Data Analysis and Decision Making - III
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Lecture 39

Welcome back, my dear friends. A very good morning, good afternoon, good evening to all of you wherever you are in this part of the globe and as you know this is the DADM 3, which is Data Analysis and Decision Making 3 course under NPTEL MOOC series. As you can see from the slide we are in the lecture number 39 which is in the eighth week.

So last but one lecture in the eighth week and this total course duration as you know is for 12 weeks spread over 60 lectures, each lecture being for half an hour and the total contact hours is 30 hours. So each week as I said, you basically take, you attend 5 lectures half an hour each and after each week you take the assignment.

So if you remember that in these last 2 lectures or the last 3 lectures in this week we were discussing about data envelopment analysis, how the concept of return to scale considering increasing return to scale, decreasing return to scale and constant return to scale can be utilized as a concept in the economic perspective where per unit increase in the input has the corresponding effects where the corresponding outputs increases at increasing rate, decreases at decreasing rate and remains constant for the case increasing return to scale, decreasing return to scale and constant return to scale.

Now having discussed all these things then we later analyzed that for the output oriented model and the input oriented model you can formulate your problem as a very simple linear programming problem. Initially it looks a little bit weird in the sense that you are basically trying to formulate either a maximization problem or a minimization problem depending on which perspective we are looking at as the input oriented or the output oriented.

And in the case of the maximization problem you want to basically maximize the efficiency wherein the numerator you have basically the output bundle that is the summation of the weights multiplied by the corresponding decision variables or the amounts and in the denominator you have the input bundle.

So if in considering that there are capital M number of inputs, capital N number of outputs and capital K number of DMUs you will basically have such capital K number of optimization problem each being a maximization of the corresponding ratio of the output bundle to the input bundle and the constraints of all these optimization problems remain the same which is corresponding to the fact that the ratio of the output to the input for the first, second, third, fourth till the capital Kth one are all less than equal to 1.

Now in this case the objective function as well as the constraints are non-linear. Hence in order to convert them to linear perspective we normalize the denominator for this maximization problem as 1. So the moment it is put into 1, you have only a simple linear programming problem as the objective function, maximization, the bundle or outputs correspondingly for each, each kth DMU. And the constraints which were basically of the less than equal to 1 ratio, once you basically convert them to the right hand side and bring it back, it is basically the bundle of outputs minus the bundle of inputs should be less than equal to 0.

That means there are capital K number of them in each of these optimization problem, and the last one which is exclusive one for each and every optimization problem basically corresponds to the fact that as you remove or try to forcefully bring the input as 1, so the input for the first DMU will be equal to 1 for the first optimization problem, similarly for second in the second optimization problem and it will go on till the kth DMU in the kth optimization problem. Then you basically solve these using the linear programming concept, have a basis starting, basic feasible solution and traverse through the corner point till you reach the optimum point.

In the case when you are trying to basically minimize the ratio, the ratio is basically corresponding to the fact that you are taking input to the output trying to minimize that, the constraints would all be for the K number of optimization problem, constraints which are capital K number would all be, the ratio of input to output being greater than 1, as exactly mentioned in few minutes back, they are non-linear in nature. So hence what you do is that you forcefully put the output bundle which is in the denominator for the objective function for all these k number of DMUs, separately you put them into one.

The moment you do that, you will basically have a minimization of the input bundle and the subject to considerations would be, because they were greater than equal to 1, so it would be input bundle minus the output bundle is greater than equal to 0. So they would be capital K number of them in each of these k number of optimization problem and the kth plus 1 constraint for all these k number of optimization problem will be unique corresponding to the fact that you are trying to basically bring the input bundle for each of them in their constraints.

So it would be the first input bundle being put into 1 would come in the first optimization problem, second input bundle being put into 1 as a normalization case as I just discussed would be come into the second optimization problem. Similarly for the kth one, it will come into the kth optimization problem. And again you solve it using the similar method of basic tableau and so on and so forth, only remember that in many of the cases you may have to add a slack of the surplus or else if required you may have to basically have this artificial variable depending on how the problem formulation has been done.

Now later on when, when we went to the concept of, I just very briefly mentioned that in many of the cases considering the problem you want to basically mix paint and prepare a manufacturing paint or even basically have logs or wood you want to bisect them or cut them in pieces and make some chairs or tables so when your decision variables are either the length of the log or the amount of paint you are going to use, or it is basically number of chairs you are going to produce or the number of tables you are going to produce or the total distance traveled by the lorry or the number of lorries you are going to employ, in many of the problems it may be seen that the decision variables would be either continuous or discrete.

In the case if it is discrete or integers, obviously in both the cases they would be greater than equal to 0. You will basically have the integer programming. The integer programming would be a pure one if all the decision variables are integers. It will be a mixed one if you have one set of decision variables which are continuous and another set of decision variables which are integers. And then we also discussed very briefly, I mentioned the name that we will basically consider the problem using the concept of Gomory Cut algorithm concept and also later on we will see another method will be the branch and bound method depending on how the branchings are done.

So for the Gomory Cut Plane algorithm we will basically give the conceptual framework and then give you the problem formulation. So another thing you remember, that in the fag end of the last class when we were discussing about the significance of the variables to be integers we saw that in one of the problem x 1 was coming out to be 4 which was basically an integer and x 2 was coming out to be 3 by 2.

Then we made a different type of searches, arbitrary hypothetical searches for different values of x 1 and x 2 and end of the whole search we saw that x 1 was 0 and x 2 was basically coming out to be 3 such that the initial value of the objective function when the case was being solved using the concept of linear programming and considering x 1 and x 2 could be continuous the value was 101.

But later on, once we put the right values considering the feasible solutions were at the corner points which were integers we found out the value to be 90. Now in order to solve such problems we use, as I mentioned we use the Gomory cut. So the Gomory cut algorithm basically works, in the conception works like this.

So you basically first solve the problem using the linear programming and then slowly relax the problem formulation in each and every stage and shift the boundaries in such a way that the non-integer values or the continuous variable values which are, which you were getting in the initial case would basically be eliminated one at a times because as the plane, in the case when you have two-dimension one it is a simple plane or a line, in a three-dimension it would be a plane, and as you go into the dimension it would be a hyperplane.

So hyperplanes, they would be taken in such a way that you will cut the planes or cut the overall feasible region in such, such, using such cuts which are known as the Gomory cuts such that you will be able to slowly solve the problem using only integer values for the decision variables. So this is how it works. I will only give you the general solution of one problem, then go into the branch and bound method and then again come back to the solution methods for the Gomory cut algorithm and the branch and bound method as we proceed.

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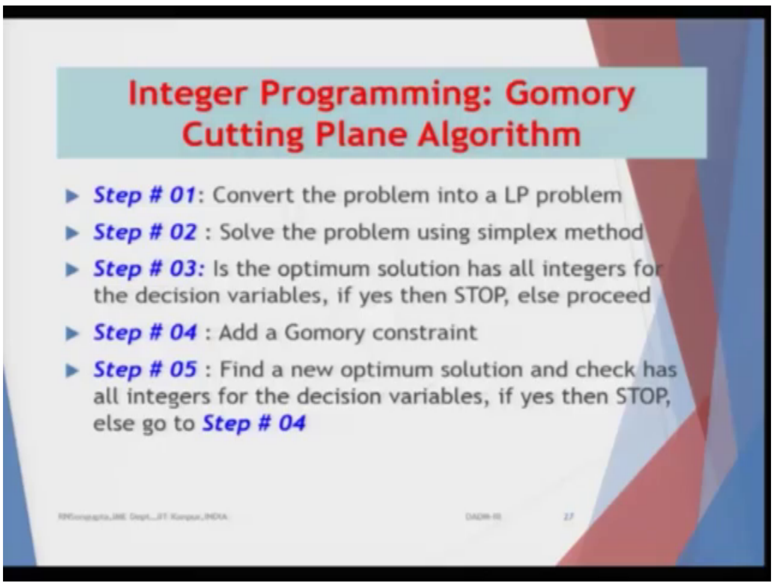

So first what we will do is that you will convert the problem into linear programming problem. Simply you would not; you would not be bothered about whether x 1 and x 2 is integers. Just solve it using the simple linear programming method, the simplest method and considering that there is a solution. Obviously there can be degeneracy, there can be unbounded solutions so on and so forth, for the case forget about that and only concentrate on the fact that there is a unique solution, you get an optimum point considering the linear programming problem has been solved.

You will convert, as I mentioned, in step 1 you will convert the problem into linear programming problem. The objective function remains the same; the constraints remain the same, only the decision variables are considered to be non-integers or continuous. You will solve the problem using the simplex method which we have been doing so long, trying to discuss time and again in the last 7 weeks in different ways.

Now you will ask the question, is that the optimum solution which you have obtained; do not be bothered about immediately about the objective function, just look at the decision variables. You ask your question is the optimum solution, does it have integers for the decision variable? Each one of them, by the way it will depend, like in many of the cases it may be possible that some of the decision variables are mentioned there continuous, some of them are mentioned as integers. So you basically check when you solve the linear programming those decision variables for which it is mentioned that they should be integers are whether they are integers or not.

If it is integer or if they are integers and all the general concept of the, or the general assumptions of the decision variables are being met in, with based on the fact that what has been stated in the initial problem, then you stop and you have basically reached the optimum solution and you basically report those decision variables and end your task. If yes, it mentions the third point or the step 3. If yes then stop else you proceed.

So what you will do is that you will add a Gomory constraint, basically cut the, or cuts or bring some planes such that the constraints would be changed accordingly. You will find an optimum solution, so for each changing Gomory constraints as you do, you will basically have a new problem, again solve it using the linear programming, get an answer, check whether the decision variables are as per the actual assumption which you have assumed, whether some of them which are integers are integers and some of them which are continuous are continuous.

If yes then stop, if no you will basically proceed accordingly and basically add Gomory cuts or Gomory constraints at each and every step till you get the solution which is optimum and obviously remember the optimum solution for the integer programming problem whether mixed or pure integer programming problem which you have solved would not be exactly the same one as you would have obtained in the case when you were considering the continuous case.

As we saw in the problem considering z is equal to 90 for the integer linear programming and 101 in the case it was simple linear programming without considering the integer for the decision variable. So let me again read step 4. Basically add a Gomory constraint, you will find a new optimum solution using the new Gomory constraint and check all integer decision variables are true or false.

I am using the word all decision variables only pertaining to the case which are to be integers. If yes then stop or else you basically go to step 4 where you add another extra Gomory constraint. That means they would keep changing and you will add Gomory constraint, solve it, check, if yes stop or else proceed till you basically get the optimum solution.

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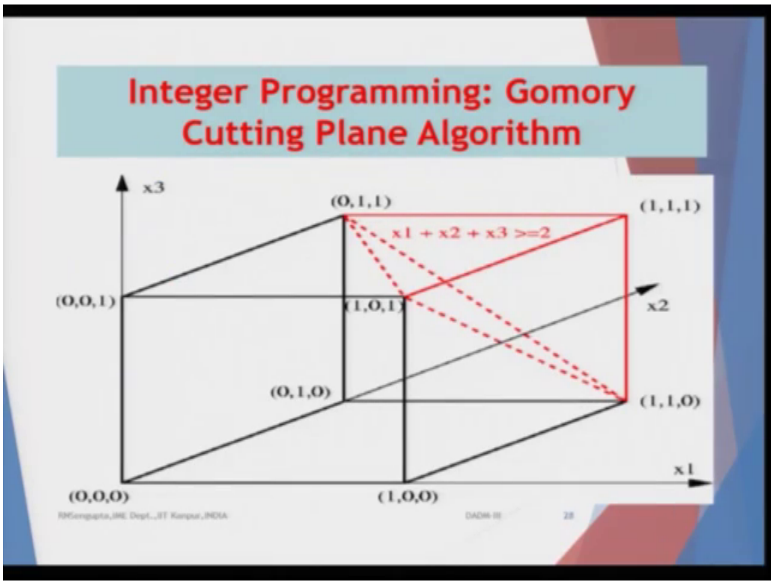

Now in this case, consider in this diagram so you have 3 dimensions, x 1 going to the right, x 2 going, so if I may see considering this point, it is going inside this computer or it is coming out from the computer depending on how you had been able to basically specify the x 2 coordinate and x 3 would be going basically vertically up.

So it is a three-dimensional figure, all of them are 90 degrees to each other, and considering this, this problem which we have, if we consider the Gomory cutting plane algorithm the Gomory cuts which we will put considering this is a cube of dimension 1 so you would basically have the origin with 0, 0; 0, 0, 0; 1, 0, 0 and so on and so forth so obviously you will basically have such 8 corners of this cube.

Now in the case if you are trying to solve a problem which is integer and the basic feasible solution is this, or the overall feasible space is this then obviously one of the corner points would be the optimum problem solution which you want to get.

But it may so also happen that you have to basically have the constraints shifted as shown by this red dotted lines which is x 1 plus x 2 plus x 3 is greater than equal to 2, such that the shifts which you make in the constraints and considering that the changes you are going to bring in the constraint using the Gomory cut would ensure that the problem solution which you are going to get would basically be all will be integers. So if I am able to come back to this diagram let me check if I have this in the last class, yes.

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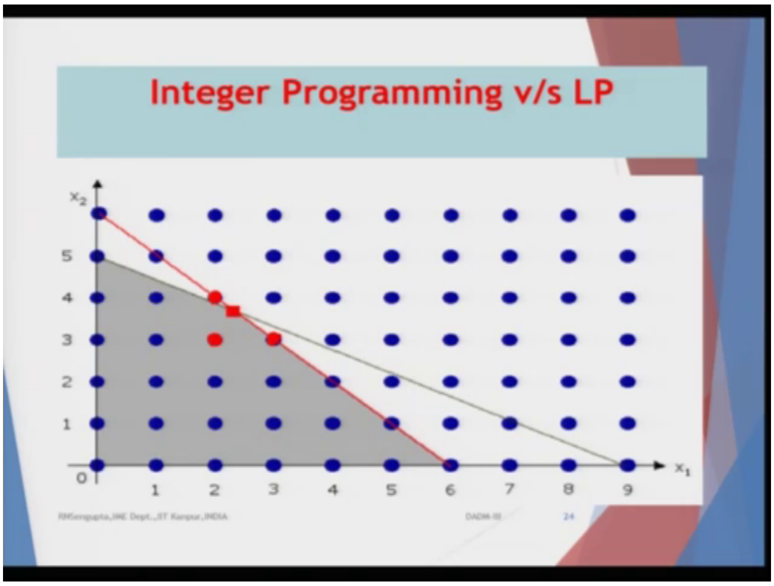

If you remember the red point which is there is basically a non-integer solution and obviously if you are able to solve the problem using the concept of linear programming is the optimum solution considering you are doing the maximization problem as the objective line more is away from the origin and goes inside the first coordinate.

But the concept is this, you want to basically add the Gomory cuts so the cuts or the constraints would change in such a way, both the green and the red one would be changed or, or changed in the sense that the addition or subtraction of the right hand side or the left hand side using the multiplicative factors would be done in such a way that the tiltation or the movements of the Gomory cut would ensure that the actual optimum solution that red point would slowly move in a way that it would be any of the corner points which is giving which are nearby it.

Obviously in the problem which you solved, remember that we had that actually rather than the points of 4 and 3 by 2, you actually ended with a answer which was 0 and 3, for the case when actual answer was 90 in case of integer programming but actual problem answer which you are getting using the linear programming problem was 101.

So this red line dot would basically be any of the points as you keep adding the Gomory cuts such that the optimum solution or the decision variables are integers and obviously the objective function will change accordingly. I will come to that problem solution within few minutes.

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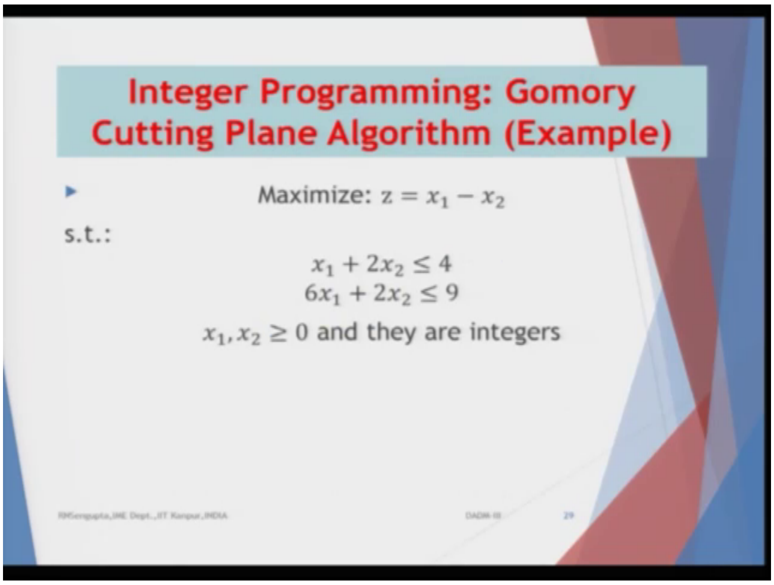

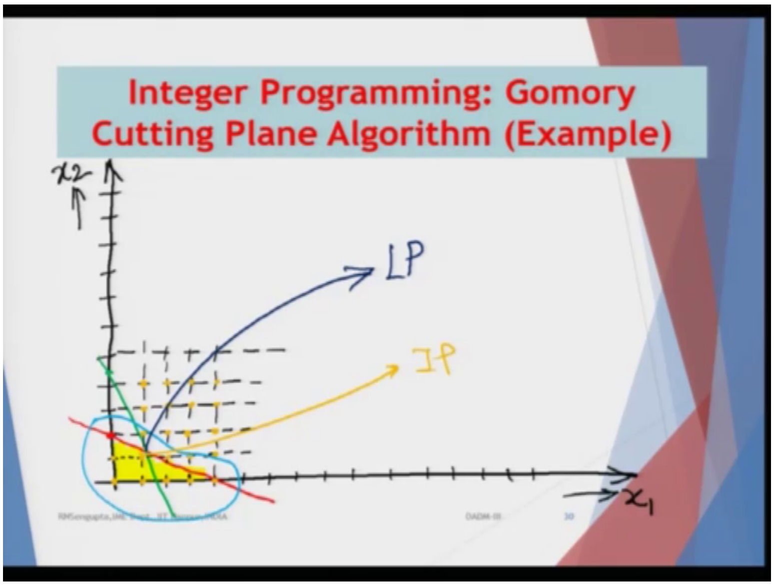

So consider the maximization problem, you have maximization of x 1 minus, x 1 minus x 2 you want to maximize that and your constraints are, remember the constraints are all of less than type. So obviously, if they were greater than type, addition and subtraction, of the slack and the surplus would just be a normal technique which we have already followed in the linear programming concept.

So the constraints of x 1 plus 2 x 2 is less than equal to 4, and the other case is 6 x 1 plus 2 x 2 is less than equal to 9. And in this case the assumptions are x 1 and x 2 are all greater than 0 and all of them are integers. That means the problem solution which you ensure would basically have x 1, x 2 as integers such that your, the constraints are also met and also you will basically have the maximization corresponding to the fact that both of them are integers.

Now let us see, before I solve the problem step by step let us see how the overall objective function and the, the constraints look like in the two-dimensional plane. So let me draw it. Just give me one minute. So I will switch over from slide to slide so it is easy for me to draw. So you have x 1 plus 2 x 2 is less than equal to 4. So let me first, this x 1 I am drawing along x axis, x 2 I am drawing on y axis so I will just, these are the units of distance along the (()) (19:42). So you have x 1 plus x 2 is less than 4. So if you put 0, 0 obviously that is satisfied.

So the points would be on to the left hand side that means, this is the maximization problem so the constraint region would be more towards 0, the feasible region. So if x 1 is 0, x 2 is 2, x 2 is 2 so let basically draw the first, x 1 is 0, x 2 is 0; x 2 is 0, x 1 is 4. So you have this, all region below that. Next point is if x 1 is 0, x 2 is 4.9, x 2 is 4.5; x 1 is 0, x 2 is 4.5. So I should use different color, 1,2,3,4, and if x 2 is 0 then you have basically x 1 as 3 by 2, x 1 is 3 by 2 which is here.

So your now area is, I have tried drawing as it is, now what I do is that these are the points. I am just drawing the integer points, I will mark it. This is the so-called dotted axis so maybe I would not be able to highlight but you will be able to get the concept. So the points, where the actual solution would be I will mark them with orange so this, so these are the actual integer points where is the actual solutions would be, would have been but these dots which are there on the right hand side are not feasible, infeasible, in infeasible region but still I am trying to basically draw it. So if you see the feasible region is basically this.

So technically the actual points which are contenders would be, this point which is possible, this point, this point, this point, this point, this point, this point and the last point, so the overall, these are the general set, these points outside the red line are not but I just drew it.

Now if I want to do the search, my search points are 1, 2, in this concept, 1, 2, 3, 4, 5, 6, 7, 8 and if I am trying to basically find out the maximization problem the line would move in such a way that it leaves the overall region and the actual objective function in the case when we have the integer programming would be, the linear programming problem would be this. But in the case when we have the problem solution as integer it will be this one and we want to basically find it out accordingly.

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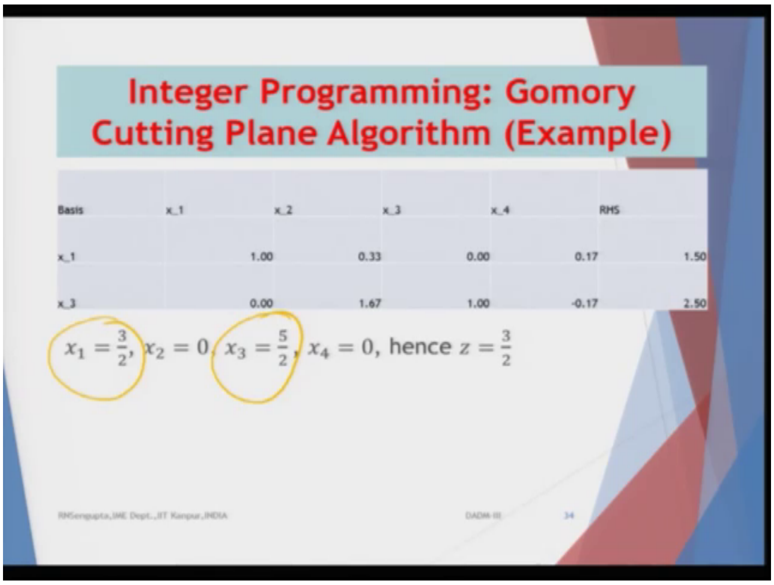

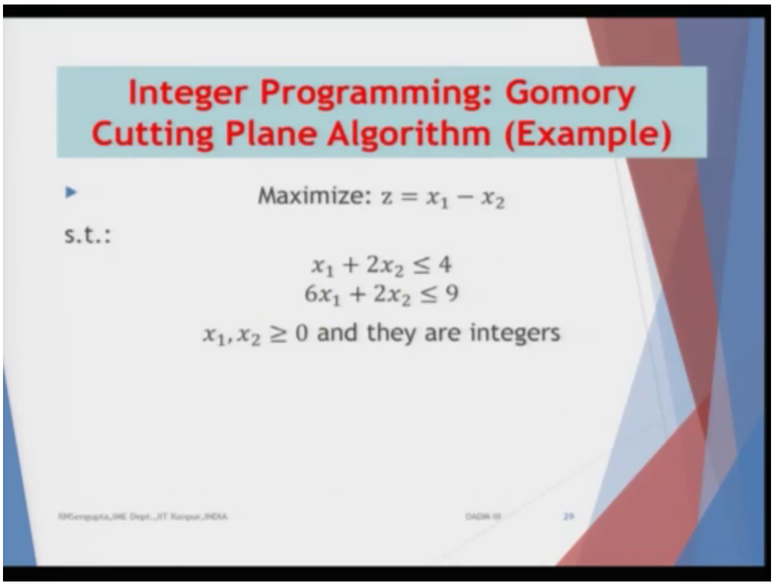

Now you basically solve the problem using simplex method and obviously there are less than type, you would be adding the slack, less than or greater than, whatever it is, you will keep adding the slacks accordingly. So consider for the first problem, the added variable is x 3. For the next constraint, for the first constraint the added variable is x 3, for the next constraint the added variable is x 4.

So the problem formulation is; I will just read it out so you will understand. So it is x 1 plus 2 x 2 plus x 3, 1 into x 3 plus 0 into x 4, why I am bringing x 4, because x 4 would be coming into the second constraint also so it is plus 1 into x 3 plus 0 into x 4 is equal to 4. Second constraint is like this. 6 x 1 plus 2 x 2 plus 0 x 3, because x 3 is for the first constraint, plus 1 x 4 is equal to 9.

And the objective function actually is x 1 minus x 2 plus 0 x 3 plus 0 x 4 because technically x 3 and x 4 are not in objective function but we have to basically format the problem in order to write the tableau accordingly. So once you have the tableau and once you solve it considering that you have, basically start with the basic feasible solution and solve it accordingly so the end result is like this. The basic solution variables are x 1 and x 3.

That means x 2 is not there, x 4 is not there and if you concentrate you have the identity matrix corresponding to the A matrix which you started, you converted into identity matrix. The values for x 1 is 3 by 5, the value for x 2 as x 2 is not there will be 0. The value for x 3 which is the, the slack slash surplus in the first constraint would be 5 by 2 and the value for x 4 which is slacks and surplus for the second constraint is 0.

So once you put these values and any can find x 3 is 5 by 2 or 2.5 as you put in this solution you get the value of z 3 by 2. That is the, and now check here very interestingly. x 2 is 0 which is fine, it is an integer. I am not bothered about x 3 and x 4 but in case if x 1 is 3 by 2, obviously it would mean that it is not as per the general concept based on which we were trying to solve the problem. You have to basically now change the, the constraints, and bring the Gomory cuts and the Gomory constraints in order to solve the problem. So we will do that in the next class and I would like to end it here. Thank you very much and have a nice day.