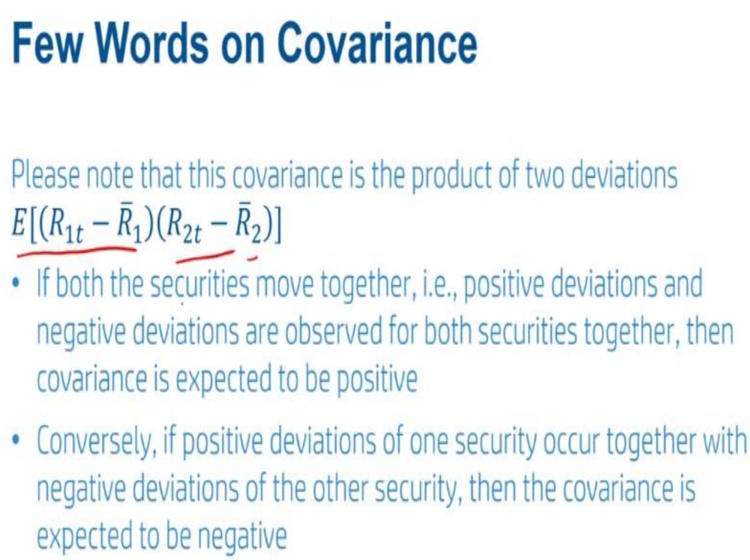

Let us have a few words on the covariance. Please note that this covariance is a product of two deviations. For example, as you can see here it is the $E[(R_{1t} - \bar{R}_1)\,(R_{2t} - \bar{R}_2)]$. Since the deviations are from two different securities, unlike the variance we may expect it to be negative as well, though that is a less practical scenario.

If both the securities move together that is large positive outcomes and large negative outcomes are observed for both of these securities together, then the covariance is expected to be high and positive. Conversely, if large positive outcomes of one security occur together with a large negative outcome of the other security, then the covariance is expected to be high and negative.

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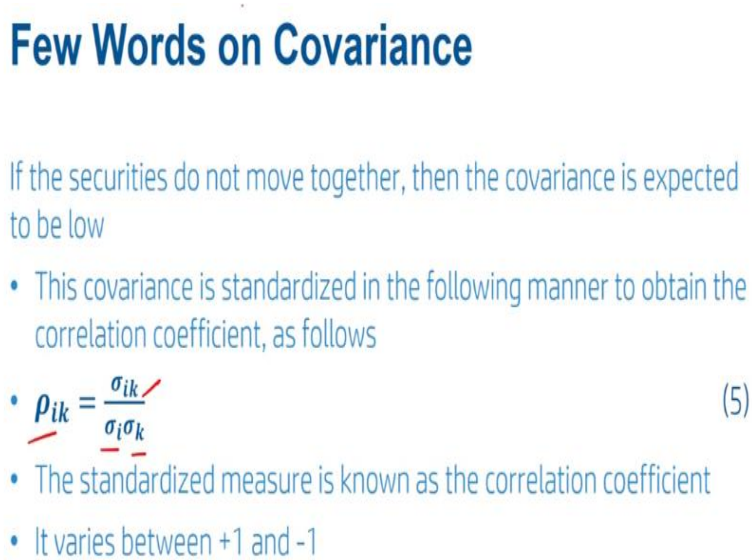

If the two securities do not move together, then the covariance is expected to be low. This covariance between security i and k is standardized in the following manner to obtain the correlation coefficient as shown here, we simply divide the covariance between security i and k that is σ_{ik} by their respective standard deviation σ_i σ_k to obtain the correlation between these two securities ρ_{ik}.

Here we have simply divided the covariance by individual standard deviations to obtain the standardized measure which is also known as correlation coefficient. This correlation coefficient is very easy to interpret and it is free of any skill bias, it varies between +1 and –1. To summarize, in this video we discussed the expected return, risk and covariance of securities in a portfolio. We also discussed that covariance can be standardized to obtain a more clean measure of correlation between securities in a portfolio and this measure ranges from plus +1 to –1.

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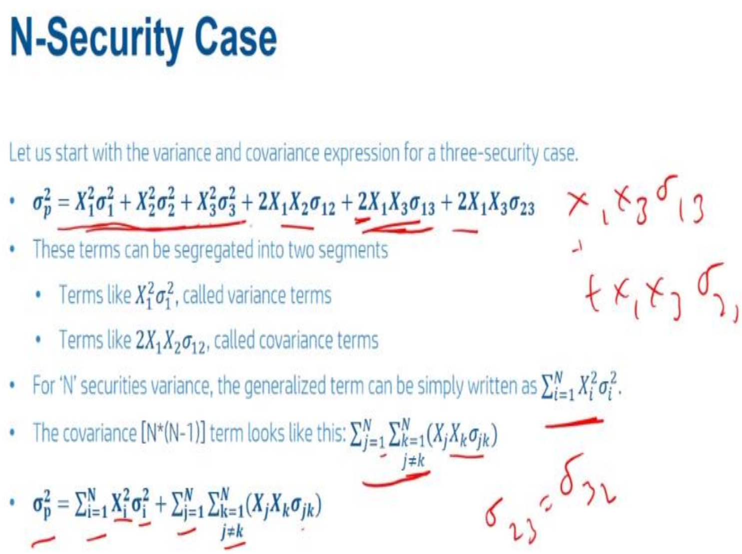

Portfolio construction recap two. In this video, we will talk about the risk of a more generic N-security portfolio case. Now, moving to a more generic case of N securities while solving these cases request computer programming, however, a small discussion will hurt nobody. If we carefully examine the terms in the equation for shown earlier, we will find that these terms can be segregated into two categories.

Terms that are similar to X1 square σ 1 square, for example, you can see here in the three-security case, you have σ p square as X1 square σ1 square + X2 square σ2 square + X3 square σ square these are the terms and another set of terms like two times X1 X2 σ12, times X1 X3 σ13 and so on.$\sigma_p^2 = X_1^2 * \sigma_1^2 + X_2^2 * \sigma_2^2 + X_3^2 * \sigma_3^2 + 2 * (X_1 * X_2 * \sigma_{12}) + 2 * (X_1 * X_3 * \sigma_{13}) + 2 * (X_2 * X_3 * \sigma_{23})$. Terms like X_1 square σ 1 square these are called variance terms and terms like X_1 X_2 these are called covariance terms.

Now, for N-security variance, the generalized term can be simply written as summation of Xi square σ square, i = 1 to N. Let us confirm the correctness of this term by applying it for two or three security case or even more. A three-security case is already provided here as you can see.

Understanding the second term, which is the covariance term, is slightly more involved. The term looks like this, it is $\sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j\ \sigma_{jk}{}_{\,j\neq k}$

Please note this j is not equal to k, for all the cases where j = k it becomes the variance terms. If you are looking for the coefficient of security two which is multiplied with all the covariance terms, please remember that j = 2, k = 3 and j = 3 and k = 2 will give the same two results that is $\sigma_{23} = \sigma_{32}$ and that is where we have a multiplication of two here. So, this would have been X1 X3 times σ_{13} + X_1 X_3 time's σ_{31}.

And that is why summation of these two will become two times X 1 X 3 σ 13. Also, the symbol j is not equal to k ensures that the k is not the same as j which otherwise would result in the variance term. So, now, we can write the full equation for the generalized N-security case very simply as follows.

$$\sigma_p^2 = \sum_{i=1}^{N} X_i^2 \sigma_i^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} X_i X_j\ \sigma_{jk}{}_{\,j\neq k}$$

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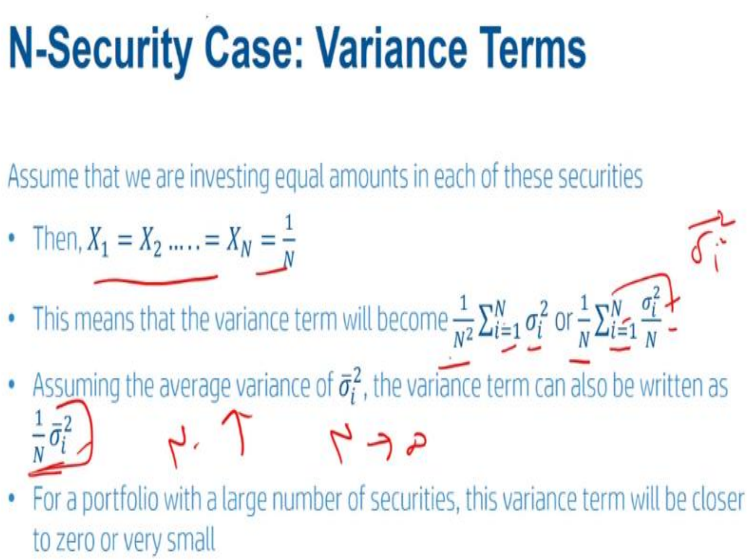

Let us examine this formula further. Assume that we are investing equal amounts in each of these securities and in that case $X_1 = X_2 = X_N = 1/N$. This means that the variance term will $(\frac{1}{N^2})\sum_{i=1}^{N}\sigma_i^2$ or we can take$(\frac{1}{N})\sum_{i=1}^{N}\frac{\sigma_i^2}{N}$ represents the average of variances or average variance that is σ i bar square.

So, we can write this term as $\bar{\sigma}_i^2$ sort of average variance and thus the summation of the variance term can also be written in the form of average variances$\frac{\bar{\sigma}_i^2}{N}$, this is average variance multiplied by 1/N. So, for a portfolio with large number of securities, this variance term will become close to 0 or very small as N increases, this overall term or summation of variances will become very small. In fact, if you assume N tends to infinity, this term will become close to 0.

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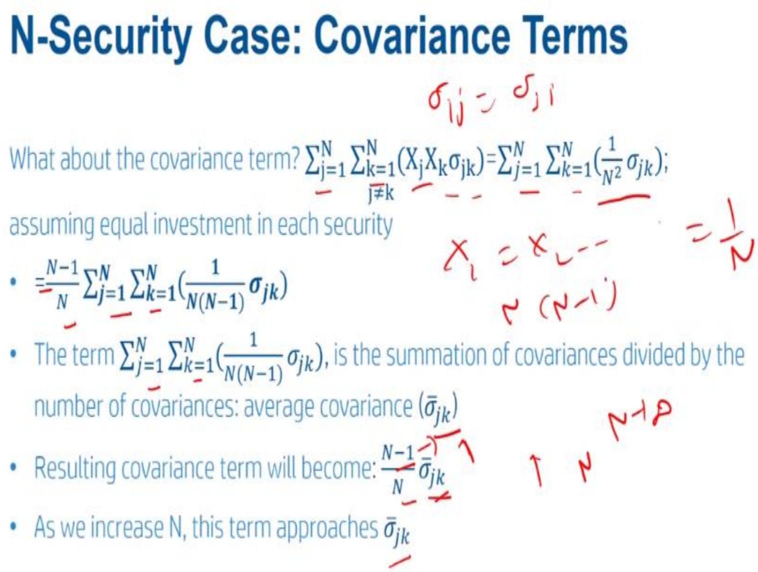

What about covariance terms, summation of covariance terms? Since there are N securities, there will be N into N – 1 covariance terms. For example, with three security there will be 6 such terms. However, since half of them are the same that is $\sigma_{ij} = \sigma_{ji}$ we use a multiple of two and only 3 terms are left, therefore if we present the covariance terms like this that is

$$\sum_{j=1}^{N}\sum_{k=1}^{N} X_i X_j \, \sigma_{jk} \quad {}_{j\neq k} \quad = \sum_{j=1}^{N}\sum_{k=1}^{N}\frac{1}{N^2}\sigma_{jk}$$

This also assumes equal investment in all the securities that means $X_1 = X_2$ and so on = 1/N equal investments and this can be further simplified like this.

$$\frac{N-1}{N}\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{1}{N(N-1)}\sigma_{jk}$$

The term $\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{1}{N(N-1)}\sigma_{jk}$ is the summation of covariances divided by the number of covariances, remember there are N into N – 1 covariance terms.

So, this is also called the average covariance or $\bar{\sigma}_{jk}$. Thus, all the covariance terms can be denoted using the average covariance term as (N – 1)/N $*\bar{\sigma}_{jk}$. In this case, please note as long as much as you can increase N, this term will approach to unity. So, if N tends to infinity, this term will approach to 1 and therefore the resulting variance or summation of all the covariance terms will approach $\bar{\sigma}_{jk}$.

Which is precisely the average covariance term and it is not equal to 0 as happened in the case of variances. So, this summation of all the covariance term with a very high value of number of securities, this approaches average covariance $\bar{\sigma}_{jk}$.

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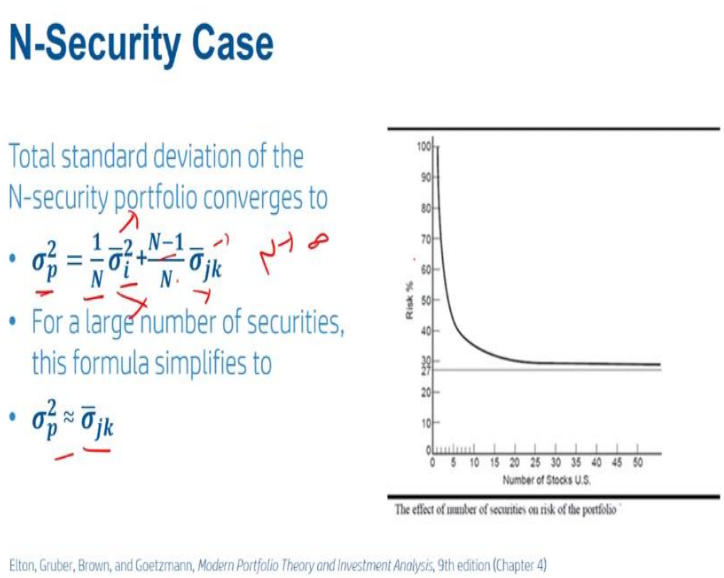

So, what happens to the overall risk of the portfolio that is summation of variance terms plus covariance term when we have large number of security? So, look at this overall total standard deviation of N-security portfolio, it converges to

$$\sigma_p^2 = \frac{1}{N}\overline{\sigma}_i^2 + \frac{N-1}{N}\overline{\sigma}_{jk}$$

which is average covariance.

Now, please note for very large number of securities that is N increasing to a very large number, this term will approach to 0 or a very small value and this N – 1 upon N will approach to unity and therefore we can say that σ_p^2 will simplify our approach to this value $\overline{\sigma}_{jk}$. This is precisely what we observe in this diagram. Let us discuss this diagram in slightly more detail.

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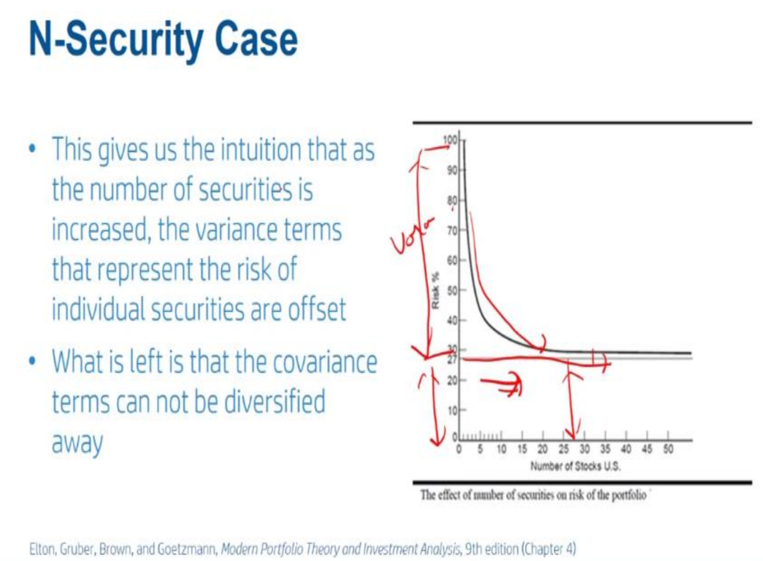

The diagram gives us the intuition that as the number of securities are increased, as you can see here the number of securities are increasing, the variance term that represent the risk of the securities are offset. For example, till here you have the variance terms component and here you have covariance component. As the number of securities are increased, this variance term drastically decreases and for even a small number of security like 30 it becomes close to 0.

So, what is left is the covariance risk that cannot be diversified away. That means this is also called the bedrock of risk as we discussed earlier and as more and more securities are added as figure shown here depicts how the specific risk or the variance term, this variance term, dies away and only the covariance risk, this risk, remains. These are the covariance terms that are left to contribute to the risk of portfolio or the average covariance.

With a fairly large number of securities, this remaining risk is equal to the average covariance of the portfolio often called as market risk or systematic risk or non-diversifiable risk. To summarize, in this video we discussed the risk or standard deviation of an N-security portfolio. We found that as the number of securities increases, the variance term or the idiosyncratic stock specific risk is eliminated.

And the only risk remaining is non-diversifiable and is on account of covariances which is driven by correlations across securities. This is the benefit of diversification, simply by adding more securities to a portfolio we are able to eliminate the stock specific idiosyncratic risk. In this video, we will introduce the Harry Markowitz mean variance framework and its role in portfolio optimization. Since 1952 Harry Markowitz's seminal work on portfolio selection, the practice of portfolio diversification is well known and documented.

In short, by choosing stocks that do not move together or with low correlation, investors can reduce the standard deviation of portfolio returns. If a sufficiently large number of past return observations are taken, their distribution is plotted, then this distribution is fairly closer to a normal distribution that we saw earlier. Though there are certain issues that tail, for example there are observations related to extreme positive and negative values.

That may cause deviations from the normal distribution and other issues such as excess kurtosis and practice, the normal distribution still is considered to define the return distributions very well and uses only two particular parameters, which is mean and standard deviation or variance. In nutshell, all the past returns can be defined by these two numbers that is mean and standard deviation and that is the reason we have discussed the normal distribution in some of the previous discussions.

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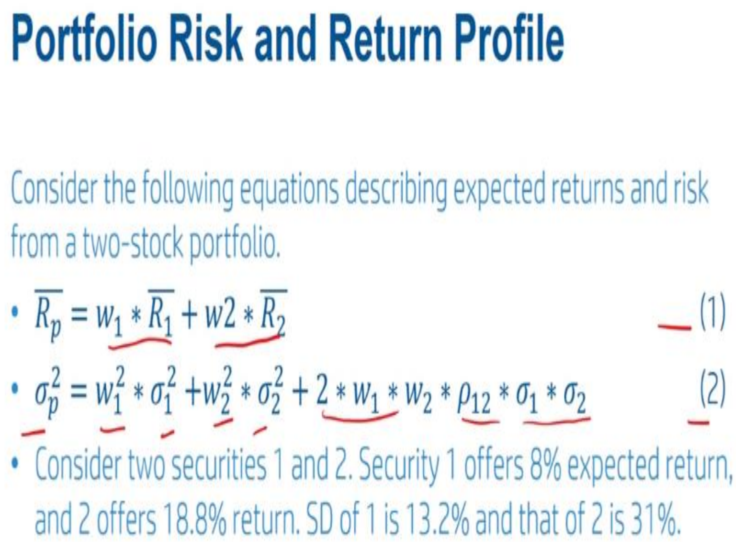

Now, let us combine two securities to construct a portfolio. Consider two securities A and B security A offers 8 percent expected return and B offers 18.8 percent return, standard deviation of A is 13.2 percent and that of B is 31 percent. A portfolio comprising these two securities has its expected return and risk are governed by these two equations, equation 1 and 2 which can be seen here.

$\bar{R}_P = w_1 * \bar{R}_1 + w_2 * \bar{R}_2$

For example, expected return of the portfolio can be easily written as w 1 times R 1 bar which is the expected return on security 1 and w 2 times R 2 bar which is equation 1 for the expected return on the portfolio, this we have seen earlier. Similarly, the variance of the portfolio is $\sigma_p^2 = w_1^2 * \sigma_1^2 + w_2^2 * \sigma_2^2 + 2 * (w_1 * \sigma_1)(w_2 * \sigma_2)\rho_{12}$, this is equation 2 which also we have seen earlier.

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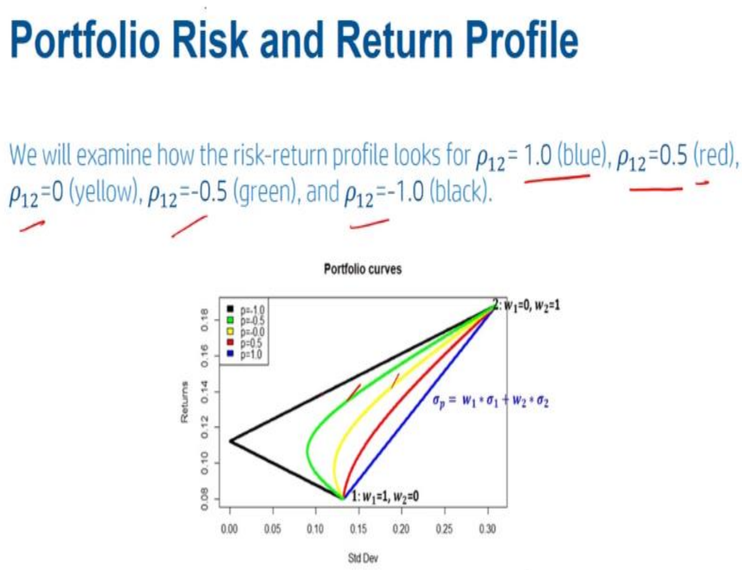

Now, we will examine how the behavior of returns and standard deviation for different levels of correlations and different investment proportions that is w_1 and w_2 for these two securities. We have plotted all the possible portfolios of correlations corresponding to $\rho = 1$ in blue, $\rho_{12} = 0.5$ in red row, $\rho_{12} = 0$ in yellow, $\rho_{12} = -0.5$ in green and $\rho_{12} = -1$ in black.

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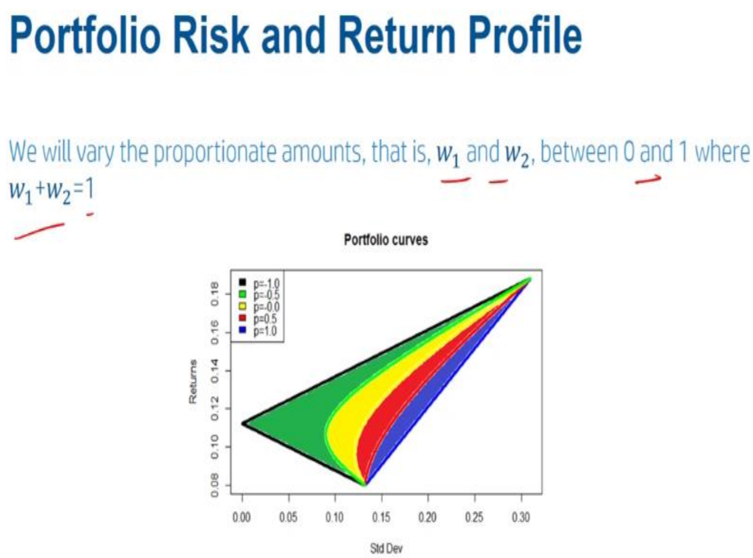

Here, each curve with a given colour indicates all the possible combinations of investment that is each possibility is a point here in either of these securities from w1= 0 to w1 = 1 where w1 + w2 = 1. So, basically we are varying the proportionate amounts that is w1 and w2 between 0 and 1 and ensuring that w1 + w2 = 1 constraint is followed. These graphs are essentially governed by the tool return and risk equations that we have already seen.

Which is $Rp = w1*R1 + w2*R2$ and $\sigma p^2 = w_1^2 \sigma_1^2 + w_2^2\sigma_2^2 + 2*w_1 w_2 \sigma_1 \sigma_2 \rho_{12}$, the equation 1 and 2 that we saw earlier. P 12 here was the correlation coefficient between the two securities 1 and 2 and then a very important parameter.

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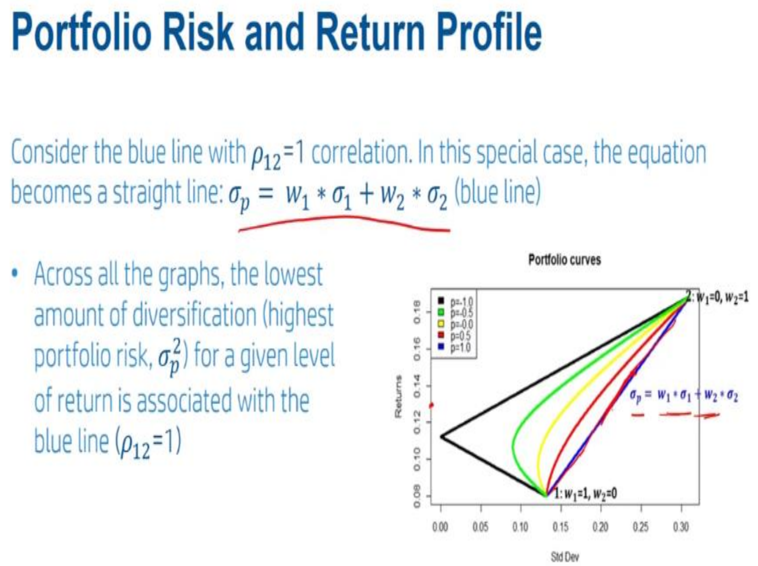

Now, please consider the blue line here with $\rho_{12} = 1$ correlation. In the special case the equation becomes a straight line with $\sigma p = w_1*\sigma_1 + w_2*\sigma_2$, this is the blue line case, this case, and it should be clear from the graph itself that across all the graphs the lowest amount of diversification that is highest portfolio risk σ_p square is achieved for a given level of return associated with this line.

By so, please notice that for this blue line for all the combination of w_1, w_2 the amount of risk for a given level of return is maximum for all the combinations. For example, if we look at this level of returns, the maximum risk is obtained on this blue line for all the different correlation combinations. We will discuss the minimum and other combinations also, but it is very easy to see here that the maximum risk is observed or obtained for this particular line.

And in this particular line, there is more diversification and the equation of the σ_p is simply the weighted average of risk between two securities that is $w_1*\sigma_1 + w_2*\sigma_2$, something similar to the weighted average of returns as we computed.

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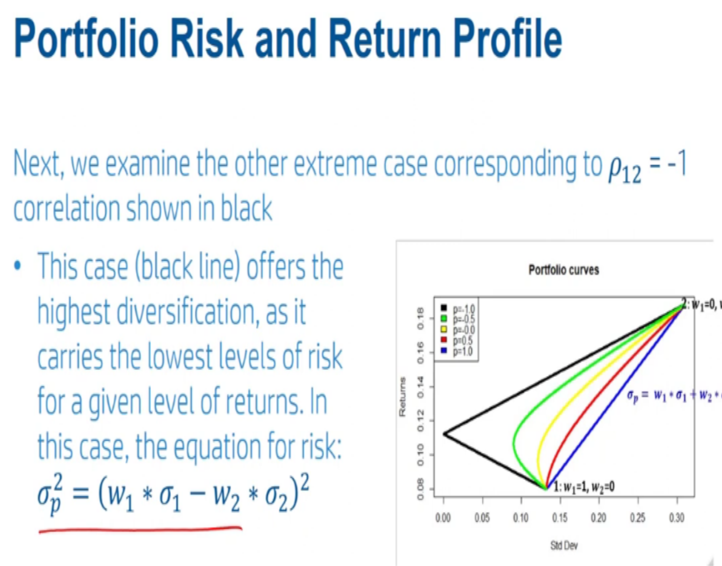

Next, we examine the other extreme case corresponding to $\rho_{12} = -1$ and that correlation is shown here in green. This is the case where highest diversification is observed as it carries the highest levels of return for a given level of risk or minimum level of risk for a given level of return. In this case, the equation for risk effectively becomes described by this equation which is σ_p^2 = (w_1*σ_1 – w_2*σ_2)^2

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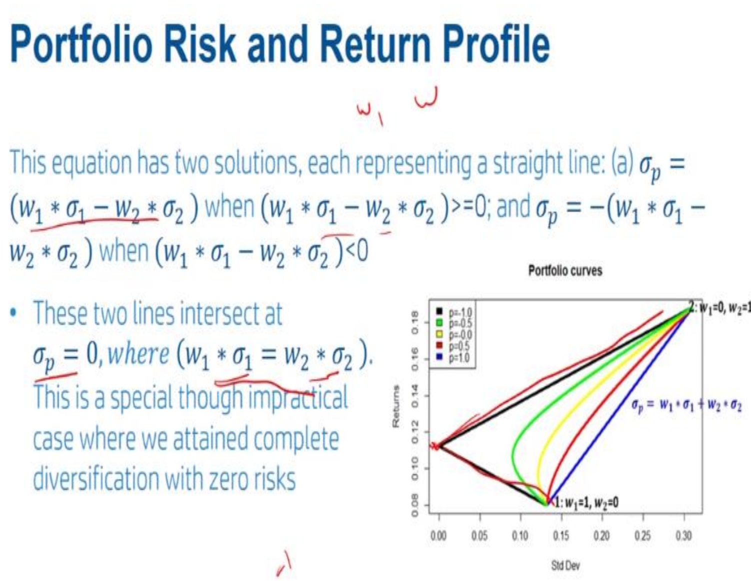

Please remember this particular equation has two solutions, each representing a straight line that is line number 1 is σ_p = (w_1*σ_1 – w_2*σ_2) which is this line and w_1*σ_1 – w_2*σ_2 which is this line. So, in this case, when σ_p = (w_1*σ_1 – w_2*σ_2) when w_1*σ_1 – w_2*σ_2 is greater than or equal to 0 and σ_p = – w_1*σ_1 – w_2*σ_2 when w_1*σ_1 – w_2*σ_2 is less than 0.

So, these two line segments can be represented by this first this black line and the other one is by this black line. So, these are two line segments and these two lines intersect at $\sigma_p = 0$ where precisely $w_1*\sigma_1 = w_2*\sigma_2$. Now, this is a very special but an impractical case where you have attained complete diversification with zero risk. Such cases of negative correlation are rarely observed and almost okay from theoretical point of view, but not observed practically.

However, if we assume them to be true in theoretical sense, then there is a possible combination of weights w_1 and w_2 such that we can obtain this condition $w_1*\sigma_1 = w_2*\sigma_2$and the risk of the portfolio such hypothetical portfolio complete risk becomes 0 or neutralize. This is a special case where minimum risk $\sigma_p = 0$ is observed.

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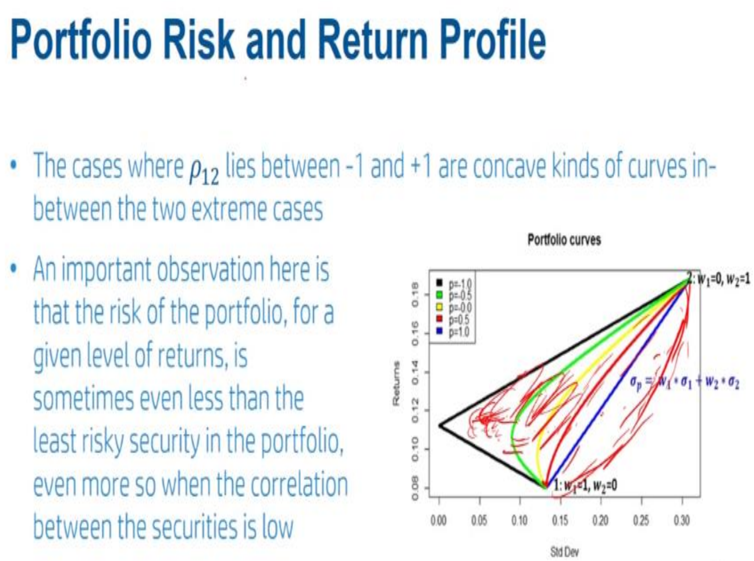

For all the other cases where ρ_{12} lies between –1 to +1 are concave kind of curves in between these two lines. For example, notice all the cases will be between this blue or black line, you will not observe any case such as this, such cases will not be observed. So, all these are concave kinds of cases between blue and black line, the extreme cases that we have discussed. Any point here would suggest even higher level of risk for a given level of return as compared to maximum correlation equal to 1 which is not feasible.

At any point beyond this black curve would suggest a risk which is lower than even this negative

–1 correlation which is also not possible. So, all the possible combinations are expected for different correlation levels are expected to lie only between these two black and blue curves and of concave nature. So, there are no convex cases like this which will be outside as the lines will strictly lie between these two extreme cases of black and blue lines.

Also imagine a concave curve, it would suggest a completely opposite situation to what we have discussed about diversification. It would suggest the existence of curves where expected returns will be lower for a given level of risk as we have considered portfolio of securities. It would suggest more risk here if it is of this nature. If the curve is of this nature, it would suggest more risk for a given level of return or less expected return for a given level of risk which is spurious and not possible.

So, our region of possibilities is strictly line between these black and blue lines. So, this is the region of possibility for all possible combination of correlations and w 1, w 2 weights. An important observation here is that the risk of the portfolio for a given level of returns is sometimes even less than the least risky security in the portfolio, so the risk is sometime even less then. For example this is the least risky security, then there are certain combination where risk is even less on the left of it.

And this is particularly the case when the correlation between the securities is very low, for example on this side here correlation is low, on the side we are expecting low correlation points. In actual terms, there may be a security that exhibits returns outside this region of possibilities, then what happens? Surely expectations cannot stay away from the actual returns for a long time if markets are reasonably efficient.

That means if any securities lying outside of this feasibility region, then market forces of arbitrage and market efficiency will drive down these points or these returns back to the normal reverse by correcting the points. So, there will be a correction witnessed in the point that will drive down the observed level of prices to align them with the expected returns.

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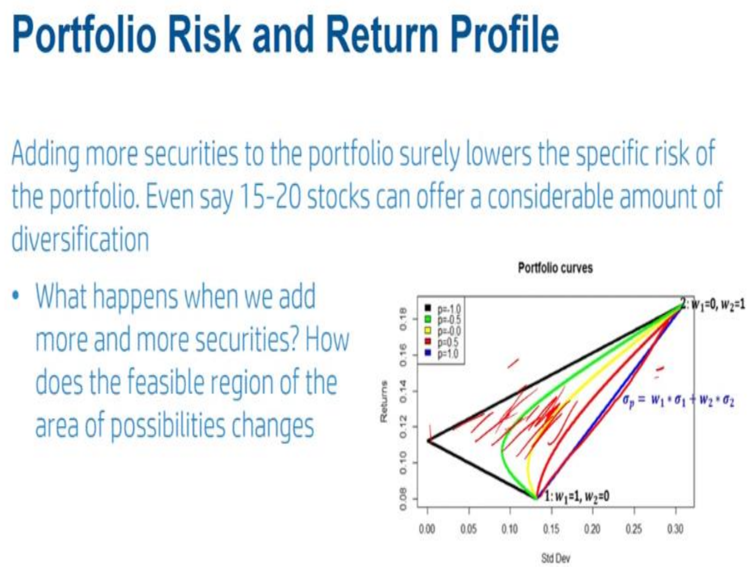

So, we have seen that adding more and more securities to the portfolio here surely lower the specific risk of the portfolio. Even with 15 to 20 stocks, we can achieve a considerable amount of diversification by neutralizing the stock specific or variance risks. What happens when we add more and more securities to this two-stock portfolio? How does the feasible region as we saw here earlier, this region changes, which is the area of all the possibilities.

In the next video we will discuss these possibilities when we add more securities to this two-stock portfolio how this feasibility region changes. To summarize, in this video we saw the mean variance framework of Harry Markowitz. We found that for correlation equal to 1, there is no diversification in risk and the total risk of the portfolio is simply the weighted average of individual risk or standard deviations of the two securities, weights being the proportionate amount invested in these.

We also saw that for a theoretical and hypothetical case where correlation is strictly equal to –1, maximum diversification is achieved which is represented here by two line segments in the black colour. Theoretically, there is a possibility of a point where overall risk becomes 0 for a given correlation equal to –1, however, that is more of a theoretical possibility. For all the possible combinations of correlation and proportionate amounts invested in the portfolio, the feasibility region lies between these two curves.

That is between this blue which is corresponding to correlation equal to 1 and the black curve which is corresponding to correlation equal to –1. So, all the feasible points lie in this region. Any other points that are outside this curve, whether on this side or this side are not possible and if there is a stock for which prices are such that that the stock lies on this point, market forces of arbitrage and market forces of efficiency will drive that point or price of that security within this region in a short frame if the markets are reasonably efficient.

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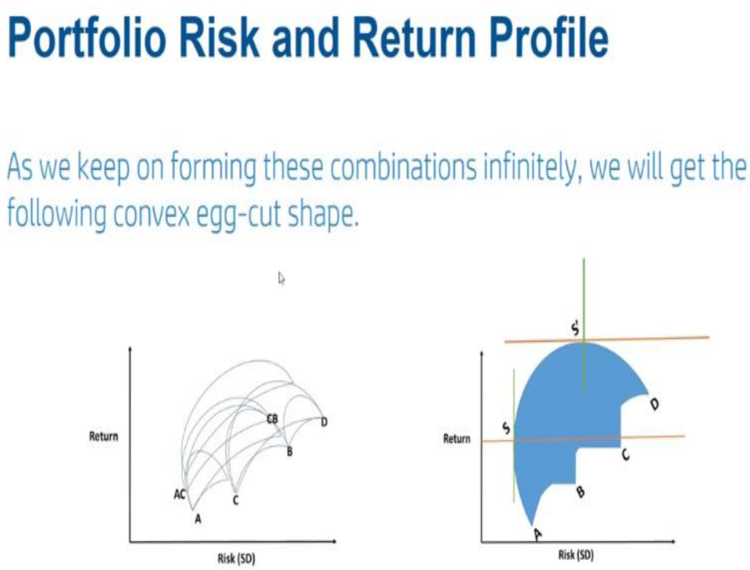

In this video, we will talk about the feasible region and various portfolio possibilities with multi-security case. We start with four hypothetical portfolios, different combinations of A, B, C, D that means A, B, C, D, AC and so on with different investment mixtures can lead to large number of portfolio possibilities as shown here. These possibilities are shown in the form of curves in the figure.

As we add more and more securities, we get the first diagram like a net cut shape. As we keep on forming these combinations infinitely, you can add more and more securities in addition to these showed securities, we will get the convex egg cut shape as provided here.

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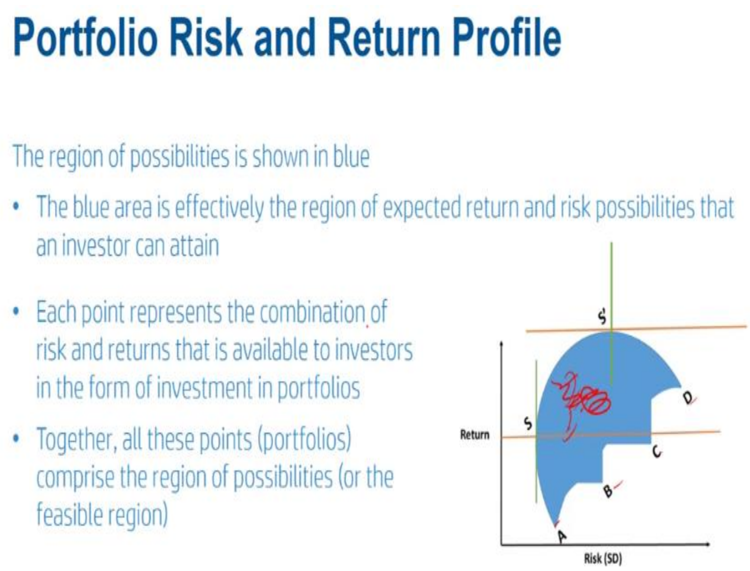

This region of possibilities is shown here in blue. The blue area is effectively the region of expected return and risk possibilities that an investor can attain. Each point here represents a combination of risk and returns that is available to investors in the form of investment into portfolios. Together, all these points or portfolios comprise a region of possibilities or the feasible region as you can see here.

This is your feasible region of all the possible different combinations. We started from four securities A, B and C and D, then we kept adding more and more securities to obtain this region of possibility.

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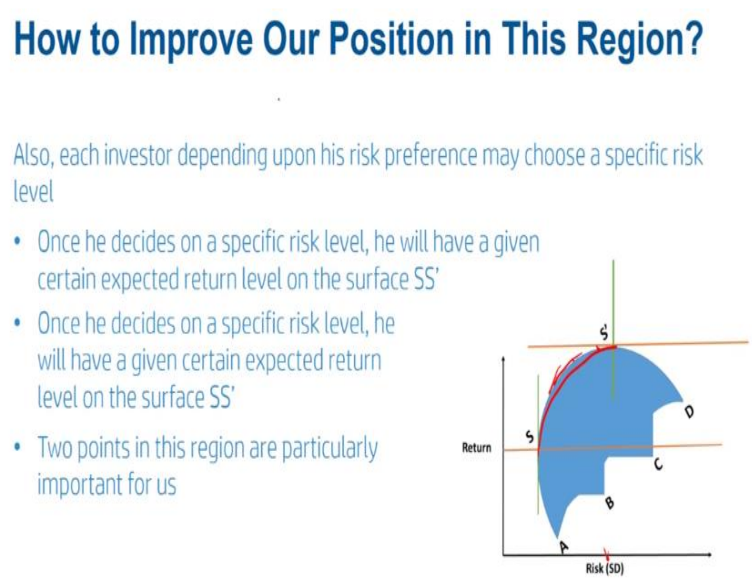

Now, on this region of possibility, our goal is clear, we want to move up or increase the returns that means we want to move in this direction where for a given risk we are increasing the expected returns and we want to move to left. That means for a given level of return, we want to reduce the risk. As we do this, we reach the top surface of the region of possibilities which is this surface or we can say S S dash on this region of feasibility.

There are no more points where we can move further left or up on this curve. So, this is the extreme available to us the best possible combination and that is why this region is called efficient frontier. All the points on this region offer the highest return for a given level of risk or lowest risk for a given level of return. For example, if you pick this level of return, the lowest amount of risk available for all the points available in feasible region would be on this that is on the surface S S dash.

Similarly, if you pick a level of risk, let us say this level of risk, the maximum level of return available could be on this S S dash line here. Now, on this S S dash or efficient frontier, each investor depending upon their risk preference may choose a specific risk level. For example, investor may choose this point if he prefers this level of risk or if he prefers a lower level of risk, they can move left or left on this particular efficient frontier.

Once the investor decides a specific risk level, they will have a given level of fixed return on the surface S S dash. So, we hope and we can clearly see here that once the investor reaches the surface of risk return, they cannot do any better by moving left or up.

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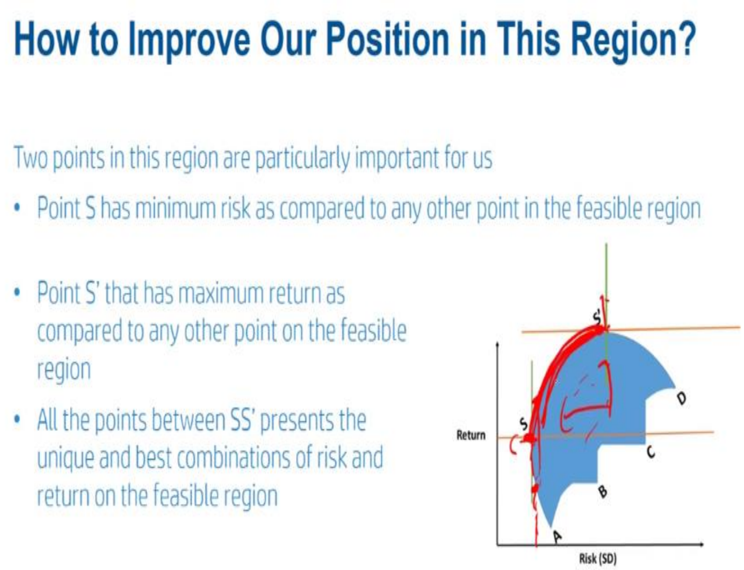

Now, if you focus on this efficient frontier which is S S dash, two points are particularly important for us here. First point S, this point is the minimum risk point. So, if you look at this efficient frontier on the feasible region, this point has the minimum risk. There is no point on the left of this in the feasible region, so this is the leftmost point and therefore represents the point of minimum risk. Similarly, point S dash which is at the top of it is the point of maximum return.

So, there is no point on this feasible region which is above this point S dash and therefore this point offers you the maximum return. All the combinations of this surface S S dash are called the efficient frontier on this entire feasible region because these points offer you the best combinations of risk and return possible in the entire feasible region. So, this becomes our efficient frontier.

One interesting point to note here or observe that if you go below this point as there are some points that offer you lower return, but these points are not efficient because for this level of risk you can also obtain this level of return. So, as compared to this point you would like to be on this point. So, in this manner the S S dash means the efficient frontier, the best combinations of risk-return that are available to you.

To summarize, in this video we saw that as we keep on adding multiple securities to a portfolio, we obtain a region of possibilities which is called feasible region as shown here in the blue. This appears to be an egg-cut shape. In order to improve our position or profile on this feasible region, we would like to move on the left to improve or lower our risk for a given level of return or we want to move up to increase our expected return for a given level of risk.

As we keep on doing that, we obtain a region called efficient frontier or a surface which is the efficient frontier which offers us the best possible combinations of risk-return scenarios. And this surface or this region which is called efficient frontier start from the point with the minimum risk, global minimum risk portfolio which is S here and the portfolio which offers the maximum expected return S dash here and this S S dash presents us the efficient frontier.

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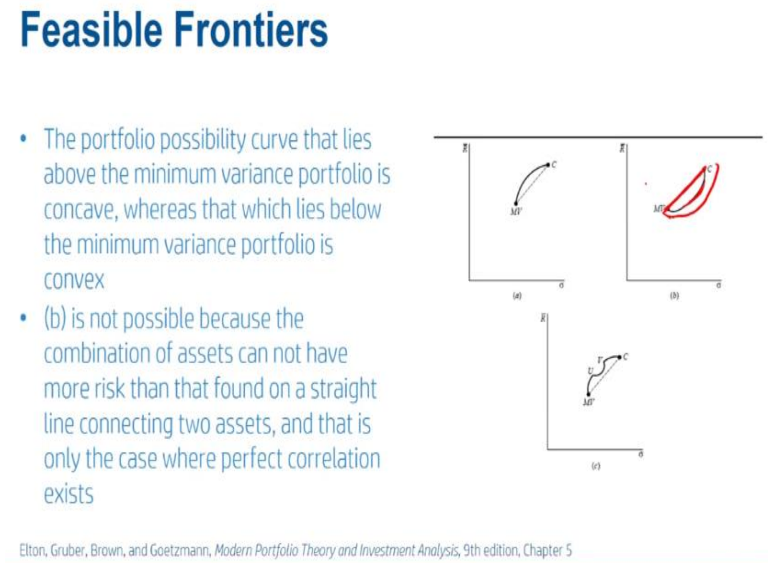

Feasible frontiers: In this video, we will examine the shape of efficient frontiers that are feasible. We will also discuss some of the shapes of efficient frontiers that are not feasible. Look at the possibility curve shown here. From what we already know, the figure b is not a possibility because it is convex in nature. Notice the convex nature of figure b here. We already know that there cannot be points to the right of this line joining MV and C.

We know that there cannot be any points to the right of line joining global minimum variance portfolio which is MV and C here which is above MV by so because for any point that lies on this curve we will have a risk which is higher than those points that have perfect correlation which is equal to 1 that fall on this line. And therefore, in that case the points on the convex curve will have more risk or higher standard deviation than the straight line.

But we already know that the straight line joining MV and C corresponds to the highest risk which is correlation equal to 1 and hence it is not possible.

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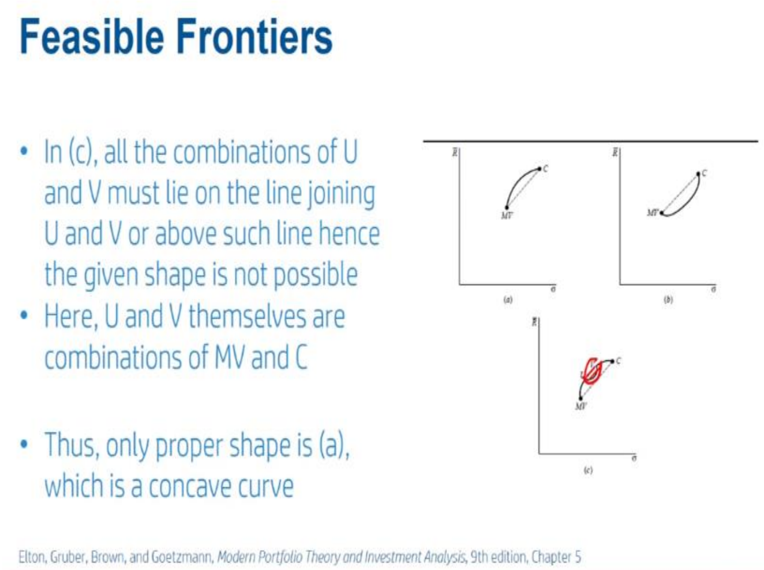

What about figure c, this figure? Please note again our reasoning remains the same, U and V are also portfolios. So, any portfolio that can be found by combining U and V would look something like this and therefore any shape which is like this convex shape like this should not be possible. So, the straight line joining this U V point will be one extreme set of portfolios with the highest risk and therefore the shape of the curve joining U V can only be at max or at best a straight line or a concave or not this kind of convex curve as we have discussed and therefore this figure c is also not a possibility.

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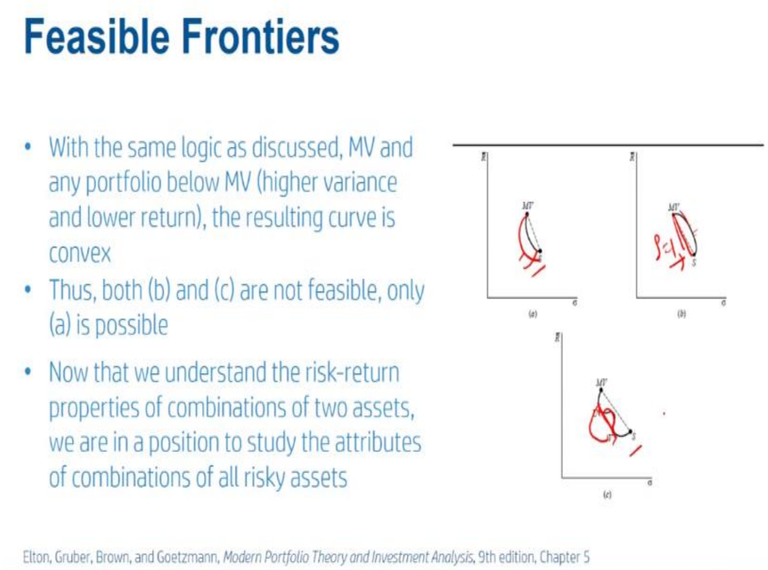

Let us also now discuss the points that are below the minimum variance frontier like here are shown in a, b and c. With the same reasons as we discussed earlier, only a is the feasible option. This one is the feasible one, the curve below the point M V should be convex like this and therefore figure b is also not possible because it would again represent those points that are having higher risk than the straight line with correlation equal to 1.

Now, if U and V are combinations of MV and S, for example, these U and V they are combination of MV and S. If the portfolios U and V are mixed the combination can be a convex curve or at max, a straight line joining U V like this. So, it can be like this or the straight line joining there. It cannot be a concave curve as shown here. So, it cannot be a concave curve and therefore c is also not a possibility. So, now we have understood the risk return profile of a combination of two assets and how feasible frontier would look like.

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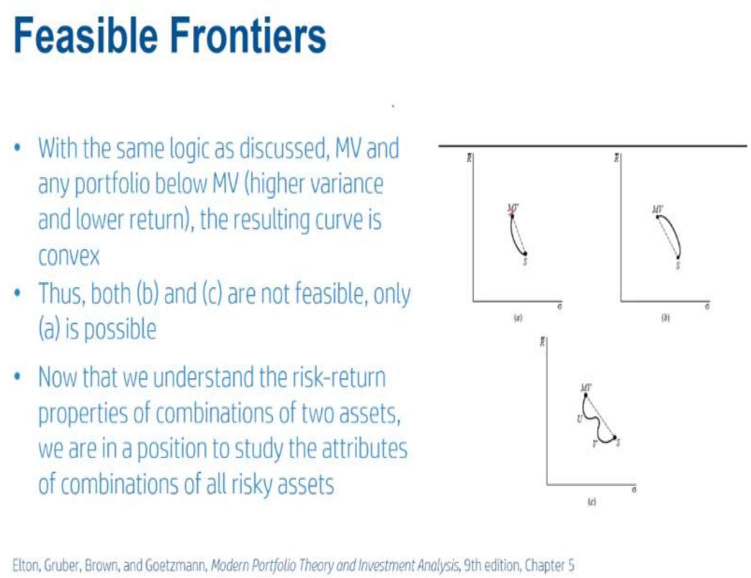

To summarize, in this video we discussed the shape of feasible frontiers. For those points that are lying above the minimum variance portfolio MV, this MV, the shape of efficient frontier should be a concave curve that is towards the left of the straight line joining point MV and S which suggests that there cannot be any portfolio with higher risk for a given level of return.

Similarly, for those points that are below MV, the curve has to be convex and not concave, the shape of feasible frontier should be convex not concave because all the concave curves would indicate those points that have higher risk for a given return than the correlation equal to 1.

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In this video, we will talk about different efficient frontiers for multi-security case, first without short sales and then with short sales.

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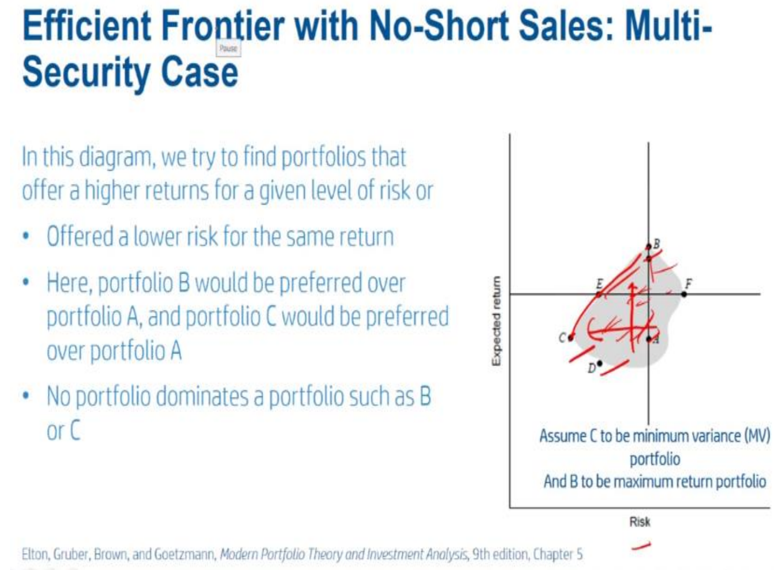

Let us start by examining a case with multiple securities that are distributed in a rather heterogeneous manner as shown in the figure here. So, here we have six securities. These are A, B, C, D, E these six securities, they are with different combinations are plotted here on risk and expected return space. Before we start our discussion, let us remember two important goals, one higher return for a given risk.

That means for a given amount of risk you would like to have higher expected returns that is you would like to move up and lower the risk for a given level of return that is you would like to move on the left. These are desirable portfolio characteristics. Now, if we compare portfolios A and B here, portfolio A and B in this possibility space which is this portfolio possibility space B of course would be preferable to A since it offers a higher return for the same level of risk as we can see here.

Similarly, C would be preferable to A because it offers same level of return with a lower level of risk. So, it may appear to us by this logic that no portfolio dominates portfolio C, B or for that matter portfolio E. So, these are the dominating portfolios. This kind of elimination process where we compare the portfolios for their risk-return profile, we can easily see that portfolio C, B, E they cannot be eliminated and therefore they are part of the efficient frontier.

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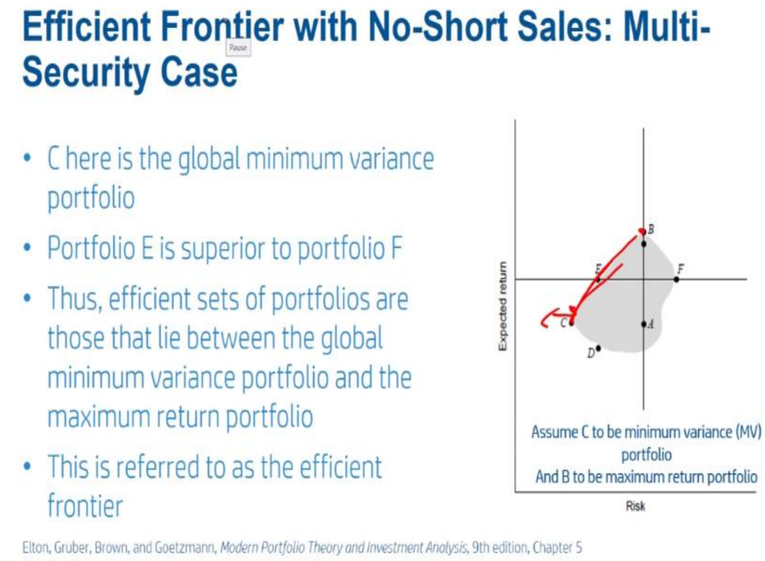

Also, we can clearly see here that C is the point of global minimum variance that means there is no portfolio that has lower risk that is on the left of this point. So, no point on this feasibility region is on the left of point C, so no other portfolio has a lower risk than this portfolio C. And therefore, we can say this surface CEB would comprise the set of portfolios that offer the highest return for a given level of risk or alternatively the lowest risk for a given level of return.

Therefore, this curvature, the set of portfolios that lie on this curvature CEB are called efficient set of portfolios or this is our efficient frontier. Here we also find that the C portfolio is the portfolio with a minimum variance and Portfolio B is the portfolio with maximum return, also we can call C as global minimum variance portfolio.

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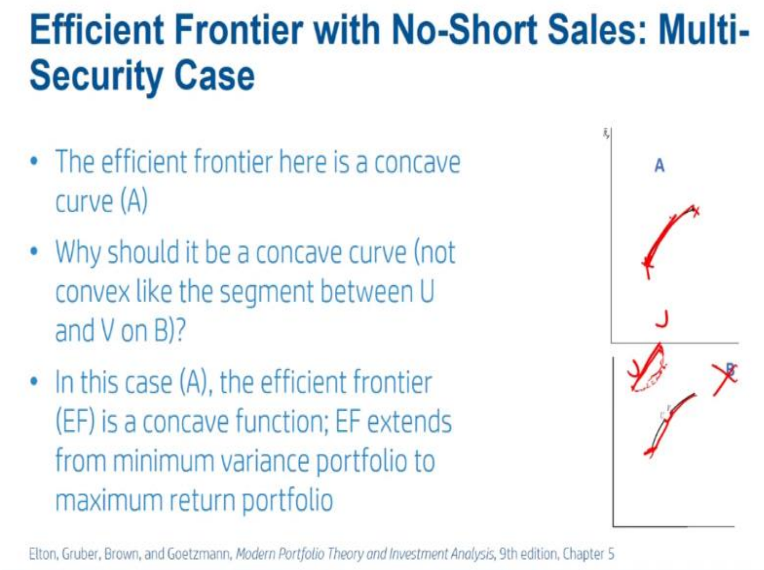

Now, if we take out of the portion of this efficient frontier, it would look something like the graph shown on A, this A curvature. We know from our previous discussions that the shape can only be concave, not convex. The lower end of this frontier represents the portfolio with the minimum variance, this one the minimum variance, the higher end of this portfolio is the portfolio with maximum return across all the portfolio that is security with the highest returns.

Please remember, in this case we cannot have a portfolio looking like B as we have discussed already. Any convex nature like between U and V is not possible why? Because U and V are also portfolios and therefore we have already discussed if all the points are above minimum variance frontier that is above this minimum variance point, any two portfolios U and V their combination can be at best a straight line.

And in that case the straight line would represent portfolios with correlation of 1 that is maximum risk. So, any portfolio on the right side of it would represent a set of portfolios that carry higher risk than those with correlation of 1 which is not possible and therefore any set of portfolios on the right side would not be possible and therefore this convex nature of curve is not possible. Therefore, B is not a feasible scenario. The only feasible scenario for efficient frontier in this case is this one.

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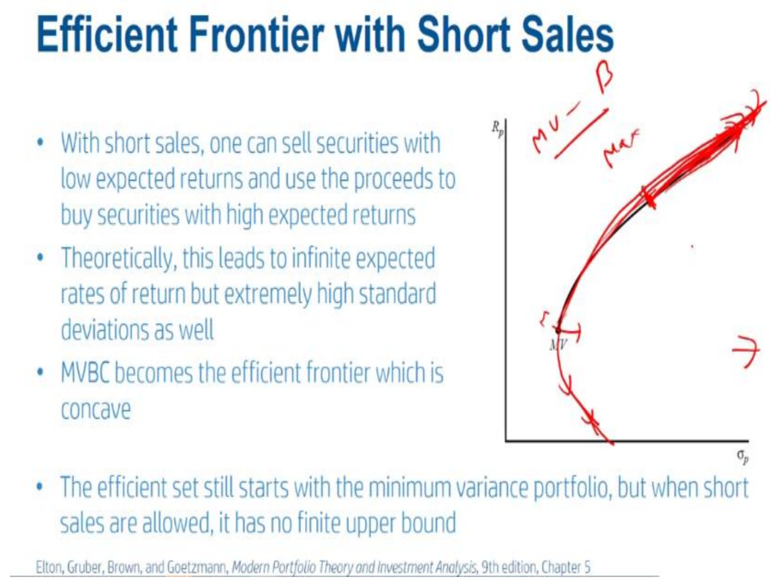

Now, let us talk about the case where short sales is allowed. Short Sale is essentially selling a security one does not own, which is equivalent to taking a negative position in the security. This negative position allows one some additional liquidity or funds that one can invest further in taking a long position or investing in any other purposes. Now, if you are making the short selling, you would like to sell securities that have lower expected returns.

And you would like to buy securities from those proceeds where you have higher expected returns and therefore those not so efficient securities which are lower in the efficient frontier here, you have this opportunity now to short sell these securities because these securities are offering us lower expected returns. This suggest that one can keep on short selling securities that are here while investing in long securities that are here and extending their position beyond B to C and further high.

Theoretically, this suggests that one can achieve infinite expected returns or very large amounts of expected returns, but at the same time when you do that, you also extend the risk of your position that is you also extend the risk of standard deviation to very large levels. And therefore, one can see that now our actual efficient frontier is not restricted at point B and one can take positions that can go beyond B and even further and further.

So now, on aggregate our position becomes MV B and beyond to C. So one can extend their portfolio possibility curve to MV B C which becomes the new efficient frontier and notice that this is a concave curve, which is similar to earlier without short sale scenario, but now with short sales. Our efficient frontier extends from MV not only moves up to the security with maximum return, but it also extend beyond that to large expected returns which have no finite upper bound.

However, please note the caveat, in order to achieve that, we are able to or we are assuming a large amount of short sales, every market have certain restrictions on short sales, but still to a reasonable extent we can assume that one can extend their efficient frontier beyond this point B which was point of maximum return to further with no upper bound. To summarize, in this video we discussed the shape of efficient frontier with and without short sales.

We found that in the absence of short sales, the efficient frontier extends from portfolio of minimum variance to portfolio with global minimum variance to portfolio of maximum return, in this case and MV to B. However, when the restrictions on short sales are removed, one can short sell securities with poor expected return and invest them in higher expected returns that are on the efficient frontier and extend their possibilities beyond this upper point B.

And in that case, the expected returns would be very higher with no upper bound on this efficient frontier. However, at the same time if one were to obtain such positions, the risk or standard deviation of such positions would also be very high.

Efficient Frontier scenarios, multi-security case 2. In this video, we will discuss the shape of efficient frontier with multi-securities and introducing riskless or risk-free instrument. We will discuss three cases. First efficient frontier with riskless lending and borrowing, then only riskless lending is allowed, not borrowing and then reckless lending and borrowing at different rates. As we can see, we gradually move in a step-by-step manner towards a more realistic assumption of different riskless lending and borrowing rates.

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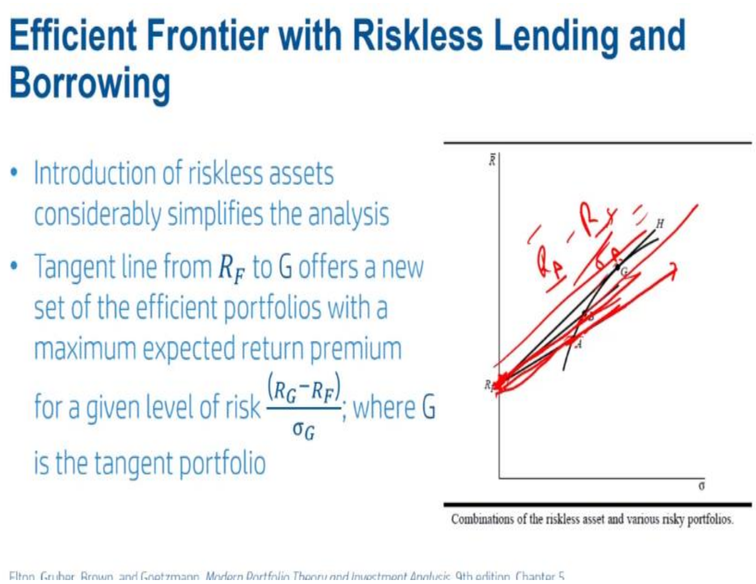

Up until this point, we were focusing on risky securities. The addition of riskless securities considerably simplifies the analysis as we will see now. Here, we are assuming the same rate for borrowing and lending that is R_F in the first case, which is this R_F, same lending and borrowing interest rate. Because the return on the security is certain whether it is borrowing or lending, practically this does not happen, borrowing is often at higher rates than lending.

However, for analysis purposes let us consider the same lending and borrowing rates for now. Now, here the intercept is R_F, this risk-free rate becomes our intercept. And if you pick randomly any point A on this efficient frontier, then the slope of this line would be $\overline{R_A}$, which is my expected return on $(R_A – R_F)/\sigma_A$which is the standard deviation or risk of security A. This is often referred to as Sharpe ratio also.

This $(\overline{R_A} – R_F)/\sigma_A$, R_A is the expected return on security and R_F is the risk-free rate. The point on the line that is to the left of A, these set of points left to the A till R_F, these are the combination of lending at risk free rate that means investment in risk free rate and investing in portfolio. So, a mixture of investment in R_F and R_A would fall on this line. Conversely, the point on the line that are to the right of A that means on this side, these are called a combination of borrowing at R_F.

And investing original wealth as well as the borrowed amount in A to result in a position that is on the right of the A on this line segment and therefore these are called the borrowing segment of the line. However, here we choose the portfolio randomly with only criteria that it is on the efficient frontier. There may be other portfolios like A, B or G and the resulting graph may look like this as we have seen here like this and like this.

Now, we have three lines indicating there is return profile corresponding to three portfolios A, B and G here. These lines extend the possibility region. For example with line B our possibility region is like this, with A our possibility region is like this and investors can hold portfolios on these lines joining risk-free rate and points A, B and G.

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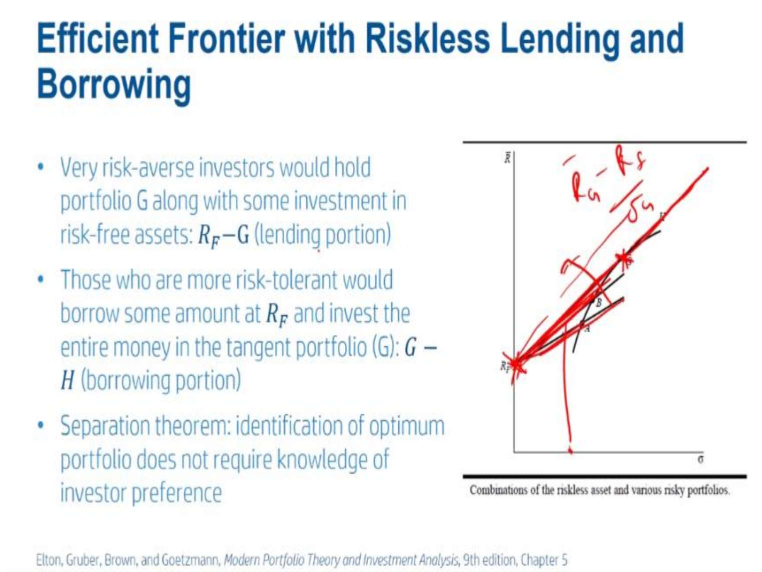

Now, the question is which portfolio is better for us? There appear to be an obvious question. For example, if you look at the points online R F to B, this R_F to B it offers the higher returns than as corresponding to R_F to A. So R_F to B offers higher return as corresponding to R_F to A for any given level of risk as we can see here. For any given level of risk R_F to B offers a high return. Similarly, the points on R_F to G offer higher returns than those on R_F to B.

So this means as we are rotating this line, we are rotating this line in the counterclockwise direction we are making steeper and steeper it suggests that the concept of tangency. It would indicate the highest we can rotate is up to the point of tangency which is here point G as shown here. At this point, the line passing from R_F attains the highest slope, remember that Sharpe ratio $(\overline{R_G} - R_F)/\sigma_G$.

The Sharpe ratio, this slope or the ratio will be highest when the point is tangency point, touching the efficient frontier tangency point. So, if the line is further rotated, then there are no portfolios that are in the feasibility region or on the efficient frontier. Therefore, this line R_F to G would offer the highest return for any given level of risk as compared to any combination of the risk-free rate asset and investment on any point of this efficient frontier.

Therefore, theoretically all the investors whether riskless or less risk preferring or more risk preferring they should all hold this portfolio G. Now those investors that are more risk averse would invest some of the amount in risk-free rate. Those who are risk averse would invest some amount here on R_F and some amount in their portfolio G so that they are on the segment R_F - G This is called the lending segment.

Similarly, those investors who are more risk preferring who would like to take more risk, their portfolio would lie on the segment called borrowing which is on the right side of G. So, they will borrow at this risk-free rate and invest not only their original wealth, but this additional borrowed amount also in the portfolio G. So, this is one of the very important tenets of portfolio problem. Here even if you do not know the investor's risk profile.

Whether he is more risk preferring or less risk preferring, once you identify the portfolio G, you can draw a risk return profile for all the optimum portfolios. This is also some* called separation theorem or two mutual fund theorem because we separate the risk profile of the investor from the investment because we select only two assets, first risk free asset and second the portfolio of assets risky assets called the market portfolio.

This portfolio G and then for all the investors some combination of R_F and G on the lending segment or borrowing at R_F and investing in G would create the desired optimum portfolio depending upon their risk preferences.

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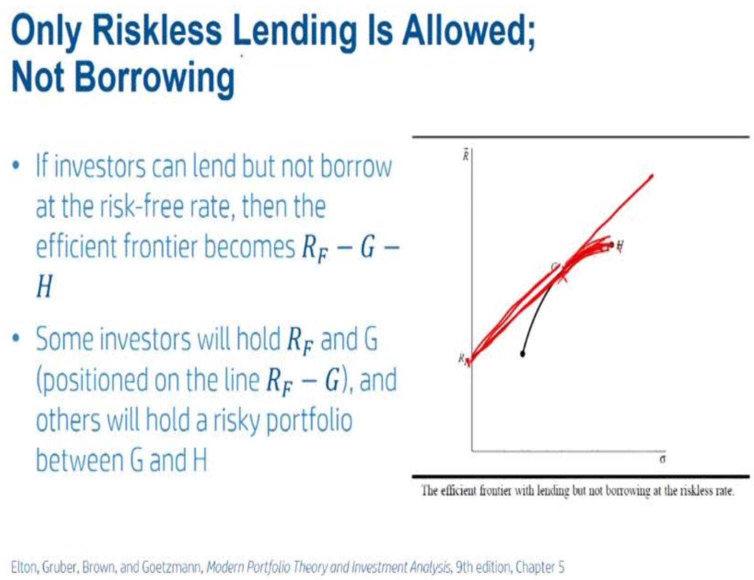

Now, let us discuss a special case where only riskless lending is allowed and borrowing is not allowed. This is a slightly more previous practical case than our previous case as generally lending at let us say government fixed deposits or various other instruments are available at risk-free lending at significantly lower rates. However, borrowing at such lower rates is not available. So, once we put this constraint the segment on the line joining R_F to G, this segment R_F to G where borrowing is not available.

So, we have only lending of this R_F to G available but not on the right side which was the borrowing segment that is not available to investors. So, now efficient frontier would look something like figure here R_F to G on the straight line and then from G onwards this is the efficient frontier. So it is like R_F G-H, G-H is on the right side of R_F - G which was part of efficient frontier in the absence of risk-free asset on the right of G-H here is the point of maximum return.

In effect, the efficient frontier becomes R_F G-H. The investment profile of risk averse investors who are placing some amount in risk free instrument and some in G the lending segment will not change. Only the risk taking investors will not be able to borrow at R_F, they will invest in portfolio on the segment G-H. The portfolios on this G-H segment are relatively riskier than portfolios on R_F G, but these portfolios also offer the higher returns possible at any given risk.

So they are offering highest return for any given risk levels. Also, please note that in this case, we need to only identify two risky portfolio G and H. So once you identify G and H, all the combinations of G and H would lie in between on this curve segment G-H since all the portfolios on curve joining G and H can be obtained by combining G and H.

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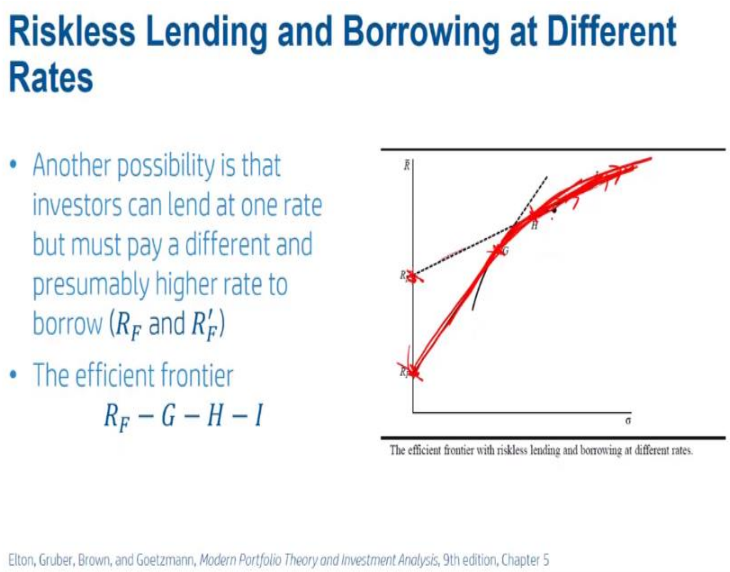

Lastly, we will discuss a special case where the riskless lending and borrowing is allowed at different rate. This is a very practical case which we encounter in daily lives. The interest rate as we borrow is usually much higher than the interest rate as we invest. For example, you can invest at R_F and borrow at $\overline{R_F}$, $\overline{R_F}$ is higher than R_F. So, the investment profiles for this case is shown here. Now, the borrowing at $\overline{R_F}$ which is higher than the lending rate R_F.

And again for the risk averse investor, the risk return profile remains unchanged. However, the region of borrowing is modified due to higher risk-free rate. So, the region of borrowing is here, this is the region for borrowing for him, a higher region. Let us say that with the new borrowing rate the point of tangency is H. So for this $\overline{R_F}$ the point of tangency is H, for the lower rate the point of tendency is G.

So, the risk taking investor have the efficient frontier as the risk portfolio lie between G-H curve so he can invest on G-H curve and after H the tangent line that joins $\overline{R_F}$ to H, so he has this G-H curve and then from H onwards. So, $\overline{R_F}$ H lines which extend from this point H, so for risk taking investor it will be something like this. So, this is G-H and the line extending $\overline{R_F}$ line extended, R dash H line extended beyond point H.

So, effectively your new investment frontier becomes starting from R_F to G, so those who want who are less risk taking they can probably invest some amount in R_F and some amount in G and obtain a portfolio on this lending segment R_F G and then if you are more frisk taking then your portfolio extend beyond G to H and then H onwards from H to further on this line segment R_F dash which is extended. So, your R_F G-H and beyond this line is your complete segment.

For those who are less risk taking they will invest partially in R_F and some amount in G, so they will obtain a portion on this R_F G portion they will invest and those who are more risk taking they will invest beyond G, G to H on this curvature and then straight line which is $\overline{R_F}$ H extended.

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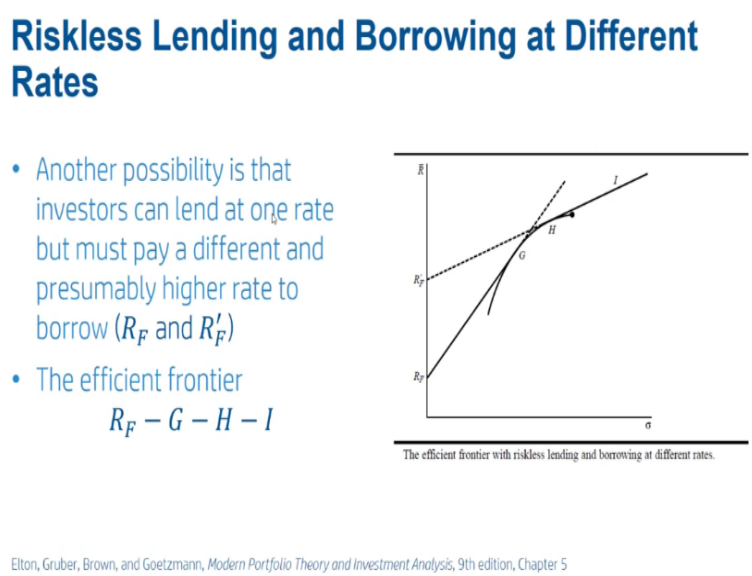

To summarize, in this video we introduced risk free or riskless instrument with efficient frontier. We discussed three cases, first with same riskless lending and borrowing rates, second unlimited riskless lending but no riskless borrowing and third the most practical case that is different risk free lending and borrowing rates where risk free borrowing is at higher rate while risk free lending is at lower rates.

We found that when risk free asset is introduced, one can find an optimum portfolio which can be available or made available to all the investors irrespective of their choices. For example, we found there was a portfolio G when riskless lending and borrowing rates were same and a combination of this portfolio G along with risk free rate can be offered to all the investors irrespective of their choices.

Those investors that are less risk taking would prefer to invest some amount in the risk free rate and some amount in the risky asset while those who are more risk taking they would like to borrow at risk free rate and invest further their original wealth as well as additional borrowed amount in the risk free portfolio. So, these two segments are called lending and borrowing segment respectively.

Lending segment for less risk taking investor who is investing in risk free asset as well as risky asset while borrowing segment for investors who are borrowing at risk free rate and invest the borrowed amount and overall wealth into this risk free portfolio to obtain a position on borrowing segment.

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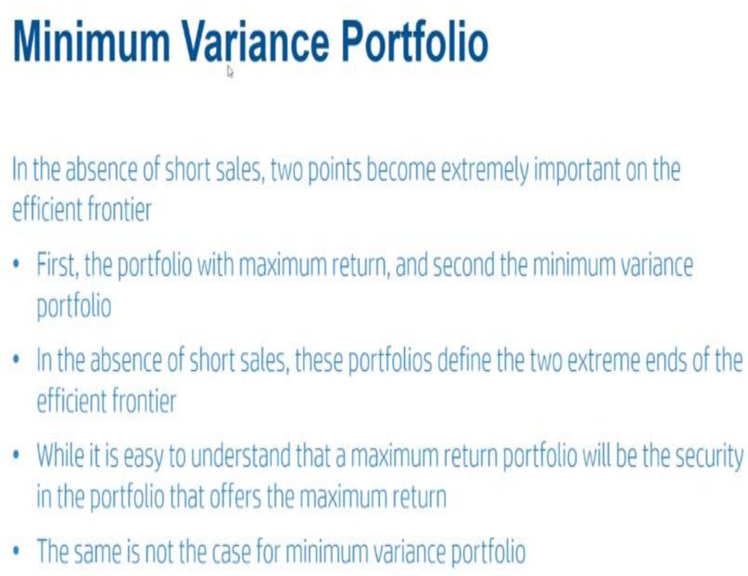

In this video, we will talk about minimum variance portfolio and some of its interesting properties.

In the absence of short sales, two points become extremely important on the efficient frontier. First, the portfolio with maximum return and second the global minimum variance portfolio. In the absence of short sales, these portfolios define the two extreme ends of the efficient frontier.

While it is easy to understand that maximum return portfolio will be the security in the portfolio that offers the maximum return since the returns are simply the weighted average of returns. The same is not the case with the global minimum variance portfolio.

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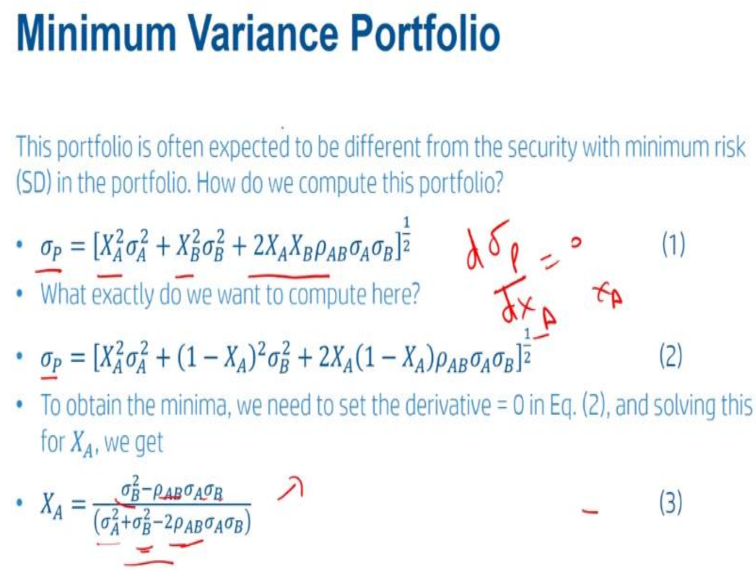

This portfolio is often expected to be different from the security with minimum risk in the portfolio. How do we compute this portfolio? We already know the generic formula for portfolio risk as shown here. The formula is simply $\sigma_P = [X_A^2\sigma_A^2 + X_B^2\sigma_B^2 + 2X_A\,X_B\rho_{AB}\sigma_A\sigma_B]^{\frac{1}{2}}$, this is for two-security case.

Now, this formula in order to come to the minima, we need to simply differentiate this expression σ P that is $d\sigma_P$, will not go into the detailed exposition, but we need to differentiate this expression with respect to dX_A and in order to obtain the minima, we need to set this derivative equal to 0 and solve for X_A. Solving for X_A, we will get something like this, a simple expression like this.

Which is $X_A = \frac{\sigma_B^2 - \rho_{AB}\sigma_A\sigma_B}{\sigma_A^2+\sigma_B^2-2\rho_{AB}\sigma_A\sigma_B}$ where ρ_{AB} is the correlation between securities A and B, σ_A is the standard deviation of security A, σ_B is the standard deviation of security B which expression is given by expression 3 here.

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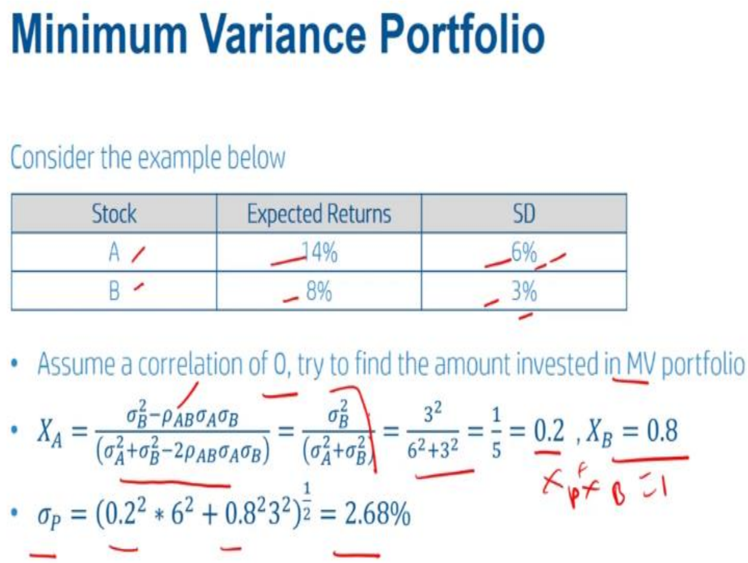

Now, let us consider a simple example here. We have been given two securities in stock A and stock B. Stock A has an expected return of 14 percent and the standard deviation that is risk of 6 percent. Similarly, stock B has expected return of 8 percent and standard deviation that is risk of 3 percent. Now, if we assume that the correlation between these two stocks is 0 and we would want to find the minimum variance portfolio for this particular correlation.

We need to solve for this expression and putting ρ_{AB} = 0 we are left with this term and given that the risk for B is 3 percent and A is 6 percent. We get this X = 0.2. Since we already know that $X_A + X_B = 1$, we get $X_B = 0.8$ and therefore we can compute now σ_P with the expression known to us. σ_P becomes $0.2^2 * 6^2 + 0.8^2 * 3^2$ which is simply nothing but $X_A^2\ \sigma_A^2 + X_B^2\ \sigma_B^2$ which is 2.68 percent. So, this is the risk of minimum variance portfolio here.

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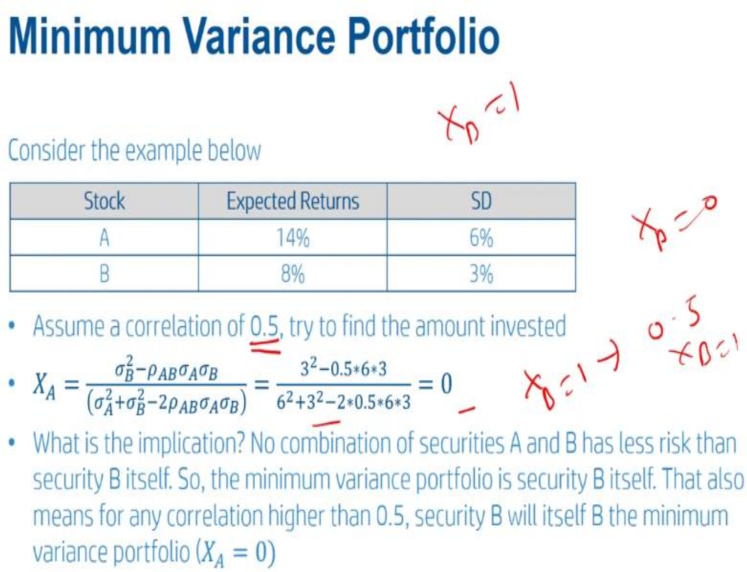

Now, let us assume a correlation of 0.5 and then try to compute this expression. Putting correlation of 0.5, we find that $X_A = 0$. What is the implication here? Since no combination of securities A and B has less risk than the security B itself, so the global minimum variance portfolio is security B itself and that is the reason when we compute global minimum variance portfolio here we get $X_A = 0$ which means X_B equal to simply 1 which is security B itself.

That also means that for any correlation higher than 0.5, the global minimum variance portfolio will be obtained for X_A that are less than 0 or in the absence of short selling for all practical purposes we get $X_A = 0$ or $X_B = 1$ that means for any correlation that are more than 0.5 the minimum variance portfolio in the absence of short selling will be $X_B = 1$ itself that is security B will remain the global minimum variance portfolio.

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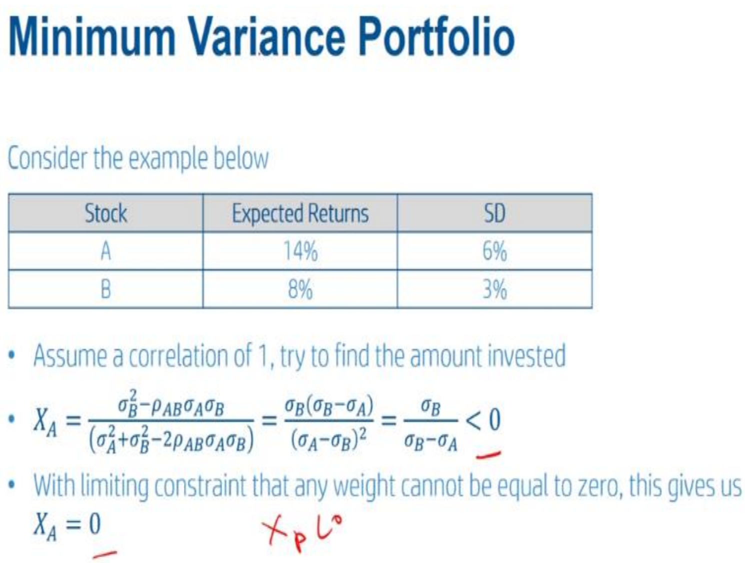

Now, let us assume a correlation of 1 exactly, $\rho_{AB} = 1$ and then try to solve this. In that case, when we do that we find X_A is less than 0. In the absence of short selling which is not a practical thing here and therefore the limiting constraint that any way cannot be equal to 0, this gives us $X_A = 0$ because we are assuming that X_A cannot be less than 0.

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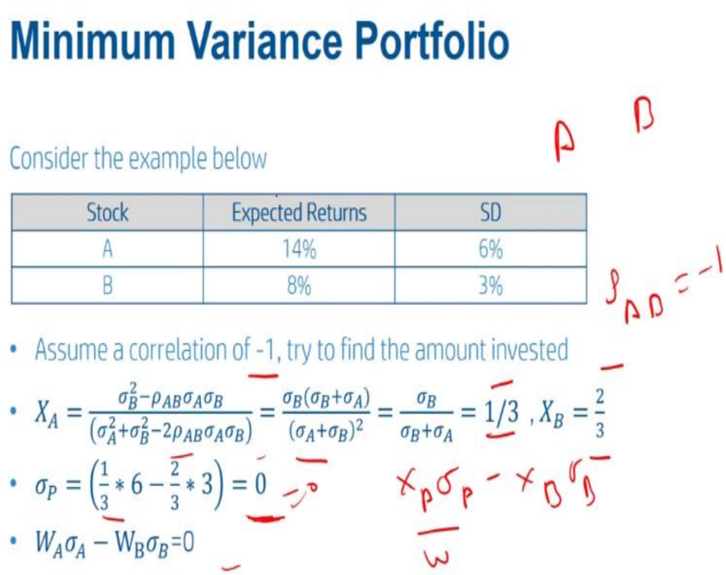

So, we are assuming a correlation of –1 and if we assume a correlation of –1, we compute this expression which works out to be $X_A = 1$ by 3 and $X_B = 2$ by 3. Solving for this we get X = 0 which is nothing but $X_A\ \sigma_A - X_B\ \sigma_B$ on $W_A\ \sigma_A\ W_B\ \sigma_B$, where W_A or X_A's are the proportionate amounts invested in security A and B. Now, this is a special case where ρ_{AB} or correlation between the securities is equal to –1.

However, that is less of a practical case because perfect negative correlations of –1 are rarely observed for long races. However, still this is a theoretically important case, but because in this case we are able to obtain complete diversification that is the complete portfolio risk has become equal to 0. So, this is a theoretically interesting case where all the risk of the security or portfolio has been diversified.

To summarize, in this video we computed the formula for minimum variance portfolio. We also computed and through examples we understood different values of this minimum variance portfolio risk for different values of correlation. We also found that at a particular correlation of –1, the overall portfolio risk can be made 0, and in that case, global minimum variance portfolio risk will become 0 as well.

We also noted that it is not necessary unlike returns where the maximum return or minimum returns are equal to simply the maximum minimum return security in the portfolio. The minimum risk portfolio can have even lower risk than the security in the portfolio with minimum risk.

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Introduction to risk-free lending and borrowing part 1. Till now we have touched upon the use of risk-free instrument and construction of efficient frontier in a more cursory manner. In this video, we will explore the application of risk-free instrument in efficient frontier construction in more detail. Let us first start by introducing the risk-free lending and borrowing at the same interest rate that is r_f. However, is it practical to assume that risk-free lending and borrowing would be available at the same rate?

What are the challenges with this assumption? For example, can you borrow from a government bank like SBI or lend or create fixed deposit with the same bank at the same rate, is it possible? While this assumption has several challenges, still this assumption has a lot of important implications when it comes to portfolio construction and understanding the efficient frontier. Let us start by assuming large number of stocks that are used to construct a feasible region of possibilities.

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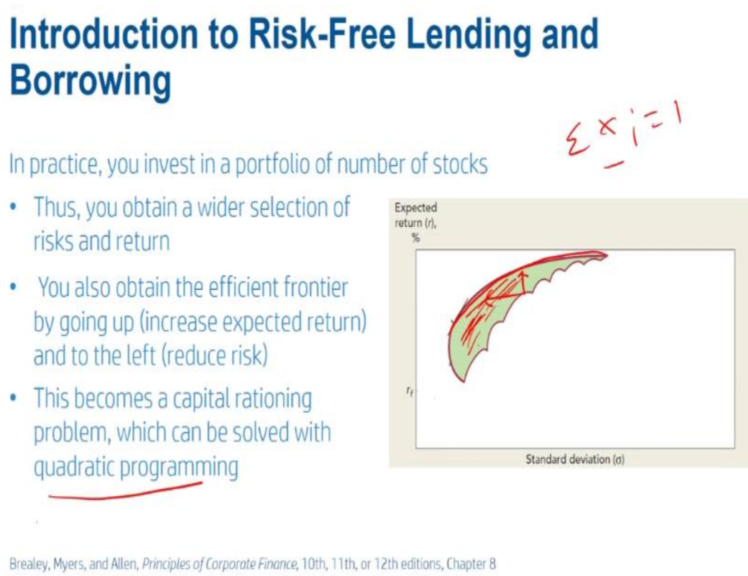

You invest in a number of stocks to construct the portfolio and the feasible region that you locked in will be something like this, the green regions obtained here and you will obtain a very wide selection of risk and return profiles in the form of this feasible region. Now, in this feasible region, you want to go up in order to increase your expected return for a given level of risk or you do want to go to left to decrease the risk for a given level of return.

And as you keep on doing that, you will obtain this efficient frontier or best set of portfolios that offer you best or optimum combination of risk and return. Mathematically, in order to solve this, you need quadratic programming. This is the sort of capital rationing problem where you have a capital constraint, for example you may have a capital constraint like $X_i = 1$ where X_i are proportionate amounts invested in different securities.

And then using this constraint you would like to solve or maximize your return for a given amount of risk with this capital rationing problem and you can solve this with quadratic programming. You will need some computer program or software, there are a lot of easily available software packages that can do this.

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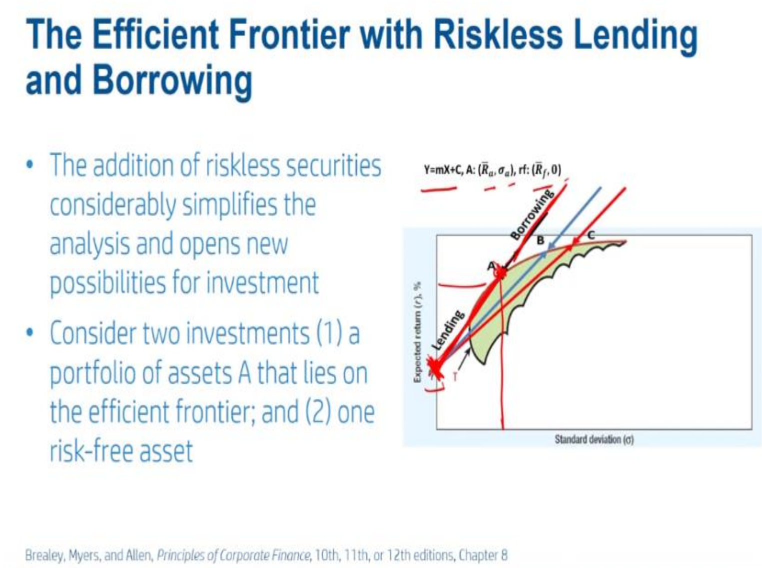

However, addition of riskless securities can considerably simplify our analysis. For example, consider two investments, one in a portfolio of asset A this risky stock and another risk-free asset may be here. When you have this, then your possibilities improve considerably. There are two segments to this investment. One, if you invest some amount in r_f and some amount in A, you obtain what is called lending segment in this region.

Generally, a person who is less risk averse would be standing here or you can borrow it r_f because we assume that borrowing and lending risk free rate are same, so you can borrow it r_f, invest your own wealth along with any additional borrowed amount in asset A to obtain a position on this borrowing segment. This is precisely the equation of a straight line like Y = m X + C which passes through two points.

First point is this point A with expected return of $\overline{R_A}$ and risk for σ_a. Another point which is risk-free asset with expected return of R_f and risk of 0 or standard deviation of 0.

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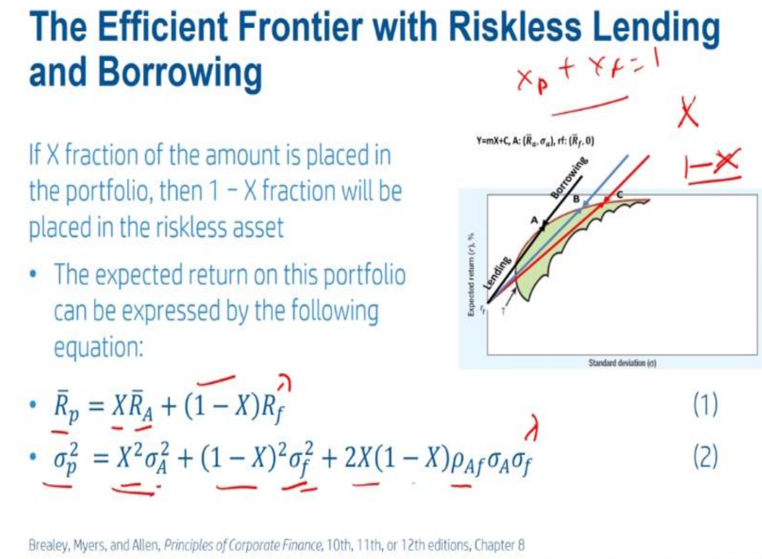

Let us assume that amount fraction X is placed in the portfolio A, X A. Let us call it call it X and then therefore remaining 1 – X amount because of capital constraint the summation of $X_A + X_f$ should be equal to 1 where X_f is the amount invested in risk free asset, they should be equal to 1. So, the amount invested in this asset is 1 – x. And therefore, the expected return on this portfolio Rp = $X\overline{R_A} + (1 - X)R_f$, R_f is the certain return on risk instrument.

Again, we can also compute the risk of this portfolio $\sigma_p^2 = X^2\sigma^2 + (1 - X)^2\sigma_f^2 + 2X(1 - X)\rho_{Af}\,\sigma_A\,\sigma_f$, but please remember here the $\sigma = 0$ because there is no risk with the risk-free instrument and therefore the amount is simply $X^2\sigma^2$.

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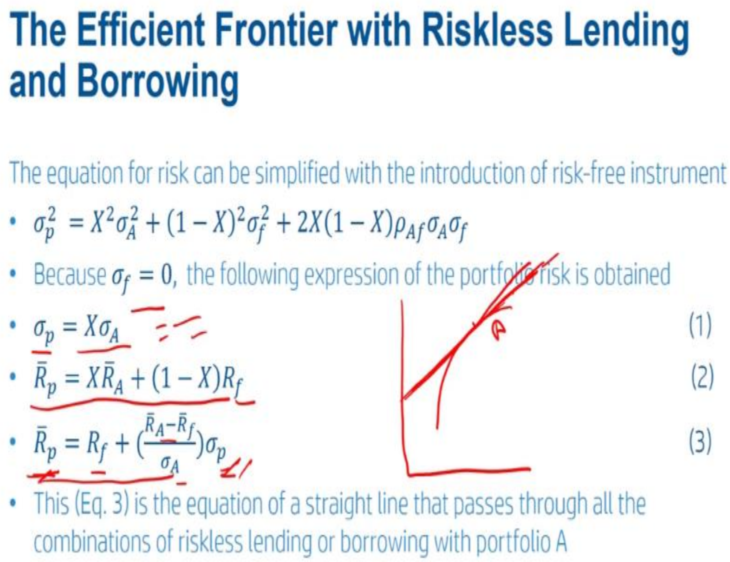

And if we can assume that, then assuming that $\sigma_f = 0$ we are left with the portfolio risk, a very considerably simplified expression because of this introduction of risk-free instrument, we have only this expression as a portfolio $\sigma_p = X\sigma_A$ and we already know that R_p can be demonstrated are described with this expression. Using these two equations 1 and 2, we can further simplify this expression in the form of $\overline{R_P} = R_f + [(\overline{R_A} - \overline{R_f})/\sigma_A] * \sigma_p$.

Now, this is a very simplified form of expression of portfolio expected return and risk and this equation is precisely the straight line that passes through all the combinations of risk-free lending and borrowing with portfolio f. So, if you remember this was that efficient frontier and any point A if you have picked here, then this would represent all the points on this line, this expression would provide you with the expected return and risk relationship which passes through this line. All the points on this line will be described by this equation.

To summarize, in this video we saw that introduction of risk-free instrument where risk-free lending and borrowing can be done at the same rate considerably simplifies the analysis of efficient frontier and we obtain a very simple expression of relationship between expected return $\overline{R_P}$ and risk of the portfolio. This is obtained because the risk-free instrument that is $\sigma_f = 0$ and because of that assumption we are able to obtain this very simple expression of expected return and risk of the portfolio. This is a more generic expression.

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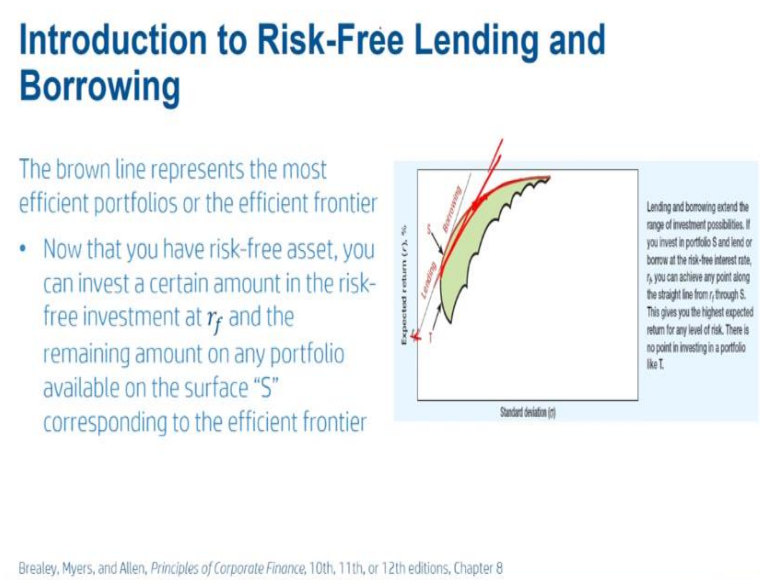

In this video, we will try to find whether there is a special case or optimum portfolio with the presence of risk-free lending and borrowing that is dominating or dominant position as compared to all the other portfolios or risky assets. Please remember as we have this opportunity to invest in risk-free asset as we saw in the previous video, we can take any position on this efficient frontier.

We saw that this brown line represents the most set of efficient portfolios or efficient frontier and now that we have this risk-free asset, I can pick and choose any point on this efficient frontier and combine it with my risky portfolio and find a number of set of opportunities, a very specific set of portfolios that depend upon the risk-free rate and the position that we are taking on this.

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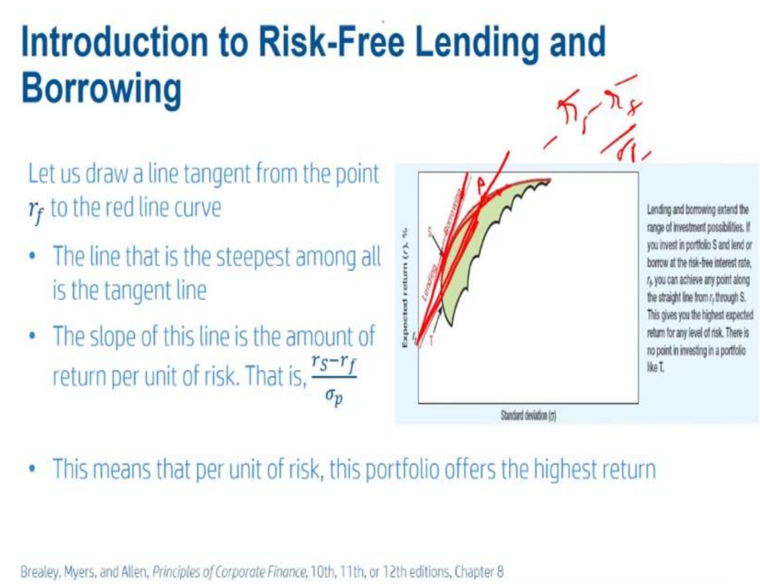

Let us find out if there is a particular portfolio on this efficient frontier that offers us the best set of combinations of risk return profiles. It would be easy to understand that if I keep on moving on the counterclockwise, I would find the steepest line to be at the tangency point, let us call this tangency point as S. Please note, if this tangent line from r_f to S is drawn, let us put it with the red line, this tangency line, this line will be the steepest line from r f to this efficient frontier.

The slope of this line can be easily defined as $(r_S - r_f)/\sigma_p$ where r_S is the expected return on security S, r_f is the return on risk-free instrument and σ_p is the risk of the portfolio that is S. Now given the fact that this slope is the highest, this is the steepest line, we can easily say that this line or this position s offers maximum expected return for a given level of risk and that is true for all the points on this line.

That they offer for a given level of risk highest amount of return possible as compared to any other point on this efficient frontier. For example, if I draw another line like this for a point let us say A, all the positions on this line would be a better combination of risk returns that means higher return per unit of risk as compared to any on this line which is joining r_f to A. Often this ratio is called this ratio $r_S - r_f$ over σ_p is a very important ratio called Sharpe ratio to measure the performance of a portfolio.

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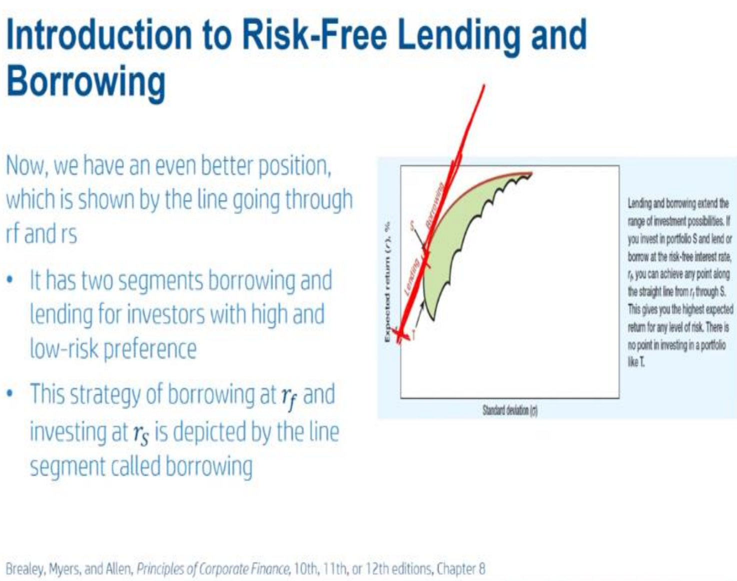

Now, we have obtained a particular portfolio or position which in combination with this risk-free instrument offers us the best or optimum set of positions or a new efficient frontier in fact, so this we can call as a new efficient frontier, which is become available to us because of this risk-free rate. There are two particular segments on this new efficient frontier that are very important to us. One is called lending or investing this one and other is called borrowing.

So, this lending segment is preferred by investors with low-risk preference, while this borrowing segment is preferred by investors with high-risk preference. What do I mean by this? So, those that are less risk averse, they would be borrowing at r_f and investing their own wealth along with this borrowed amount in S to obtain a position on this borrowing segment, which extends from S and beyond on the straight line.

While those who are high-risk averse and do not prefer risk much, they would invest partially their wealth in r_f and partially in S to obtain a position on this segment r_f S which is called lending or investing.

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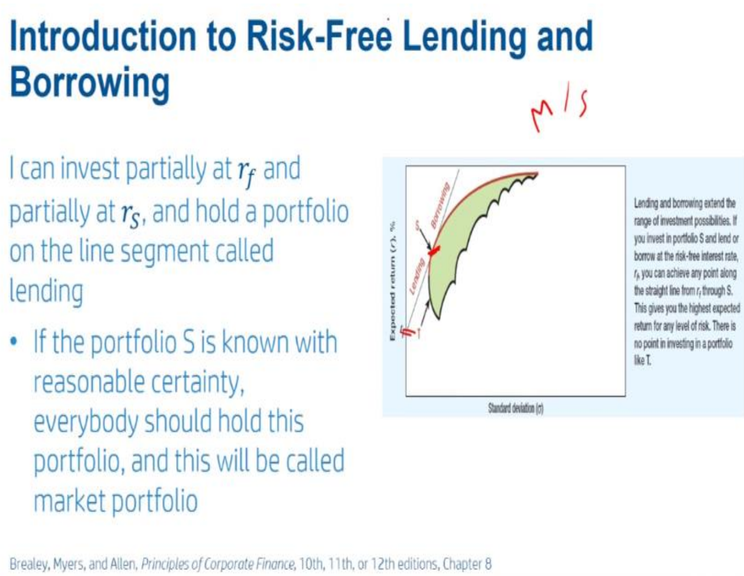

Now, please remember this kind of position on lending and borrowing segment is freely available to all and if this portfolio s is known with certainty, then everybody would be holding this portfolio s, nobody would hold any other portfolio, but some proportion of investment in s and some proportion of investment on borrowing in r_f. And therefore, since everybody is holding these two portfolios only, the only set of risk-free assets are held are that in portfolio S and therefore this portfolio s is also often referred to as market portfolio, some* denoted by M or some* by S. This is a market portfolio which is held by everybody.

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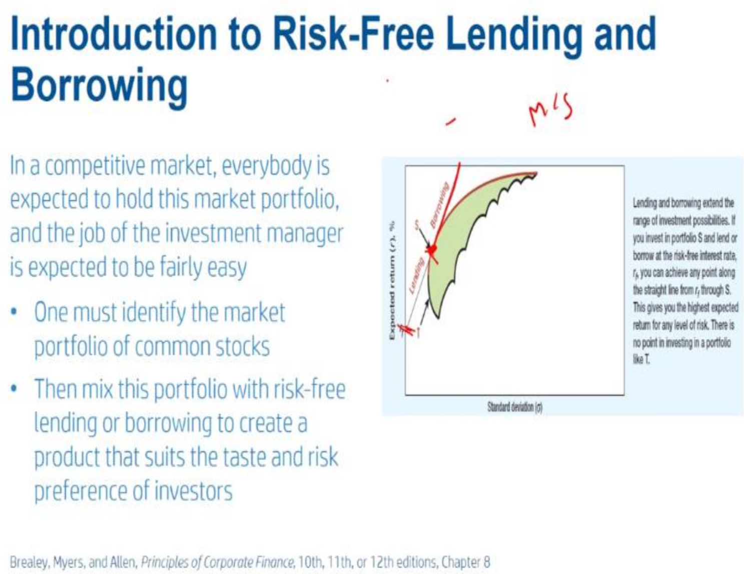

Why we are making this assumption that financial markets are often considered to be very efficient and competitive? So, there is no reason for us to believe that anybody or somebody may have a particular advantageous information for long *, that means everybody will have some similar information and all of them will want to hold the same market portfolio M or what we call S and therefore the job of investment manager becomes very easy, he is to find this market portfolio.

Once he has identified this market portfolio of common stocks, he need to mix it with r_f depending upon the risk preference of individuals. Those who are more risk averse for them some investment in r_f and some investment in S, while those who are more risk taking some borrowing at r_f and the borrowed amount plus own wealth can be invested in S to obtain a position on this borrowing segment.

So mixing this market portfolio or this portfolio S along with risk-free asset in different combination can generate various combinations of portfolios that may suit the tastes and risk preferences of different profile of investors and that is why this is often referred to as two-fund theorem or separation theorem that means the decision to select this portfolio S is independent of investor's risk preference and risk profile.

This is a very important result. So whether investors are risk taking lions or risk fearing chickens, they can be provided their suitable instrument just by mixing this one portfolio S along with the risk-free instrument. To summarize, in this video we saw that when the risk-free instrument is available, one particular portfolio which is the tangency portfolio becomes the optimum position for all individual investors.

And therefore, a fund manager can mix this particular optimum portfolio with a risk-free instrument to provide various combinations of portfolios. This includes portfolios on lending segment for those who are more risk averse and portfolios on borrowing segment by borrowing at rf and investing all in this asset to those who are more risk taking. And therefore, the fund manager can separate the decision of investing from the risk profile of individual investors. This is often referred to as separation theorem or two-fund theorem.

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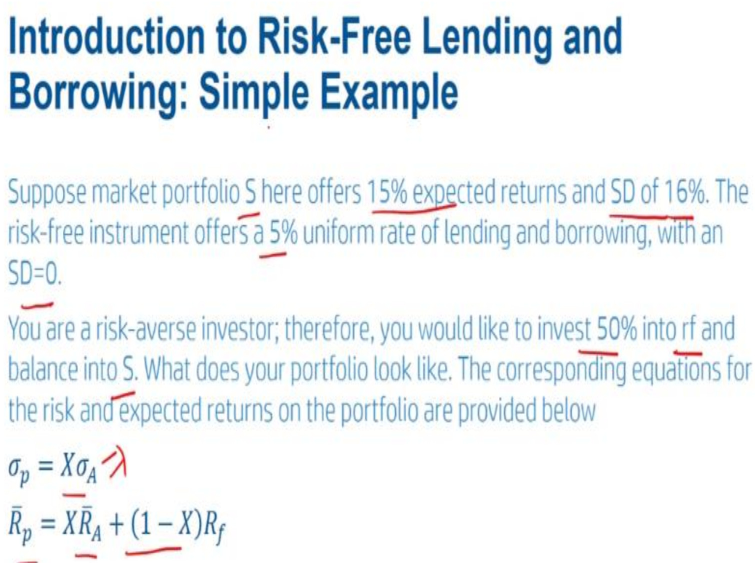

In the previous videos, we have understood the introduction of risk-free lending and borrowing can simplify the analysis considerably and we find an optimum portfolio that is suitable for all. In this video, we will understand the implications of that optimum portfolio with the help of simple numerical examples. Suppose the market portfolio that you have identified the best optimum portfolio S offers 15 percent expected return and the standard deviation of 16 percent.

Also the risk-free instrument available to you offers your 5 percent return of lending and borrowing with a standard deviation or risk of 0. Now, you are a risk-averse investor, therefore you would like to invest 50 percent of your money into r f which is risk free and remaining 50 percent into a portfolio S. Now, what does your portfolio look like? Let us try to find out the profile of your investment with the help of following equations of risk and return.

Already we have seen that the risk of our portfolio would be $\sigma_p = X^*\sigma_A$, while that expected return of the portfolio is $X\bar{R}_A + (1 - X)R_f$.

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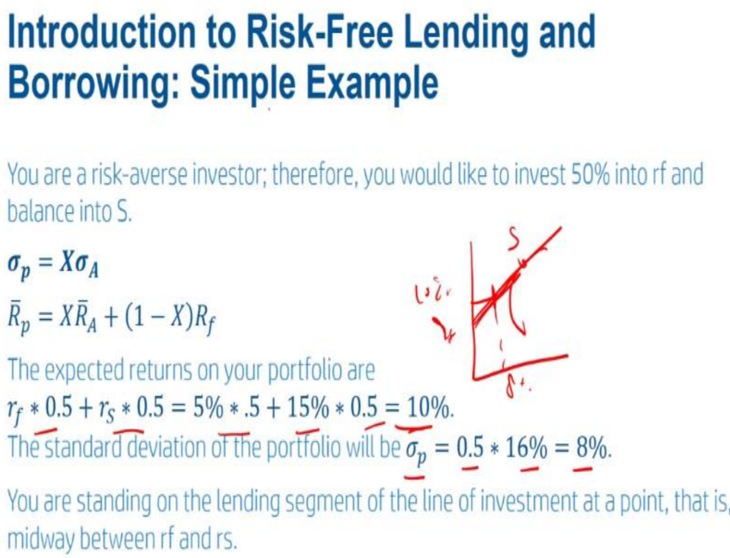

Now, let us consider some values here. So, given that our investment is 50-50 percent in two risk-free instrument and risk-free portfolio, resulting expression becomes r_f *0.5 + r_S*0.5 which is equal to 10%, 5% *0.5 + 15%*0.5 = 10%. The standard deviation of our portfolio σ_p = 0.5 * 16% = 8%. And therefore, at this stage we are standing on that lending segment of portfolio.

And this is the tangency portfolio S, this is r_f, then right now we are standing somewhere in the middle of this which offers us an expected return of 10% and risk of 8%. So, this is midway between r_f and S.

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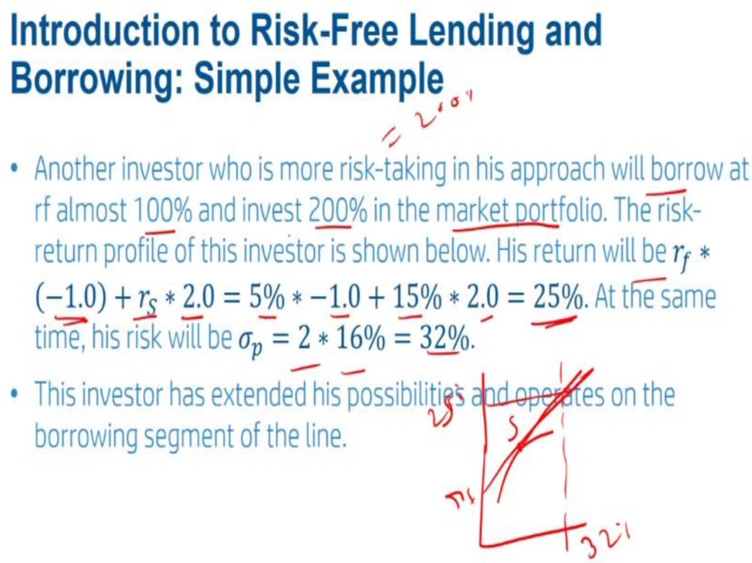

Now, consider another investor who is more risk taking in his approach. This risk taking investor would like to rather borrow at r_f almost 100% which is equal to his initial wealth and invest the total amount that is 100 of borrowed money plus 200% of initial wealth that is overall 200% in this market portfolio S and therefore we can easily compute the risk return profile of this investor in the following manner.

First, his return will be r_f*(–1), this –1 represents the 100% borrowing, + r_S*2 which represents 200% investment in market portfolio. The resulting profile becomes 5%*–1 + 15%*2 = 25%. Now, this is quite a large expected return that he is getting from his portfolio, a very high return. But at the same time, his risk is 2*16% = 32%.

So, while this investor has extended his possibilities and he is obtaining a very high amount of expected return, he is standing on the borrowing segment and his risk has also increased considerably. So his position if this is the tangency portfolio S, this is r_f, then his position is almost here. So while he is getting that higher expected return of 25%, at the same time he is also facing a risk of 32% standard deviation.

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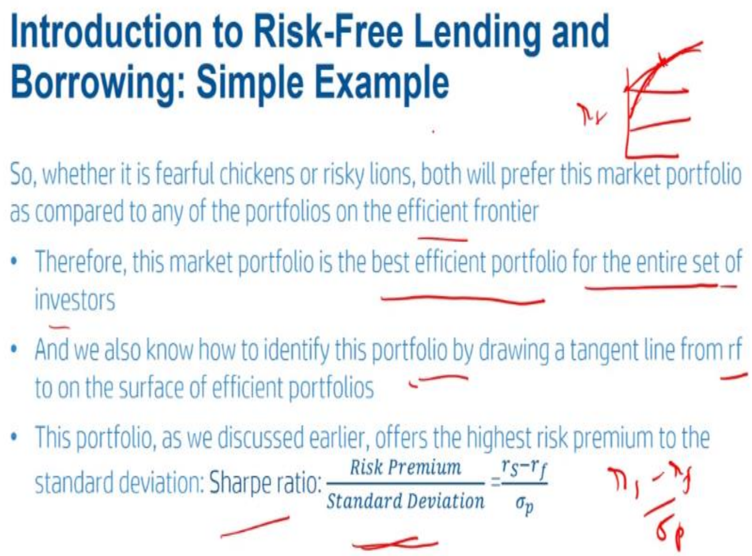

So, we saw whether one is a risk fearful chicken or less risk taking person or a risky lion that means more risk taking person, both of them will prefer to invest in market portfolio than any other combinations available on the efficient frontier, this tangency portfolio what we are calling as market portfolio. And therefore, this market portfolio is the best efficient portfolio from all the entire set of investors. For all the investors, this is the best portfolio and identifying this best portfolio is quite easy.

We need to draw a tangent line from the risk-free instrument r_f to the efficient portfolio. The original efficient portfolios that we identified these efficient portfolios we need to draw a tangent line from this risk-free instrument and this tangent line will give us that market portfolio. This portfolio offers the highest risk premium that means highest value of the Sharpe ratio which is $(r_S - r_f)/\sigma p$, this Sharpe ratio or the slope or the amount of risk premium per unit of risk is offered by this market portfolio is highest with reference to the slope of this line.

To summarize, these numerical we saw whether an individual is risk preferring person or less risk preferring person, one particular portfolio which is the market portfolio or tangency portfolio is the most preferred for all of us. And therefore, a combination of this market portfolio with that risk-free instrument provides us with our required portfolio with the best combinations of risk return.

The slope of this portfolio or the per unit of risk premium for a given unit amount of risk is best across all the set of portfolios available to us on the efficient frontier. And therefore, this market portfolio or optimum portfolio helps us solve the portfolio problem.

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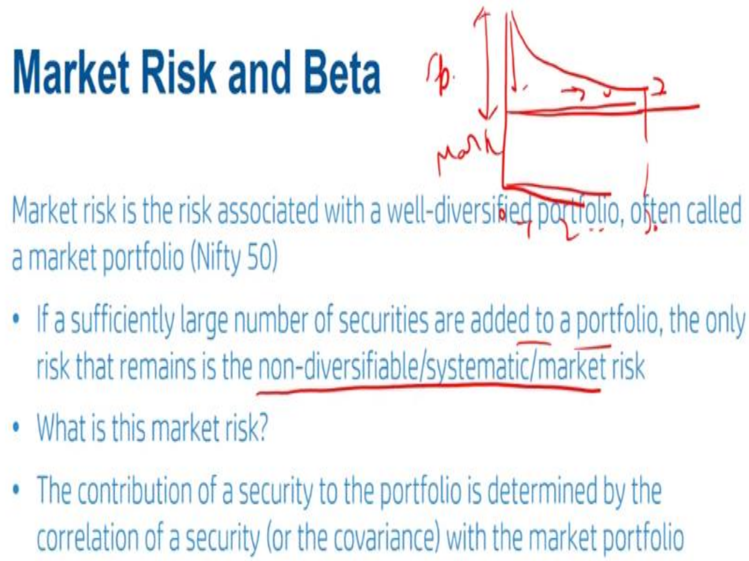

In this video, we will talk about market risk and a very important measure of market risk which is β, which is the sensitivity of a security to market risk. With the help of a simple example, we will also try to understand the computation of betta. Usually, market risk is associated with a well-diversified portfolio. This is so because for a well-diversified portfolio the diversifiable or stock specific risk is eliminated and only it is a systematic non-diversifiable market risk that matters.

For example, a portfolio like Nifty 50 or NYSE index, S and P 500 for such portfolios only market risk matters. So, if sufficiently large amounts of securities are added to a portfolio, the only risk that matters is the non-diversifiable systematic or market risk. What is this market risk? Remember the diagram that we saw earlier. So, market risk is the bedrock of risk. There are two components, one is idiosyncratic stock specific risk let us call it as and market risk, which is driven by the correlation across securities.

Now, as you keep on adding securities 0, 1, 2, 3 after sufficiently large number of securities are added to the portfolio, then this stock specific risk is eliminated and it tends to go to very close to 0, even with 30-50 securities it can be completely eliminated, while the market risk is not eliminated with this diversification of adding more securities and it sustains. However, it is important to know when a new security is added to a portfolio, what is the contribution of the security to the portfolio?

And this contribution of the security to a portfolio is determined by the correlation of that security with the market portfolio. Market portfolio is a portfolio of stocks that carries sufficiently large number of stocks from the market that represents the market and eliminates most of the diversifiable risk.

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This correlation between the security and market portfolio is often measured through β i. This is called β of the stock or sensitivity of the security to the market portfolio. For example, if the market moves by 1 percent up and security moves by 1.5 percent up, then in that case you will say that the β of the security is 1.5. If market moves by 1 percent or a particular amount and security moves by exactly the same percentage, let us say in this case 1 percent then it is said that security has a β of 1 and it has the same risk as that of the market.

Another interesting case is that a security is insensitive to market. Although it is more of a theoretical case, generally securities move to some extent, some way large, some way small, but they definitely move when market shifts. But theoretically, if a security is not at all sensitive to market and therefore whether markets move up or down it does not move, then we say that β of that security is equal to 0 and that security does not have any market risk or systematic risk.

One good example of this would be government securities because government securities are risk free and they are not affected by the market risk or they do not move with the market. So, essentially in summary, this β represents the sensitivity of the security to market movements and therefore this β is a good measure of securities' contribution to portfolio risk because it is not eliminated. When you add the security in the market portfolio, this risk is not eliminated.

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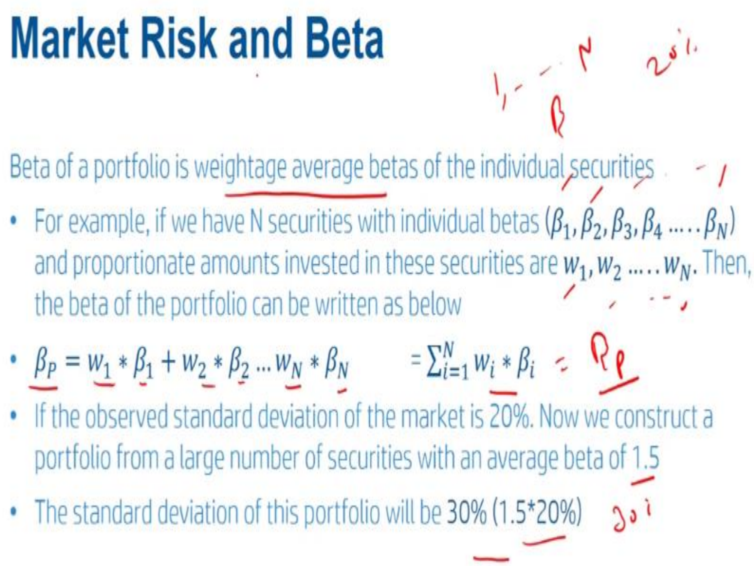

What about the risk of a portfolio or beat of a portfolio? Mathematically, β of portfolio is weighted average of β_s of individual securities. For example, if you have a N stock portfolio 1, 2, N then if individual β_s are β_1, β_2, β_3 and so on up till β_N, their proportionate amounts invested in these securities are w_1, w_2, and so on up till w_N then the β of the portfolio can be simply written as β_p = w_1*β_1, w_2*β_2 and so on up till wN*βN.

So, summation wi βi is the β of the portfolio and this is the β of the portfolio. For example, if you observe that the market has a standard deviation of 20 percent and you construct a portfolio which has a β of 1.5, then the standard deviation of this portfolio would be 30 percent which is 1.5 * 20 percent. So, in summary the β of portfolio is simply the weighted average of βs of individual securities.

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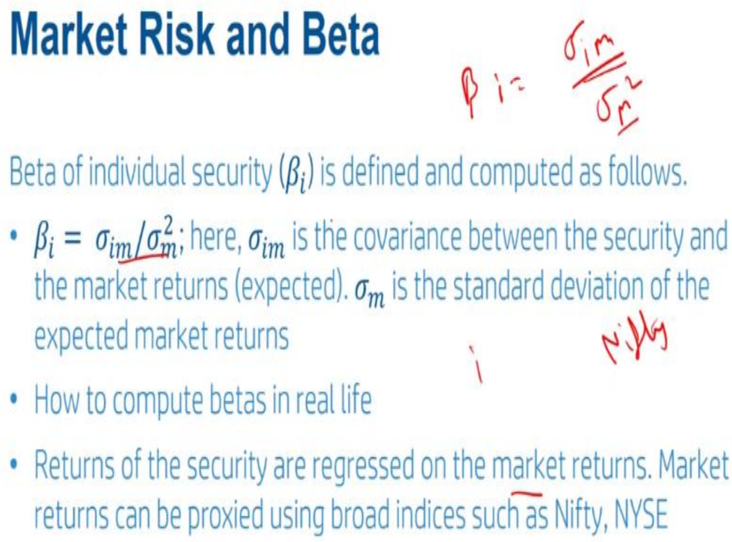

Now, how to mathematically compute β of a security or a portfolio? So, β of a security or portfolio i is simply the ratio between covariance of that security and market which is σ_{im} divided by variance of market. So, this formula represents the mathematical computation. It comes from the regression model as we will see shortly, but this is the ratio of covariance between the security and market divided by the variance of the market.

Now, how to compute βs in real life from the given data? So, security returns are regressed on market returns, for example a security i would be regressed on market returns where market can be proxied like indices such as Nifty or NYSE and in this kind of regression model, the slope of the variable market that is Nifty or NYSE is called β.

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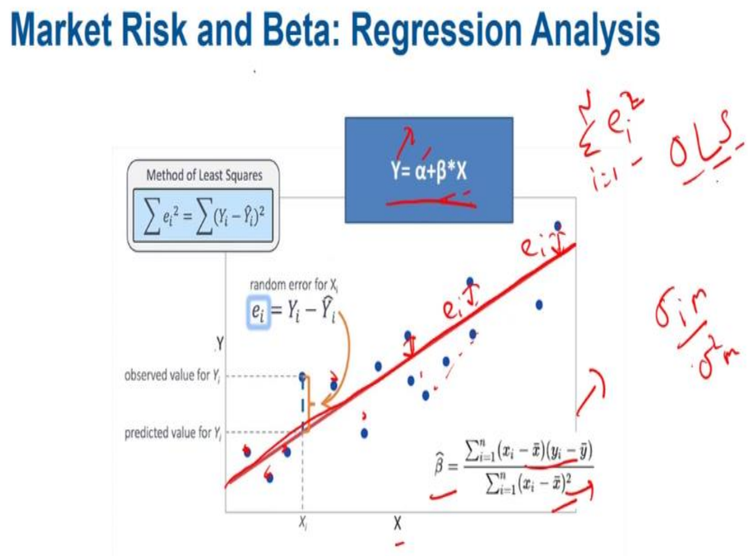

This computation of β requires us to understand the regression analysis a little bit briefly. So, the model that we are running is Y = α + β*X where Y is the return on the security i, it can be any security or a portfolio and α is the constant intercept term here and β is the slope of market variable X. Now mathematically as in a regression model, it works like this you have a scattered plot where you have different return observations plotted along two axes X and Y, where X is the security return, Y is the market return.

So, if these blue points represent those observations, then you fit a line using ordinary least squares method. Notice as the OLS works, you try to minimize this error, this difference, sum of squares of these errors, these are error terms ei's. The perpendicular between fitted line and the observed point you take all these errors ei's and you try to minimize the summation of square of these error terms that is you try to minimize the summation of these error terms.

And when you minimize in that process, you obtain the fitted coefficients, you obtain these αs and βs by fitting this line which has minimum squared residuals and this line is called OLS fit, ordinary least square fit model. In this process, the coefficient β you get is represented by this formula mathematically, which when translated to our contests or our background of market risk becomes σ_{im} which is the covariance between the security and market divided by variance of market σ^2.

So, this is the formula for β where it comes from using regression analysis when an OLS ordinary least squares line is fit between security returns and market returns regression.

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Example: Beta Computation

	A	B	C	D	E	F
Period	R_m	R_1	$R_m - \bar{R}_m$	$R_1 - \bar{R}_1$	$\sigma_m^2 = (R_m - \bar{R}_m)^2$	$\sigma_{im} = (R_1 - \bar{R}_1) * (R_m - \bar{R}_m)$
1	-1.00	3.60	-1.30	-0.10	1.69	0.14
2	-6.00	3.20	-6.30	-0.50	39.69	3.18
3	10.00	4.48	9.70	0.78	94.09	7.53
4	10.00	4.48	9.70	0.78	94.09	7.53
5	-3.00	3.44	-3.30	-0.26	10.89	0.87
6	-11.00	2.80	-11.30	-0.90	127.69	10.22
7	8.00	4.32	7.70	0.62	59.29	4.74
8	-6.00	3.20	-6.30	-0.50	39.69	3.18
9	10.00	4.48	9.70	0.78	94.09	7.53
10	-8.00	3.04	-8.30	-0.66	68.89	5.51
Avg.	0.30	3.70			63.01	5.04

$\sigma_{im} = 5.04, \sigma_m^2 = 63.01, and\ \beta_i = \frac{\sigma_{im}}{\sigma_m^2} = 0.08$

Let us do a simple numerical example here to understand this process. Let us say we have these return observations given to us, market return observations and security return observations for security 1, let us call them R_1 and R_m for market returns. First, we will compute the difference between deviations from mean like $R_m - \overline{R_m}$. So the mean of R_m is zero 0.3. So –1 – 0.3 is –1.3 and similarly we compute deviations for all the observations.

Same goes for the security returns. Its average or mean is 3.7. So for example, first observation 3.6 – 3.7 which is –0.1. And similarly, we will compute the deviations for all the points.

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Example: Beta Computation

	A	B	C	D	E	F
Period	R_m	R_1	$R_m - \bar{R}_m$	$R_1 - \bar{R}_1$	$\sigma_m^2 = (R_m - \bar{R}_m)^2$	$\sigma_{im} = (R_1 - \bar{R}_1) * (R_m - \bar{R}_m)$
1	-1.00	3.60	-1.30	-0.10	1.69	0.14
2	-6.00	3.20	-6.30	-0.50	39.69	3.18
3	10.00	4.48	9.70	0.78	94.09	7.53
4	10.00	4.48	9.70	0.78	94.09	7.53
5	-3.00	3.44	-3.30	-0.26	10.89	0.87
6	-11.00	2.80	-11.30	-0.90	127.69	10.22
7	8.00	4.32	7.70	0.62	59.29	4.74
8	-6.00	3.20	-6.30	-0.50	39.69	3.18
9	10.00	4.48	9.70	0.78	94.09	7.53
10	-8.00	3.04	-8.30	-0.66	68.89	5.51
Avg.	0.30	3.70			63.01	5.04

$$\sigma_{im} = 5.04, \sigma_m^2 = 63.01, and\ \beta_i = \frac{\sigma_{im}}{\sigma_m^2} = 0.08$$

Then we will also compute the standard deviation of market which is deviation squares, for example square of this term, deviation square. And then summation of all these deviations will be averaged out by dividing them with the total number of observations which is 10 here. So we get the 63.01, which is the summation of this upon n. Similarly, we will compute these multiples R m – R m bar into R 1 – R 1 bar.

We will get this number and then multiply this to get this number and this is multiplied by this to get this number and so on. We will get all the 10, $R_1 - \overline{R_1}$ into Rm – $\overline{R_m}$ and then divide it by the total number of observations that is 10 to get this number. Now the ratio of this, this is our covariance between security and market, and this is our standard variance of market. If we divide the σ_{im} / σ_n^2, we get the β measure.

That is 0.08 that is 5.04 divided by 63.01, we will get 0.08, which is a measure of β of the security or portfolio. To summarize, in this video we understood the concept of β which is the sensitivity of security to the market movements and a very important measure of risk of that security. This represents the sensitive of the security to the market movements and therefore the contribution of the security to the portfolio risk.

We also understood that this β can be calculated by regressing the security returns on market returns and the slope coefficient on the market variable is this β. Mathematically, this is the ratio between covariance between security and market divided by variance of market, this is a β for the security or portfolio. To summarize this lesson, for a portfolio with a large number of securities only systematic or market risk that is relevant.

Idiosyncratic stock specific risk is eliminated due to diversification. When two securities are perfectly correlated that is their correlation equal to 1, no diversification is achieved. When two securities are perfectly negatively correlated, maximum diversification is achieved. As we keep on adding more and more securities, the region of all possible risk return scenarios is obtained which is often called feasible region.

On this feasible region, we would like to go up that is increase the expected returns and go to the left that is decreased the risk. When short selling is not allowed, a set of best efficient portfolios from minimum variance portfolio to maximum return portfolio are obtained that dominate all other risk-free profiles and often referred to as efficient frontier. When short selling is allowed, an extended feasible region is obtained. The efficient frontier is also extended to the top right.

In the presence of risk-free security, a new efficient frontier is obtained which is a tangent line joining risk-free security to the tangency point. On this new efficient frontier, the line segment toward the left of the tangency point is called the lending segment, which is a mix of investment into risk-free security and tangency portfolio. The line segment towards the right of the tangency point is called the borrowing segment that is borrowing at the risk-free rate and investing the complete amount into the tangency portfolio.