

Artificial Intelligence (AI) for Investments
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Lecture-31

In this lesson, we will introduce arbitrage pricing theory as application of index models in portfolio management. We will discuss a simple graphical proof of APT. Next, we will discuss various approaches to test APT. We will also try to understand whether CAPM becomes inconsistent in the presence of APT. Lastly, we will discuss the application of these asset pricing models such as APT in active and passive fund management. We will go through a few interesting examples to demonstrate the application of these models in fund management industry.

Arbitrage Pricing Theory (APT)

CAPM had its genesis in the mean-variance analysis

- Investors choose the optimum diversified portfolio on an efficient frontier based on the expected return and variance analysis
- The arbitrage pricing theory (APT) of Ross (1966, 1977) employs a multifactor (alternatively called multi-index) approach to explain the pricing of assets
- It relies on the single/multi-index approach to provide the return-generating process

Arbitrage pricing theory APT. In this video, we will introduce arbitrage pricing theory of ROS. Before we start with APT, please remember that CAPM had its genesis in the mean variance analysis. That is, investors choose the optimum diversified portfolio on an

Arbitrage Pricing Theory (APT)

Using the return-generating process, APT derives the definition of expected returns in equilibrium with certain assumptions

- At the heart of this approach is the arbitrage argument (and thus the name), similar to that employed in the CAPM
- Two items with the same cash flows cannot sell at different prices
- APT is more generic than CAPM in the sense that it does not assume that only expected return and risk affect the security prices

efficient frontier based on the expected return and variance analysis.

In contrast, the arbitrage pricing theory of ROS which is APT employs a multi-factor approach also called multi-index model approach to explain the pricing of assets. It relies on the single and multi-index approach to provide the return generating process. Using this return generating process, APT derives the definition of expected returns in equilibrium with certain assumptions. The definition is as good as these assumptions are held. At the heart of this approach is the arbitrage argument and therefore the name APT or arbitrage pricing theory.

This APT is based on the law of one price. That is two items with the same risk profile and cash flows cannot sell at different prices. This is at the heart of APT. APT is more general in the sense as compared to CAPM. It is more general that unlike CAPM, it does not restrict the forces that affect pricing to expected mean and variance.

Arbitrage Pricing Theory (APT)

The assumption of homogenous expectations remains

- Instead of a mean-variance framework, we make assumptions about the return-generating process
- APT argues that returns on any stocks are linearly related to a set of indices

So, it is more generic in that sense. Nonetheless, the assumption of homogeneous expectations is also there, which is similar to CAPM. That assumption remains. The assumption of the mean variance framework is replaced by the assumption about the process generating stock returns and that process comes from the single and in multi-index models that we have already seen. For example, APT argues that returns on a stock

Arbitrage Pricing Theory (APT)

APT argues that returns on any stocks are linearly related to a set of indices

- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \dots + b_{ij}I_j + e_i$, where
- a_i is the expected level of return on the stock "i" if all indices have a value of zero.
- I_j is the value of the j th index that affects the return on stock i .
- b_{ij} is the sensitivity of stock i 's return to the j th index.
- e_i is a random error term with a mean of zero and variance equal to $\sigma_{e_i}^2$
- Essentially, the above-mentioned equation describes the process that generates security returns

are linearly related to set of indices like I_1 , I_2 and so on I_j .

Here these I s can be indices like oil and gas sector index or banking sector index or broad macroeconomic influences like market influences. So, R_i equal to α_i plus b_{i1} into I_1 plus b_{i2} into I_2 and so on up till b_{ij} into I_j plus e_i . This equation is precisely the return generating process that we are talking about. α_i here is the expected level of return on stock i if all the indices have zero value of the sulphide which is also stock specific term. I_j are the values of j th index for example I_1 , I_2 and so on that affects the return on stock i .

$$R_i = \alpha_i + b_{i1}I_1 + b_{i2}I_2 + \dots + b_{ij}I_j + e_i$$

Now these are the broad influences that affect a large number of stocks at the same time in a similar manner. b_{ij} is the sensitivity of stock i to the j th index. It is something similar to beta that we have studied in the context of CAPM. So, for example, b_{ij} here is the sensitivity of stock i to influence ij . Again, e_i is the random error term with a mean zero and variance equal to $\sigma^2 e_i$.

Arbitrage Pricing Theory (APT)

APT argues that returns on any stocks are linearly related to a set of indices

- $R_i = \alpha_i + b_{i1}I_1 + b_{i2}I_2 + \dots + b_{ij}I_j + e_i$ — CAPM / APT
- For the above model to be more accurate, the following assumptions are made
- $E(e_i e_j) = 0$; for all i and j where $i \neq j$ — e_i is i
- $E[e_i(I_j - \bar{I}_j)] = 0$ for all the stocks and indices — e_i is i
- It is an extension of a multi-index family of models

This e_i or the residual or error term is what reflects the idiosyncratic or stock specific risk which is captured by this variance $\sigma^2 e_i$. So, essentially this above mentioned equation that we can see here this equation describes the process that generates security returns. So, as we have said already this APT theory argues that returns on any stocks are linearly related to a set of indices like I_1 , I_2 and so on and this relationship is reflected in this process which is shown here by this equation R_i equal to α_i plus b_{i1} into I_1 and so on as return generating process. So, for the above model to be accurate description of reality there are certain assumptions and the assumptions are not alien to us. For example, expected value of e_i into e_j equal to zero which means e_i and e_j are correlated.

$$E(e_i e_j) = 0$$

$$E[e_j (I_j - \bar{I}_j)] = 0$$

Here i is assumed to be not equal to j that means these E_i and E_j are stock specific components they are not supposed to be correlated. Any correlation between E_i and E_j would be sort of violation of these index models. The idea is that any variation in stock which is common across different stock is captured by these broad market-bound basis. So, there should not be any correlation left between E_i and E_j and should be captured by this model itself. And therefore, there should not be any correlation explicitly between these residuals stock specific component of variance.

$$E[e_j (I_j - \bar{I}_j)] = 0$$

If you remember we call the variance of these E_i s as sigma square E_i which is the stock specific component of variance or variance which is specific to stock. We also assume that expected value of E_i into I_j minus \bar{I}_j is zero that means the correlation between residual and indices is also zero. So, this E_i is purely stock specific and should not be having any relationship with these stock indices. This is E_i stock specific and these I_j s are indices that are broad market-bound indices affecting a large number of securities. So, the stock specific component which is this E_i is supposed to be uncorrelated with the indices I_j s.

A Simple Proof of APT

Suppose the following two-index model describes the returns

- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + e_i$; also consider that $E(e_i e_j) = 0$
- Here, each index represents a certain systematic risk
- Now, if the investor holds a well-diversified portfolio, only the systematic risk – represented by the indices I_1 and I_2 – will matter
- The residual risk captured by $\sigma_{e_i}^2$ will be close to zero
- The sensitivity of the portfolio to these two components of the systematic risk is represented by b_{i1} and b_{i2}

$$R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \dots + b_{ij}I_j + e_j$$

This is to a large extent ensured by regression model. So, these models are estimated

with the help of some kind of regression. It is assumed and by the design of regression model to a large extent this property is held. And we would have noticed by now that this is a familiar expression this equation which is a part of multi-index family models. So, to summarize we have understood that this model is basically the definition of multi-index family of models.

Essentially APT is the description of expected returns when the process that generate these returns can be defined by single or multi-index model as seen here. The precise contribution of APT is in reaching the equilibrium expected returns from a given single or multi-index model family return generating process. Lastly, in this video we also discussed or compared made a comparison between CAPM and APT. A simple proof of APT part 1. In this video, we will discuss a simple visual proof of arbitrage pricing theory APT.

Let us consider a simple two-index model shown here that is R_i equal to $\alpha_i + \beta_{i1}I_1 + \beta_{i2}I_2 + e_i$. In this model, we also know based on the assumptions of $E_i e_j$ that expectations of $E_i e_j$ is 0 that means E_i and e_j are not correlated. Here each index I_1 and I_2 represents a certain systematic risk like market risk or industry sector specific risk. Now if an investor holds a well diversified portfolio, then only systematic risk like I_1 I_2 should matter and idiosyncratic stock specific risk which is driven by this e_i residual is 0 that means the variance of this e_i is $\sigma_{e_i}^2$. The risk component is supposed to be idiosyncratic and stock specific so it should be close to 0.

And therefore the sensitivity of the portfolio towards these two components I_1 and I_2 that represents systematic risk which is also called β_{i1} β_{i2} . This is something similar to beta in the context of capital. Here each index I_1 I_2 represented a certain systematic risk. It may be related to for example oil and gas firm or a firm in energy or the risk of recession or risk of FMCG firm. And if the investor held a well diversified portfolio only the systematic risk represented by I_1 I_2 will matter.

The residual risk as we discussed $\sigma_{e_i}^2$ will be close to 0. Now let us take a simple example here. Consider three well diversified portfolios as shown here A portfolio B and C their expected returns are given to us and their sensitivities towards these indices I_1 and I_2 are also provided to us. For example expected return of security C

is 10% its sensitivity towards index 1 is 0.3 which is the beta or B_{i1} and its beta or B_{i2} with respect to security index i_2 index i_2 is 0.

Please note that these returns that we are discussing are returns at equilibrium that means there is no arbitrage. Also please remember our discussions over CAPM where the sensitivity towards a single index which was market portfolio in the context of that single index in CAPM that single index was market portfolio. Although securities that were in equilibrium as per CAPM they were defined by a line called SML security market line which had two axes R for expected return and B_1 which was the sensitivity with respect to market or you can call it beta. Now in this case since there are two indices and we have two sensitivities B_1 and B_2 therefore we can safely assume that these portfolios will lie on a plane like this where one axis is expected return axis on security i one is B_1 and one is B_2 or you can call them B_{i1} or B_{i2} as well. If we believe in that plane the generic equation of any plane which passes through three axis can be those axis are B_{i1} B_{i2} and expected return \bar{R}_i that plane can be easily described by this equation.

A Simple Proof of APT

Consider the three well-diversified portfolios shown below

Portfolio	Expected Return (%)	b_{i1}	b_{i2}
A	15	1.0	0.6
B	14	0.5	1.0
C	10	0.3	0.2

- The generic equation for a plane is as follows: $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$
- Can we solve for the values of λ_0 , λ_1 , and λ_2 using the values provided in the table?

The mathematical proof of this equation is not discussed here \bar{R}_i equal to λ_0 plus λ_1 times B_{i1} plus λ_2 times B_{i2} . Here λ_0 λ_1 and λ_2 are unknowns we have a equation three unknowns and three observations we have A B and C. So therefore we can find a deterministic solution we can find a solution for this equation using the information from these points and we can solve for the values of λ_0 λ_1 and λ_2 . Let us do that. Once you solve the equation of plane using these three information point you get this equation \bar{R}_i equal to 7.

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$$

75 plus 5 into B_{i1} plus 3.75 into B_{i2} this is our equation of plane. This can be easily solved with any simple calculator financial calculator or excel it is quite easy. We need to formulate three equations each one equation for each of these points and then solve for

unknowns λ_1 λ_2 which are obtained here. Now consider a third portfolio E

A Simple Proof of APT

Consider the three well-diversified portfolios shown below

Portfolio	Expected Return (%)	b_{i1}	b_{i2}
A	15	1.0	0.6
B	14	0.5	1.0
C	10	0.3	0.2

- The returns are provided at equilibrium: No arbitrage
- Remember our discussion of CAPM with sensitivity towards a single index (market portfolio) where all the securities in equilibrium were lying on a straight line (two axes: R and b_1)
- Here, since we have two sensitivities (two betas with respect to each axis), we can safely assume that these three portfolios will lie on a plane (three axes: R , b_{i1} , and b_{i2})

with expected return of 15 percent b_{i1} of 0.

6 and b_{i2} of 0.6. Let us compare this portfolio E with another portfolio D that places one third of the amount in portfolio A one third in portfolio B and one third in portfolio C. How do I compute my expected return on portfolio D and its b_i 's noting that it invests one third in all the securities A B and C it is quite easy knowing that this is a one third portfolio of ABCD we can provide sorry ABC we can provide one third of the expected returns which is 13 percent and we can similarly we can compute the averages of b_{i1} and b_{i2} which is shown here for example for this portfolio we can call it BP1 since the portfolio BP1 is 0.6 which is nothing but a simple average of these three values and similarly for b_{i2} BP2 we can compute the simple average of these three values to get this value 0.6 although we could have already computed this with the help of this equation of plane by substituting values we could have computed the expected return easily but a simple average would also do so the expected return is 13 percent and sensitivities are 0.

6 and 0.6 with respect to index I_1 and I_2 . Let us compare this portfolio D with portfolio E notice that portfolio E has same b_i 's that means b_{i1} and b_{i2} are same as 0.6 as portfolio D but the expected return it offers is higher than portfolio D which was 13 percent and therefore it appears that portfolio E is undervalued because it offers a higher return as compared to portfolio D despite the fact that the risk profile is same since these

A Simple Proof of APT

We get the following equation: $\bar{R}_i = 7.75 + 5b_{i1} + 3.75b_{i2}$

Portfolio	Expected Return (%)	b_{i1}	b_{i2}
A	15	1.0	0.6
B	14	0.5	1.0
C	10	0.3	0.2

- Now, consider a third portfolio E with expected returns of 15%, $b_{i1} = 0.6$ and $b_{i2} = 0.6$
- Compare E with another portfolio D that places one-third in A, B, and C
- What is my expected return and sensitivities of D, and are there arbitrage opportunities?

are diversified portfolios only the systematic risk will matter which are represented by B_{i1} and B_{i2} which are the sensitivities of portfolios I and D with respect to the only two possible indices as assumed I_1 and I_2 . Since E offers a higher expected return by the arbitrage argument or law of one price these two portfolios cannot sell for a different price for a very long time or have expected returns that are different from each other which in case of portfolio E is 15 percent and in case of portfolio D it is 13 percent that means portfolio E is undervalued and offers a higher expected return. So the arbitrageurs or in general you can say investors as soon as they observe this kind of mispricing they would buy E they will go long in E buy it and sell D short which guarantees a riskless profit of 2 percent where riskless because the risk profile of portfolio E and D as represented by B_{i1} and B_{i2} is identical so when they go long in E and short in D they essentially eliminate this risk that is coming from B_{i1} and B_{i2} and therefore as more and more people will buy E, E will start to fall its expected return will fall its price will rise and relative the price or expected return related to portfolio D will change for example relative to D its expected returns will fall and relative to D its price will rise till the time that both of them will reach the equilibrium plane.

For example if E is away from plane and D lies on the plane and E is much further away then E will be driven towards this plane and will meet point D within a very short while. It will continue to fall until it reaches this plane which is defined by points A, B and C. We have already identified a point which is point D corresponding to point D and E will fall on this point. This plane that we are discussing it is made on three axis. Axis 1 is expected return \bar{R}_i , 1 is B_{i1} which is the risk or sensitivity with respect to index i_1 and B_{i2} which is the risk or sensitivity with respect to index i_2 .

A Simple Proof of APT

Solving for D, we get the following values

- $b_{p1} = \frac{1}{3} * (1.0) + \frac{1}{3}(0.5) + \frac{1}{3}(0.3) = 0.6$

- $b_{p2} = \frac{1}{3} * (0.6) + \frac{1}{3}(1.0) + \frac{1}{3}(0.2) = 0.6$

- $\bar{R}_D = \frac{1}{3}(15) + \frac{1}{3}(14) + \frac{1}{3}(10) = 13$

- D has an identical risk profile offered by a lower return

- We could also have computed the expected return on \bar{R}_D using the equation of the plane

- $\bar{R}_D = 7.75 + 5b_{p1} + 3.75b_{p2} = 7.75 + 5 * 0.6 + 3.75 * 0.6 = 13$

$$b_{p1} = \frac{1}{3} * (1.0) + \frac{1}{3}(0.5) + \frac{1}{3}(0.3) = 0.6$$

$$b_{p2} = \frac{1}{3} * (0.6) + \frac{1}{3}(1.0) + \frac{1}{3}(0.2) = 0.6$$

$$\bar{R}_D = \frac{1}{3} * (15) + \frac{1}{3}(14) + \frac{1}{3}(10) = 13$$

So any securities whether it was undervalued or overvalued at equilibrium it will be driven towards this equilibrium plane if it is overvalued then it will be below and if it is undervalued it will be above. It will be driven towards this plane by arbitrage argument because if it is not on the plane this will lead to arbitrage opportunity and such securities that are undervalued or overvalued they will converge to this plane. To summarize in this video we discussed the simple visual proof of arbitrage pricing theory. We saw that arbitrage pricing theory is derived by law of one price or arbitrage argument that is if a security is underpriced it will fall above that equilibrium plane. If it is overpriced it will fall below that equilibrium plane.

This equilibrium plane is defined by three axis. What are these axis? These axis are return expected return axis \bar{R}_i , B_{i1} sensitivity towards index $i1$ and B_{i2} sensitivity of the security towards index $i2$. By arbitrage argument or law of one price if a security is mispriced and not on this plane whether up or down it will be driven by arbitrage trading activity it will be driven towards this plane let us say if it is underpriced and it falls above its expected returns are higher then because of arbitrage argument it will be driven towards this plane. If it is overpriced it will be below this plane and driven upwards towards this plane because of arbitrage opportunity and ultimately within a short frame it will lie on this arbitrage equilibrium plane.

A Simple Proof of APT

The general equation of the plane in return, i.e., b_{i1} and b_{i2} space, is shown below

- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} \Rightarrow$ APT
- This is the equilibrium model provided by APT when the returns are generated by a two-index model
- Here, λ_1 and λ_2 are the increases in returns for one unit increase in b_{i1} and b_{i2}
- Essentially, λ_1 and λ_2 reflect the returns for bearing the risks associated with the indices I_1 and I_2

A simple proof of APT part 2. In this video we will conclude our discussion on the proof of APT with our broad understanding of various properties related to PPT and the equilibrium model definition. In the previous video we discussed that the general equation of plane includes three axis which are return B_{i1} B_{i2} space which is shown here

\bar{R}_i equal to λ_0 plus λ_1 into B_{i1} plus λ_2 into B_{i2} which is precisely the equation of this plane equilibrium plane. This definition is what is the contribution of APT to the asset pricing model as defined by multi index and single index model. So, this equilibrium model is precisely provided by APT where the returns are generated by two indices I_1 and I_2 and λ_1 and λ_2 here are the increase in returns or you can call them risk premia for one unit increase in or one unit change in B_{i1} and B_{i2} . So, these are the risk premia associated with indices I_1 and I_2 where sensitivities with respect to these indices for a given security i is B_{i1} and B_{i2} .

A Simple Proof of APT

Consider a zero b_{ij} portfolio with no sensitivity to either index

- If it has no risk, then it should offer a risk-free return $\lambda_0 = R_F$
- In case the riskless rates are not available, then instead of R_F , we denote it by \bar{R}_Z , i.e., the return on the zero-beta portfolio (what is a zero-beta portfolio?)
- Imagine a portfolio that mimics index 1 and, therefore, has $b_{i1}=1$
- Also, it is not sensitive to I_2 and, therefore, has $b_{i2}=0$

So, to summarize λ_1 and λ_2 essentially reflects the returns the extra returns for bearing the risk that are associated with indices I_1 and I_2 . Let us consider a 0 B_{ij} portfolio that means a portfolio which has no sensitivity to either of these two indices I_1 and I_2 that means it does not have any risk. Since it does not have any risk therefore it should only offer a risk free rate which is R_F . Now, in this previous expression we have already seen that this B_{i1} is 0 B_{i2} is 0 then in that case \bar{R}_i equal to λ_0 and we are saying that λ_0 is equal to R_F . So, \bar{R}_i is equal to R_F which is same as λ_0 .

A Simple Proof of APT

For this portfolio, the equation $[\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}]$ becomes

$$\bar{R}_1 = R_F + \lambda_1 \text{ and } \lambda_1 = \bar{R}_1 - R_F$$

$$\text{Similarly, } \lambda_2 = \bar{R}_2 - R_F$$

The above analysis can be generalized to a j index case shown below

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$$

$\lambda_0 = R_F$ and $\lambda_j = \bar{R}_j - R_F$ where the return-generating process can be described as

$$R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \dots + b_{ij}I_j + e_i$$

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$$

$$\bar{R}_1 = R_F + \lambda_1$$

$$\lambda_1 = \bar{R}_1 - R_F$$

$$\lambda_2 = \bar{R}_2 - R_F$$

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$$

So, we have the value of λ_0 one parameter. However, many times risk less rates are not available. So, you tend to use 0 beta portfolios which have no sensitivity towards any indices. So, that is why we are calling them 0 beta. The 0 beta portfolio will have a beta of 0 that means B_{i1} of 0 B_{i2} of 0 with respect to indices I1 and I2.

So, these will be proxied if risk less rates are not available. Now, let us imagine a portfolio that mimics index I1 or index 1 that means it has only one sensitivity which is with respect to index 1 and that is B_{i1} . So, B_{i1} equal to 1. It has no sensitivity with respect to the other index which is I1 and I2 and therefore B_{i2} equal to 0.

A Simple Proof of APT

The above analysis can be generalized to a j index case shown below

- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$
- $\lambda_0 = R_F$ and $\lambda_j = \bar{R}_j - R_F$
- The derivation assumes here that both the indices are orthogonal
- In practical situations, there are always correlations between the risk factors represented by two indices
- Researchers orthogonalize both indices to remove any common component. In that case, the new indices may not be well-defined

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$$

Let us discuss the properties of this portfolio. In the equation that we saw earlier for two index equilibrium returns \bar{R}_i equal to λ_0 plus $\lambda_1 B_{i1}$ plus $\lambda_2 B_{i2}$. In this particular case where B_{i2} equal to 0 and B_{i1} equal to 1 this portfolio becomes \bar{R}_i equal to R_F plus λ_1 . Remember λ_0 equal to R_F . So, we substitute that value and therefore here we get λ_1 equal to \bar{R}_1 minus R_F . Similarly, we can also obtain λ_2 equal to \bar{R}_2 minus R_F by simply suggesting that there exists a

portfolio with only sensitivity towards this index I_2 , B_{i2} equal to 1 which is the sensitivity with respect to index I_2 and all the other B_{ij} as 0.

For example, if there are only two indices as we have assumed then B_{i1} equal to 0. So, using that property we can find λ_2 equal to \bar{R}_2 minus R_F . This analysis can be generalized to a J index case also. For example, the equilibrium model for a J index case can be written as this \bar{R}_i equal to λ_0 plus $\lambda_1 B_{i1}$ so on up to $\lambda_j B_{ij}$. Here, if we keep assuming that λ_0 equal to R_F and λ_j equal to \bar{R}_j minus R_F where the return generating process is given by this equation, we can define APT very simply by this equation.

This is our APT equation where λ_0 equal to R_F λ_j , generic term for λ_j is \bar{R}_j minus R_F and this return this equilibrium model, this APT equilibrium model is driven by this multi index return generating process. So, this is our multi index return generating process which leads to this equilibrium model as defined by APT. So, with this simple derivation, we find this equilibrium model and these values for λ s that are risk premia for different thesis. This derivation assumes that all these indices in this case if there are two indices, then both the indices are orthogonal, if there are multiple indices or these indices are orthogonal. When we say orthogonal that means all these indices I_1 , I_2 and so on are not correlated with each other.

However, in practical situations, there are always correlations between these indices or risk factors. For example, if one of the risk factor or index I_2 is oil and gas index, chances are it will be heavily correlated with one of the other index let us say I_1 is market index. So, chances are these two will be definitely correlated. Researchers try to orthogonalize these indices mathematically by removing any common component and create new indices I_1 dash and I_2 dash. However, these I_1 dash and I_2 dash may not have the exact concrete meaning that was earlier.

For example, if I_1 was earlier market index, I_1 dash may not necessarily be proxy of market or I_2 dash, for example, I_2 was earlier oil and gas industry index, then I_2 dash may be slightly different in its behavior as compared to oil and gas industry, maybe something different. So, these indices, these new indices that are orthogonalized not correlated. So, mathematically they have a good property, but they are not well defined.

To summarize the APT model, we started with the simple CAPM. In a straight line CAPM, there was only two axis return and beta axis.

We translated this and found two coordinates corresponding to each point that denoted a portfolio. Now, we translated this to a two index APT model where two indices were I_1 and I_2 and there was one return index \bar{R}_i and two beta indices, beta axis corresponding to each of the I_1, I_2 indices, we had two beta axis and therefore we had to use three coordinates one corresponding to the return and two corresponding to B_{i1} and B_{i2} , betas. This led to the definition of plane and that plane also defines our efficient frontier. So, in the context of CAPM, we had the efficient frontier defined by only two coordinates. Here we have three coordinates, one corresponding to return and one, two corresponding to two beta axis, which define the efficient frontier as well.

If any portfolio which is represented by a point is above or below this plane, that means that security or portfolio is under or overpriced. And therefore, this violates the law of one price. That means at equilibrium, all the securities that have similar risks should fall on the same point. In this case, that point becomes a plane so they should fall on that equilibrium equity plane. If this law of one price is violated, then arbitrageurs and other market participants may conduct a riskless arbitrage, they will buy the undervalued security and sell the overvalued security.

A Simple Proof of APT

If the law of one price is violated, then arbitrageurs may conduct risk-less arbitrage by selling (or buying) the under (or over) priced portfolio and taking a counter position in the portfolios that are fairly priced

- This will drive the prices of the inefficient portfolio towards this plane, that is, efficient frontier or efficient plane
- The implication of this riskless arbitrage is that all portfolios in the equilibrium would lie on this plane, that is, an efficient frontier
- That is, in the space defined by three coordinates: expected return, b_{i1} , and b_{i2}

And as more and more participants keep on doing that, these securities will drive towards that equilibrium plane and become fairly priced. So, if they are over or above or below this plane, they will be driven towards this equilibrium plane and become efficiently priced because of this arbitrage activity. The implication of this riskless arbitrage is that in equilibrium, all the security should lie on this plane. And this plane is

precisely defined by three coordinates expected return, BI_1 and BI_2 . This is in the case when the process that is generating return is assumed to be driven by two indices I_1 , I_2 .

A Few Important Points About APT

In the context of CAPM, it was needed to identify the "market portfolio," and, therefore, all the risky assets

- While testing CAPM, one can always question whether all the securities are truly captured in the risky assets
- Therefore, have we achieved the true market portfolio?
- However, in the context of APT, arbitrage conditions can be applied to any security or portfolio
- Thus, it is not necessary to identify all the risky securities and market portfolio

If the process is a return index model I_1 , I_2 , I_j and so on, then the return generating process will lead to an APT model which will also reflect the sensitivities towards all these j indices BI_1 , BI_2 and so on up till BI_j . A few important points about APT. In this video, we will talk about some interesting and contrasting aspects about APT that differentiated from other asset pricing models such as CAPM. In the context of CAPM, we need to identify the market portfolio and identification of market portfolio requires identification of all the risky assets that comprise market portfolio. While testing CAPM, one can always come back and question whether all those securities are truly captured in this risky asset.

That means have we identified all the risky assets? This is a critical question and difficult one at that. So, have we truly achieved and constructed market portfolio? This question needs to be answered while identifying and working with CAPM. However, in the context of APT, arbitrage conditions can be applied to any set of securities or portfolios. And therefore, when it comes to APT, it is not necessary to identify all the securities in the market and therefore it is not necessary to identify the market portfolio.

That makes APT a very generic one. So, for example, APT can be tested for a very small number of stocks. For example, a simple index like Nifty 50 which comprises 50 stocks can be utilized to test APT. So, only 50 stocks making up Nifty 50 can be used

and tested for APT. Given this advantage, many studies and research papers argue that us while tests of CAPM are simply nothing but test of single factor APT where that single factor is market, market factor. Because one can always question the validity of market portfolio which is a set of all the risky asset in that given set of securities.

A Few Important Points About APT

APT can very well be tested for a small number of stocks, for example, all the 50 stocks making up the “Nifty-50” index

- Given this advantage with APT, many studies argue that the tests designed for CAPM are actually the tests of single-factor APT
- Therefore, they utilized a limited number of securities, which arguably may not capture the entire market

So, one can say that if we do not have all the securities whatever form or shape we tested that was single factor APT where we had the market factor as part of that single index return generating process that we have discussed. Because these studies utilize a limited number of securities which arguably may not capture the entire market. One caution needs to be highlighted here. The systematic influences that are used as a part of return generating process that we saw which is R_i equal to α_i plus $b_{i1}i_1$ plus $b_{i2}i_2$ and so on plus error term. All these indices or systematic influences that one has used should be adequately described.

That means there can be an issue when you have large number of securities. Possibility is that there are large number of indices or systematic influences that may affect the set of securities and therefore identifying all the relevant set of indices or systematic influences or factors may be an issue while testing APT. To summarize, in this video, we discussed that APT is extremely generic in nature. APT allows us to describe

A Simple Proof of APT

The straight line in the CAPM had two axes: return and beta axes

- Thus, two coordinates corresponding to each point denotes a portfolio
- In the context of a two-index APT model, we have one return and two beta axes (for each index)
- Thus, three coordinates that define the plane also define the efficient frontiers
- If a point is above (or below) this plane, this means that the security is under (or over) priced with respect to one or both of these indices
- Thus, it violates the law of one price

equilibrium with the help of single and multi index models as return generating process. However, APT does not tell us what are these relevant influences or indices that should be used to model a set of securities. We do not know the lambda, the risk premia and the influences i_1 , i_2 and so on.

A Few Important Points About APT

- The only caution needed here is that the systematic influences (or indices/factors) affecting these sets of stocks that are tested for APT should be adequately described
- This can be an issue when we have a large set of securities. Then, finding an adequate number of indices (or systematic influences) may become a challenge

Generally, researchers identify these influences through data generated models like factor analysis. For example, these risk factors like inflation risk, market risk and all. Once the mathematical model has provided with set of indices, these are tested for their correlations with real macroeconomic factors such as market risk, inflation risk and so on. However, the model itself does not provide these definitions. One does not have the direct specific economic rationale for a given factor that is thrown to us by mathematical models and that requires a lot of efforts in identifying the relevant influences.

A Few Important Points About APT

APT is extremely general in nature

- It allows us to describe the equilibrium in terms of a single/multi-index model
- However, it does not define what would be the most appropriate multi-index model
- We do not know λ 's or i 's
- They are generated from the data available (e.g., through factor analysis)
- For example, what risk factor a given I_j indicates (inflation risk, market risk, etc.) that is not provided by the model
- So, one does not have the direct specific economic rationale for a given factor

In this video, we will examine how to test the APT. As per the APT, the multi factor return generating process is provided here. In this model, the corresponding APT model is provided here. This model $R_i = \lambda_0 + \lambda_1 B_{i1} + \lambda_2 B_{i2} + \dots + \lambda_j B_{ij}$ is the equilibrium return as per the APT. In order to test this APT model, one needs to identify these indices i 's.

These are risk factors or broad market wide influences. Once these I_j s are identified either through some mathematical procedure or maybe through a priori theoretical underpinning. Once these are identified, one can obtain sensitivities that is B_{ij} . B_{ij} here is the sensitivity of a security i to a risk factor I_j . One identifies these B_{ij} s which is the sensitivity of a given security to the risk factor.

Testing the APT

The multifactor return-generating process is provided below

- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \dots + b_{ij}I_j + e_i$
- The corresponding APT model is shown below
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$
- In order to test the APT, one has to identify I_j s, that is, risk factors
- Subsequently, one can define the sensitivity of a given security b_{ij} to this risk factor
- Unfortunately, APT does not offer a direct economic rationale or description of I_j s
- What do we know about b_{ij} , I_j , and λ_j ?

$$R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \dots + b_{ij}I_j + e_i$$

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$$

However, APT does not offer any economic rationale for these I_j s. And therefore, in order to define the complete model, we need to understand what is B_{ij} here, what is I_j , what is λ_j and what are their roles in this return generating process and APT model. Each form or a security has a unique sensitivity B_{ij} . Each form or security i has a unique sensitivity B_{ij} for a given index I_j . Thus, this B_{ij} is a security specific attribute.

For example, beta of a firm or dividend yield. This is specific to security and act as a sensitivity towards an index I_j . The value of I_j which is the broad market wide influence does not change with security. It is same for all the securities. For example, let s say I_j is the bank index.

This index will remain same for each of the security. These I_j 's or risk factors are supposed to be systematic influences that affect a large number of securities at the same time in a similar manner and therefore these influences act as a source of covariance between these securities. λ_j here is the extra risk done, extra expected return required because of the sensitivity of security to the j th attribute. So, this is the extra return or return premia that is offered because a security is sensitive to a given index. For example, if one looks at CAPM, in CAPM, this B_{ij} would be something similar to the beta which is the sensitivity to market.

I_j is the market index here. For CAPM, I_j would be something similar to market index and λ_j which is the risk premia that is additional return is R_m minus R_f which is because the extra return that is offered on account of this sensitivity beta towards market index. These are well defined for CAPM. However, for APT, these are not well defined in the model and one has to test the model below to establish APT. However, in order to test this model, you need these I_j 's and B_{ij} 's so that you can test this model.

Testing the APT

Each firm has a unique sensitivity b_{ij} for each index I_j

- Thus, b_{ij} is a security-specific attribute (such as dividend yield) or security-specific sensitivity to an index
- The value of I_j is the same for all the securities
- These I_j 's are systematic influences affecting a large number of securities and, therefore, are the source of covariance between those securities
- λ_j is the extra-expected return required because of the sensitivity of a security to the j th attribute of the security

Once you have I_j 's and B_{ij} 's, then only you can obtain λ_j 's. So, this testing, testing of this APT model requires estimates of B_{ij} and λ_j which essentially also requires inputs of I_j 's. As per the modern market theory, most of the tests of APT use this kind of return generating process, this model. They use a predefined set of indices like I_1 , I_2 and so on I_j 's and obtain sensitivities B_{ij} by estimating this model where I_1 , I_2 and I_j 's and so on are predefined. Once B_{ij} 's are identified, then the following equation is used to obtain the estimates of λ_j and thus completely identify the APT model. So, first we start with I_j 's indices, then we obtain sensitivities B_{ij} 's through some kind of regression estimation and once B_{ij} 's are estimated, then putting these values of B_{ij} , one estimates the return generating process and therefore the λ_j 's.

In this model, you can keep on adding more and more risk factors that is i_j 's and keep on increasing the explanatory power of the model. However, after a certain time, the incremental or marginal contribution of explanatory power by adding more factors is very less. So, it is a trade-off between how much explanatory power you want and how many factors you want to introduce. Introduction of large number of factors can make model less parsimonious and also introduce noise in the model. Moreover, effectively these tests are not only tests of APT, but they are essentially joint test of APT as well as the factors or influences or portfolios that are i_j 's that are involved in the model.

So, it is a sort of joint test between APT as well as the i_j 's or the factors. Because there is no generalizable theory that explains all the factors, a number of mathematical approaches are available. For example, factor analysis approach or specifying the attributes of security like dividend deal, beta and so on. Specifying the influences like some industries in the sector specific influences that affect the return generating process that is single or multi-index models or specifying a set of portfolios that capture the return generating process.

Testing the APT

Most of the APT tests use the following equation on a set of predefined indices to obtain b_{ij}

- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \dots + b_{ij}I_j + e_i$
- Then, the following equation is used to obtain the estimates of λ_j 's and thus the APT model
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$
- In this manner, one can keep identifying risk factors until a sizable portion of expected returns are identified
- Effectively, these are joint tests of APT as well as the factors/influences/portfolios considered in the model

$$R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \dots + b_{ij}I_j + e_i$$

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$$

So, there are certain approaches. We will discuss these approaches in next set of videos. To summarize, in this video, we discussed that in order to test the APT, we need to identify i_j 's. Since APT does not offer any conventional or generalizable theory to obtain these i_j 's, one needs to rely on some kind of mathematical procedure or a priori defined

factors such as market or industry specific factors. Once these i_j 's are specified using the multi-index return generating process, we identify b_{ij} 's, sensitivity of securities to these i_j 's. Once we have these b_{ij} 's, we identify λ_j 's that are the risk premia that go into the APT model which is $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$ and so on which leads to identification of complete APT model. There are certain approaches like factor analysis approach specifying the attributes of security, specifying the influences or specifying a set of portfolios that are often employed to test the APT.

Testing the APT

For CAPM b_{ij} (sensitivity to the market, beta), I_j (market index), and $\lambda_j(R_m - R_f)$ were well-defined

- For APT, these are not defined in the model
- One has to test the model below with the observed returns
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$
- This requires estimates of b_{ij} and λ_j

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$$

These tests are essentially not only test of APT but sort of joint test of APT along with the factors that are employed in the model or return generating process. Generating APT Factor Analysis In this video, we will discuss a very important method to generate APT factors with the help of factor analysis. A slightly purer and advanced method to generate factors and factor sensitivities is conducted through factor analysis. As a part of factor analysis, one obtains mathematically determined i_j 's or factors and b_{ij} 's or sensitivities in a manner that reduces the covariance of residuals. Remember, these single or multi-index return generating process have residuals that are stock specific components and it was assumed that components residuals $e_i e_j$ should have minimum or

Testing the APT



Since there is no generalizable theory that explains all the factors, the following methods are used to provide a broad set of factors in the APT model

1. Factor analysis approach
2. Specifying the attributes of the security
3. Specifying the influences (factors) affecting the return-generating process
4. Specifying a set of portfolios that capture the return-generating process

almost zero correlation or covariance with each other.

The way factor analysis works here, it tries to minimize this correlation and generate a set of factors ij 's and their factor loadings which are also referred to as sensitivities b_{ij} . So, these factor loadings are referred to as factor sensitivities. Now, one can keep on adding factors here or these ij 's in the model in order to increase the explanatory power of the model so that these factors can explain the covariance matrix or in simple terms the explanatory power of the model and decrease the residuals. However, this presents a trade-off in a sense that more and more factors will have lower or lower marginal contribution to the explanatory power, but at the same time may increase or involve certain noise, introduce certain noise to the model. Once we are done with the factor analysis, we obtain ij 's, factors and b_{ij} 's factor loadings in the parlance of factor analysis or sensitivities of the security.

Testing the APT: Factor Analysis

Post factor analysis, the following equation is used to obtain the estimates of λ_j 's and thus the APT model

- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$

I_j / b_{ij}

- The challenges with the factor analysis are discussed as follows
 - Like any similar analysis, the estimates of I_j 's and b_{ij} 's are subject to the error of the estimate
 - The factors produced in the analysis have no meanings
 - For example, the signs of factors and three betas (and therefore, the lambdas) can be reversed with no change in the resulting expected return

Now, once you have the b_{ij} 's using the APT model that we discussed earlier, we can obtain lambda j 's or return premia in the following form. Through this model, we estimate lambdas once we have b_{ij} 's or factor sensitivities. However, this analysis also involves certain challenges as we can see here. First and foremost, the estimates of ij 's and b_{ij} 's that are provided by factor analysis, they are subject to the error of estimate, which is a part of any statistical analysis of this kind. Moreover, APT does not provide any universal theory of these factors and therefore the factors produced here, these ij 's do not have any special macroeconomic meaning.

Testing the APT



Specifying the attributes of the security

- If we can establish, a priori, that a certain set of attributes of security that affect the return →
- Then, the extra return required on account of these attributes can be measured through the following equation: $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$
- Here, b_{ij} s would represent the level of an attribute (j) associated with the security " i " associated with each characteristic
- λ_j would represent the extra return because of the sensitivity to that characteristics

For example, once you have ij 's and bij 's from factor analysis, you can very well change their sign or multiply and discount their magnitude. For example, ij 's can be doubled and bij 's can be halved without creating any effect on the mathematical analysis or ij 's can be put negative and bij can be put negative, signs can be changed without affecting the model or analysis. So, in a sense, these ij 's and bij 's do not have any economic interpretation that is driven by factor analysis. To summarize, in this video, we discussed how we can obtain ij 's and bij 's as a part of factor analysis. However, we also noted that these estimates are subject to error of estimation and second, these estimates do not have any special economic meaning as provided by factor analysis that economic intuition or interpretation has to come from theory or outside factor analysis.

Testing the APT, Specifying the attributes of the security. In this video, we will discuss one more method of testing the APT which is through specifying the attributes of the security and thereby defining the APT model. To begin with, if we can establish that there are certain attributes of a security that can affect the returns expected from that security, then one can specify and define APT model more easily. These attributes, for

Testing the APT

Specifying the attributes of the security: $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$

- “ $n\%$ increase in dividend of the portfolio is associated with $\Delta\%$ increase in the expected returns.”
- Once these b_{ij} s are directly obtained, risk premiums for these attributes are computed using the APT model
- These attributes directly affect the expected returns
- Once major firm attributes and the corresponding risk premiums (λ s) are identified, the equation $[\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}]$ can be estimated to define the APT

example, beta of a security or expected dividends from a security, such attributes do affect expected returns and once we specify these attributes, for example, we can specify the expected level of dividends from security as b_{ij} . Once you specify the level of attributes, here b_{ij} 's are not necessarily the sensitivities but certain specific attributes of a security and then extra return required on account of these attributes can be measured or estimated with the help of this APT equation.

Again, remember b_{ij} here is a level of attribute associated with security for that characteristic. For example, a security may have 5% of dividend level or a security may have certain level of liquidity and the return associated expected return associated with that level of liquidity will become the lambda corresponding to that attribute and therefore lambda j would represent certain extra return because of the sensitivity of that security towards that j th attribute. More precisely, while specifying the attributes of the security and estimating the lambdas and b_{ij} s through this APT equilibrium model, let's look at this statement. $N\%$ increase in the dividend of the portfolio is associated with delta percent increase in the expected returns. Here, dividend is that security specific characteristic which also leads or determines the expected return from the security. So, we can quantify a certain level of dividend or the change in level of dividend with a certain change in expected returns.

Once these b_{ij} s are directly obtained in the form of security specific characteristic, then λ s or risk premiums of these security specific characteristics or attributes can be estimated very easily from the APT model through a methodology such as regression. We must remember that these attributes are important because they directly affect the expected returns. Once some of the major attributes of the securities and corresponding risk premiums that is λ s are identified, we can easily establish or define this APT return equilibrium equation which defines the expected return from the security as per the APT model. To summarize, in this video, we discussed that there are certain major

Testing the APT

I_1, I_2, I_3

Specifying the influences (factors) affecting the return-generating process

- Another set of tests involve time-series regressions of the individual portfolios to examine their sensitivities (b_{ij}) towards these macroeconomic variables
- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + \dots + b_{ij}I_j + e_i$
- In the second stage, cross-sectional regressions are performed using all the portfolios to determine the market price of risk (λ_j)
- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$

attributes of a security that determine expected returns of that security.

These attributes can be utilized as b_{ij} s in the APT model. Once these b_{ij} s are obtained, we can obtain the estimate of λ s or return premiums for that security as defined in the APT model and thereby, state establishing and defining the complete APT model. Testing the APT by specifying a set of systematic influences or portfolios. In this video, we will discuss an interesting method of testing and specifying APT with the help of a set of portfolios or a priori systematic influences. In this method, we try to specify a set of influences or factors that affect the return generating process.

Testing the APT

Specifying the influences (factors) affecting the return-generating process

- Another alternative is to determine and pre-decide the set of risk factors (influences) that affect the return-generating process
- A set of economic variables that affect the cash flows associated with the security
- For example, inflation, term structure of interest rates, risk premia, and industrial production

These influences are determined and pre-decided which affect the return generating process. These are set of economic variables that can affect the cash flows associated with the security. Some of the most prominent influences or factors include inflation, term structure of interest rates, risk premia and industrial production. As a part of this method, first, we a priori we specify these influences which are I_1, I_2 as we discussed these can be inflation, inflation, industrial production and so on. Once these influences or factors are decided a priori, pre-decided, then using this return generating process, this model R_i equal to $\alpha_i + b_{i1}I_1$ and so on. Using this model in a cross sectional regression that means for each security i a cross sectional regression over time is run equal to 1, t equal to 2 and so on.

With security specific time series regressions, one obtains the sensitivities of the security with respect to these indices b_{i1}, b_{i2} and so on. Once we have these sensitivities of security with respect to these indices through these time series regressions for each cross section equal to 1, t equal to 2 for each cross section for a number of securities, we obtain the estimates of λ . So, while obtaining the estimates of λ , we conduct cross sectional regressions, where in each cross section a set of securities are employed, their sensitivities b_i 's are known, they are obtained from time series regressions as we discussed earlier in a time series. Now, through cross sectional regressions, we estimate these λ 's from the given or identified b_i 's.

Testing the APT

Specifying the influences (factors) affecting the return-generating process

- For example, ONGC will be definitely affected by the crude-oil prices
- So, a crude oil price index or any broad energy index can provide one risk factor, that is I_j
- Using these indices, the return-generating process can be employed to estimate the betas (b_{ij})
- Once the betas are obtained, the APT model can be used to obtain risk premiums ($\lambda_j: R_j - R_f$)

Thus, once we have the estimates of lambdas and bi's, we have completely specified the APT model. For example, let us take case of ONGC. Now, crude oil prices affect ONGC and a number of similar securities that are part of oil and gas industry. And therefore, any broad energy index that captures the movement in crude oil prices can provide one such risk factor I_j related to oil and gas industry. Similarly, various such indices such as inflation, interest rates and so on can be decided and in the return generating process, these I_j 's can be specified to obtain estimates of sensitivities of the security towards these factors I_j 's. So, these b_{ij} 's are the sensitivities of the security towards these pre decided or specified factors such as oil and gas index, inflation and so on.

These are time series regressions of individual securities over time, as we discussed equal to 1, equal to 2 and so on for each of the security. Once we have estimates of these b_{ij} 's, we can rely upon the APT model through cross sectional regressions. In cross sectional regressions at each cross section equal to 1, let us say or equal to 2, in each cross section, multiple securities with their b_{ij} 's, we have estimates of b_{ij} 's for all the securities, we estimate lambdas from the model APT model, which is $\lambda_0 + \lambda_1 v_1 + \lambda_2 v_2$ and so on up to $\lambda_j v_j$, we specify and obtain the estimates of lambda. This lambda is the risk and once we have the estimates of lambda and sensitivity, we have the APT model completely specified. Another very similar method is to employ a set of portfolios that capture these influences, risk influences or factors as we discussed earlier, these risk influences that affect the return generating process, not directly but through certain portfolios, we try to capture these risk influences.

For example, difference in returns between small and large stocks can be one such factor which captures the influence of size factor, this is often referred to as size factor, which is the difference in large and small stock returns or difference in high book to market to low book to market stocks. This is often referred to as value versus growth factor. So, you have difference in long term corporate long term government bond, which is the greatest factor. So, similarly, you can have portfolios that capture the risk of these influences or factors that can be employed instead of predefining or determining a factor itself.

Testing the APT

Specifying a set of portfolios that capture the return-generating process

- Another option is to construct a set of portfolios that capture the influence of risk factors affecting the return-generating process. For example
 - Difference in the returns on small and large stock portfolios
 - Difference in returns on the high book-to-market and low book-to-market stocks
 - Difference in the returns on long-term corporate and long-term government bonds
 - Difference in the returns on long-term corporate and long-term government bonds

So, this is another method where you can specify portfolios and use their returns as i_j 's, i_1 , i_2 and so on. Once you have these i_1 , i_2 , you can estimate the sensitivities that are b_{ij} 's through cross time series regressions. And once you have the estimates of b_{ij} 's in cross sectional regressions, you can estimate λ 's or the risk premium. To summarize, in this video, we discussed how to estimate APT model by either specifying the set of influences or risk factors or specifying the portfolios that capture these influences or risk factors. Once we have identified these portfolios or influences, we have estimates of i_1 's, i_j 's. Using these i_j 's in time series regression, we obtain the sensitivities of securities towards these portfolios or risk influences.

These are b_{ij} 's. Once we have the estimate of these b_{ij} 's, we can estimate λ 's in the APT model thus completely specifying the APT model. APT and Kappan single market index. In this video, we will try to answer the question whether Kappan becomes inconsistent in the presence of APT where single market index is available. To answer

this question whether Kappan becomes inconsistent in the presence of APT, we start with a simple single index case.

Here, this single index is market portfolio like NIFTY50 or S&P500. Here, the return generating process that we have discussed for a single index case takes the following form. R_i which is the return on security i , α_i plus β_i times R_m plus e_i . e_i here is the residual or error term. Now, if we go back to our earlier discussions about APT, we said that this return generating process can be taken to the equilibrium return form by APT when it is written like this. \bar{R}_i which is the expected return equal to λ_0 plus λ_1 into b_{i1} plus λ_2 into b_{i2} and so on up till λ_j into b_{ij} .

Notice there is no residual term because this is equilibrium equation of return as defined by APT. Here, λ_0 if R_f risk free rate is available then λ_0 equal to R_f , λ_j equal to \bar{R}_j minus R_f . Now, λ_j here is the risk premium with respect to index j . If there is only one single index available and risk free rate R_f is also available therefore, this single index where the single index is market index and in the presence of risk free rate, we can derive the APT expression as follows. Here, \bar{R}_i equal to R_f plus β_i times \bar{R}_m minus R_f where \bar{R}_j becomes \bar{R}_m where \bar{R}_m is the return on market.

This is the equation for expected return as per APT model which is also similar to that provided by CAPM. This equation, this derived APT equation suggests that when single index return generating process is the true depiction of APT, CAPM is indeed consistent with APT. But what about multi index forms? So, in the next video, we will talk about multi index case whether APT is consistent with CAPM or not. To summarize, in this

APT and CAPM

Does CAPM become inconsistent in the presence of APT?

- We start with a simple single-index case, where this index is a market portfolio (or market index like Nifty-50)
- The return-generating process is of the following form
- $R_i = \alpha_i + \beta_i R_m + e_i$

video, we discussed that if single index model which is market model is there, CAPM and APT are not necessarily inconsistent. APT and CAPM multi index case.

In this video, we will examine whether APT and CAPM become inconsistent in the presence of multi index return generating process. Let us start with a simple return generating process with two indices. The process looks like this. R_i equal to α plus B_{i1} into I_1 plus B_{i2} into I_2 plus A_i . A_i here is the residual. The corresponding APT equilibrium model for this return generating process is \bar{R}_i equal to R_F plus λ_1 into B_{i1} plus λ_2 into B_{i2} where assumption is that risk free rate R_F is available.

APT and CAPM

$\bar{R}_i = R_F + \lambda_1 b_{i1} + \lambda_2 b_{i2}$ can be effectively written as

- $\bar{R}_i = R_F + b_{i1}\beta_{\lambda_1}(\bar{R}_m - R_F) + b_{i2}\beta_{\lambda_2}(\bar{R}_m - R_F)$
- $\bar{R}_i = R_F + (b_{i1}\beta_{\lambda_1} + b_{i2}\beta_{\lambda_2})(\bar{R}_m - R_F)$
- Define $\beta_i = (b_{i1}\beta_{\lambda_1} + b_{i2}\beta_{\lambda_2})$
- Then, we obtain the CAPM form as follows: $\bar{R}_i = R_F + \beta_i(\bar{R}_m - R_F)$
- This can be extended to multiple factors (indices) as well

$$\bar{R}_i = R_F + \lambda_1 b_{i1} + \lambda_2 b_{i2}$$

$$\bar{R}_i = R_F + b_{i1}\beta_{\lambda_1}(\bar{R}_m - R_F) + b_{i2}\beta_{\lambda_2}(\bar{R}_m - R_F)$$

$$\bar{R}_i = R_F + (b_{i1}\beta_{\lambda_1} + b_{i2}\beta_{\lambda_2})(\bar{R}_m - R_F)$$

$$\beta_i = (b_{i1}\beta_{\lambda_1} + b_{i2}\beta_{\lambda_2})$$

Please remember here, λ_j is the price of risk for a portfolio where B_{ij} equal to 1. That means the portfolio has a sensitivity of 1 for one index and 0 for all the other indices. And for that index j , λ_j equal to \bar{R}_j minus R_F , that price of risk. If you assume that CAPM holds, then it also holds for all the securities as well as portfolios like I_1 , I_2 that is broad market wide portfolio such as I_1 , I_2 which are the risk factors in the

model as well. And since we assume that this CAPM holds for portfolios I1, I2 as well, industry portfolios like I1, I2 may have some sensitivity to the market portfolio.

Let's call this sensitivity beta lambda j. If you remember, the risk premium in simple CAPM was beta times R_m bar minus R_F when the sensitivity to the market was beta i. So in this case, where sensitivity is beta times beta lambda j, the risk premium lambda j can be written as beta times lambda j into R_m bar minus R_F . Assumption here is that the industry or in I1 portfolios like I1, I2, they are also sensitive to market. And therefore, if the return generating process for two indices is this one, R_i equal to alpha i plus B_{i1} into I1 and so on, the corresponding equilibrium model for this return generating process can be written as R_i bar equal to R_F plus lambda 1 into B_{i1} plus lambda 2 into B_{i2} , where lambda 1 equal to R_1 bar minus R_F . But we said that we also believe in CAPM and therefore R_1 bar minus R_F can be written as beta times R_m bar minus R_F , beta lambda 1 times R_m bar minus R_F in fact.

Similarly, this is for index I1 and therefore similarly for index I2, we can write the value of lambda 2 as R_2 bar minus R_F , which is equal to beta lambda 2 into R_m bar minus R_F . This is for index 2. With these expressions, we can simplify our APT model.

APT and CAPM

Now, refer to our earlier discussions on APT, where we said that the above return-generating process could be written in terms of sensitivities of the securities to index and the price of risk in the following form

- $\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$ with $\lambda_0 = R_F$ and $\lambda_j = \bar{R}_j - R_F$

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij}$$

Our original APT model is R_i bar equal to R_F plus lambda 1 B_{i1} plus lambda 2 B_{i2} . We can simplify by substituting the values of lambda 1. Lambda 1 was beta lambda 1 into R_m bar minus R_F and lambda 2 was beta lambda 2 times R_m bar minus R_F . Simplifying this expression, we can take out R_m bar minus R_F , which is common to both these terms and therefore the resulting expression is R_i bar equal to R_F plus this combined term into R_m bar minus R_F . Let's simplify this and call it beta i. And this results in an expression

which is very similar to CAPM, which is \bar{R}_i equal to R_F plus β_i , where β_i is this times \bar{R}_m minus R_F .

APT and CAPM

$$\bar{R}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \dots + \lambda_j b_{ij} \text{ with } \lambda_0 = R_F \text{ and } \lambda_j = \bar{R}_j - R_F$$

For a single index case, that is, market index, and in the presence of a risk-free rate, the above expression becomes

$$\bar{R}_i = R_F + \beta_i(\bar{R}_m - R_F): \text{ this is the expected return form provided by CAPM}$$

This suggests when a single-index return-generating process is true depiction, the CAPM is clearly consistent

But what about multi-indices?

And this analysis can be extended to multiple factors or indices as well. Based on this, we can say that the APT solution, the APT solution to the return generating process, even with multiple factors is consistent with CAPM. So even if we assume that CAPM holds, it is not inconsistent with multiple indices as well. So with multi-index return generating process, the APT model that is derived with multiple indices that explain the covariances or com movements between the stock returns, even there, if we assume CAPM holds, there is nothing that violates that assumption. That means CAPM is consistent with APT even when the return generating process is one with multi indices.

To summarize, in this video, we examined whether CAPM and APT are consistent in the presence of multi-index generating process. We found that CAPM and APT can remain consistent with each other even in the presence of multi-index generating process. Application of asset pricing models in passive management. In this video, we'll discuss the applications of asset pricing model in the context of passive management. A very simple and intuitive application of APT models is to construct a portfolio of stocks that closely tracks a certain index. This index can be a market index like Nifty Fifty or a certain industry sector specific index like Bank Nifty, which represents the risk of

APT and CAPM

The return-generating process in the context of two indices becomes

- $R_i = a_i + b_{i1}I_1 + b_{i2}I_2 + e_i$
- The equilibrium model for this return-generating process with a risk-less asset becomes: $\bar{R}_i = R_F + \lambda_1 b_{i1} + \lambda_2 b_{i2}$
- Recall that λ_j is the price of risk for a portfolio that has $b_{ij}=1$ for one index and zero for all the other indices: $\lambda_j = \bar{R}_j - R_F$
- If we say that CAPM holds, it holds for all the securities as well as portfolios

banking stocks.

$$R_i = a_i + I_1 b_{i1} + I_2 b_{i2} + \dots + e_i$$

$$\bar{R}_i = R_F + \lambda_1 b_{i1} + \lambda_2 b_{i2}$$

APT and CAPM

If we say that CAPM holds, it holds for all the securities as well as portfolios

- Therefore, this industry portfolio may have some sensitivity to the market portfolio, that is, β_{λ_j}
- Recall that the risk premium was $\beta_i(\bar{R}_m - R_F)$ when the sensitivity to the market was β_i
- Then, the effective risk premium for this index λ_j becomes $\beta_{\lambda_j}(\bar{R}_m - R_F)$

The core idea behind this kind of passive management is to use a rather small number of stocks to construct a diversified portfolio. If you have large number of stocks that will involve a lot of transaction costs, so you tend to use small number of stocks and you try to track a certain index like market index. However, please remember it is difficult to track an index which has large number of stocks. One cannot track all the stocks in a certain index or market. So the strategy here is to hold only to that extent the number of stocks that is sufficient to diversify the portfolio.

APT and CAPM

Therefore, the APT solution, even with multiple factors, is consistent with CAPM

This means that despite the fact that multiple indices (risk factors) explain the covariance between the returns, the CAPM holds

However, there's one challenge with this kind of strategy. If there are multiple indices that affect your portfolio and your model is not well matched, for example, if you have a certain portfolio where a number of indices are influencing that portfolio and you miss one of the important indices, then the shocks to those indices may appear in your portfolio or in your model as residuals EI. That is residuals EI. Now if your portfolio is

not explicitly matched to some of these indices, then let's say there is a shock to one of these indices which you have not matched explicitly to your portfolio and the moment in your portfolio, the price moment of your portfolio may appear as over or under performance depending upon the nature of shock in terms of performance evaluation.

Passive Asset Management

Passive management

- A simple application of APT is to construct a portfolio of stocks that closely tracks an index
- The index that represents a risk factor (Bank Nifty represents the risk of banking stocks)

That is your portfolio performance is exposed to these changes. Let us discuss this aspect in more detail. The benefit of using multi indices instead of single indices. For example, if your portfolio has five indices, energy, banking, inflation, cyclical stocks and government bond portfolio. Compare this kind of portfolio matching strategy. Instead of five indices, you can also select only one index which is market portfolio.

Passive Asset Management

Passive management

- The attempt is made to use a rather lesser number of stocks
- A large number of stocks would incur significant transaction costs
- In order to track the market index (market portfolio), one cannot hold all the stocks in the markets

Both of these portfolios will capture the sensitivity to market risk. Since many of these indices that we discussed like energy, banking, inflation, they tend to capture the impact of market movement as well and therefore affect the risk of market. So both of these portfolios, the multi index and single index, will capture the sensitivity to market risk. However, if there is an oil price shock or some unexpected changes in inflation, the market portfolio with its sensitivity matched to Nifty Fifty or some kind of broad market index may not be very efficient in tracking that index. This is so because while constructing the market portfolio, you are indifferent whether the stocks in your portfolio are sensitive to inflation or unexpected changes in oil price movements. And therefore, if

there are any shocks, if let's say the stocks in your portfolio are indeed sensitive to oil price index or inflation index, your portfolio which is explicitly designed and match its sensitivity with market portfolio may not be very efficient in tracking those shocks and their impact on your portfolio.

Passive Asset Management

Passive management

- One attempts to hold only to the extent the diversifiable risk can be offset →
- Those indices for which portfolio sensitivity is not matched, if receive unexpected shocks (like oil price shock), may appear as the residual risk in the model
- That is, our portfolio may be exposed to these changes

Because for the simple reason that objective of your portfolio construction was just to construct a portfolio with its sensitivity matched to some kind of market index like Nifty Fifty and no diversifiable risk. And therefore, while analyzing the performance of this market index matched portfolio to another portfolio which is matched with multi index APT kind of model that is explicitly matched to the sensitivity of oil price index or let's say inflation index, the evaluation of performance can be very different. So that portfolio which is only matched to market index can incorrectly or inefficiently give you a sense of over or under performance which actually is not the case. That is that additional over or under performance is driven by one of the indices that is not matched properly or its sensitivities are not properly matched. If you would have matched the sensitivity of that portfolio with that additional oil and gas or inflation index, then probably the performance would have appeared appropriate with the risk of portfolio and not as a over or under performance.

To summarize, in this video we discussed how multi index and single index models can be applied in the context of passive portfolio management. In this video, we'll discuss the application of asset pricing models in the context of active portfolio management. As a part of active portfolio management, one would like to hold a particular portfolio which is well diversified but at the same time one would also like to take bet on certain risk

factors. For example, let's say you hold a market portfolio or a portfolio that mimics market indices like Nifty Fifty or S&P 500. Still, you would like to take some active bets on sectors.

Passive Asset Management

Passive management

- Compare this to holding only the market portfolio (Nifty)
- Both of these strategies will capture the sensitivity to market risk, as all the portfolios (except government bonds) may reflect, to some extent, the risk of market

For example, let's say you believe that currently the oil and gas sector is undervalued and the prices may go up in future. For example, there is some kind of regulation which is going to benefit which is coming and going to benefit this particular sector. And therefore, in order to benefit from this kind of bet, what would you like to do is you would like to add more securities or more stocks that are sensitive to oil and gas index and the prices of these securities are likely to go up. However, at the same time you'd also not like to lose your beta or change your beta of your portfolio vis-a-vis market index that means market beta.

Passive Asset Management

Passive management

- The benefit of using multi-indices instead of a single market index can be explained here as follows
- Consider five indices, including the energy portfolio, banking, inflation, cyclical stocks, and government bond portfolio

So, you will add oil and gas sector sensitive stocks in a manner that your market beta remains indifferent. For example, if it was earlier 0.5, you'd like it to remain intact while adding certain oil and gas stocks and removing certain other stocks so that net market beta remains constant. That is one. Second, as the gains are materialized, for example, as the market has risen, oil and particularly oil and gas sector has risen and the prices of these securities have risen, you have benefited, gains are realized and now you can

liquidate those holdings. And now you can liquidate those these additional securities from oil and gas sector and go back to the original position while your beta still remains, market beta still remains intact. In this fashion, you can apply these multi index or single index model to take active bets in the market while at the same time benefiting from them and maintaining your portfolio diversified.

Active Asset Management

Active management

- In active management, one continuously holds on the market portfolio and makes calculated bets on different risk factors
- For example, if one believes that oil prices can go up – this means that currently the stocks that are sensitive to this risk are underpriced and will go up in future

In summary, in this video, we discussed the application of multi index and single index model in the context of active portfolio management. While CAPM has its genesis in the mean variance framework, APT relies on the arbitrage argument. APT utilizes the return generating process provided by the single and multi index models to generate the equilibrium asset pricing model. Under the APT, riskless arbitrage drives prices towards the equilibrium plane. The equation of this plane is determined by the systematic risk influences affecting the set of securities under consideration.

APT can be tested with the help of a factor analysis, specifying the attributes, specifying a set of systematic influences and specifying a set of portfolios. In the presence of APT, CAPM does not necessarily becomes invalid as long as the APT factors are influenced

Passive Asset Management

Passive management

- However, if there is a certain oil price shock or unexpected changes in inflation, the market portfolio with its sensitivity matched to Nifty may not be very efficient in tracking the index
- This is because one is indifferent to holding stock from different industries (e.g., oil stocks) in constructing the Nifty, as long as she is able to replicate a market portfolio with no diversifiable risk

by the market factor and have a well specified beta with respect to the market factor. Some of the most widely employed applications of asset pricing models include active and passive management and factor investing. Thank you.

Active Asset Management

Active management

- Then, one can increase the sensitivity of his portfolio by adding additional stocks from oil companies and others to the extent that increases the sensitivity to this risk index
- Once the price increase has materialized, one can go back to holding the market portfolio by selling the additional stocks and realizing the gains