Advanced Green Manufacturing Systems
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Lecture – 05
Mathematical modeling example

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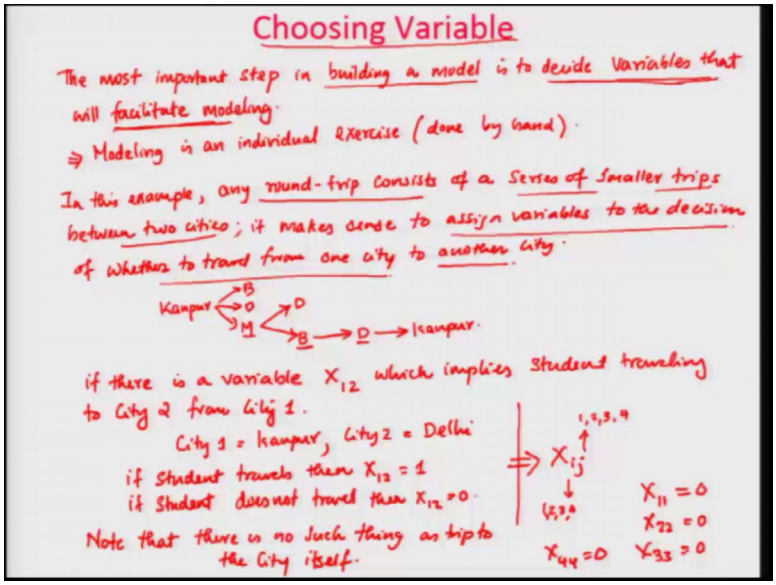

So, continuing onto the simple example. As we said we were trying to find out how our student will travel from cheapest possible from Kanpur, and visit three colleges cities. Now, let us start focusing on the first aspect of it choosing variable ok. So, the most important step, the most important step in building a model; in building a model is to decide on, is to decide variables that will facilitate modeling ok.

So, the most important step, when you want to build a model, the most important step is to desire the variable that will facilitate modeling ok. In many cases, when you try to build models as I remember ok, modeling is usually done by hand modeling, modeling is an individual exercise done by hand or done by expert ok. So, the individual does the modeling.

So, the choice of the variable is to typically left to that individual. And this choice of the variable or the decision on which variable to use influences the, it facilitates the

modeling process ok. In this example, any round-trip consists of a series of smaller trips between two cities. So, any round-trip in our model consist of a series of small trips between two cities. What we do is it makes sense, it makes sense to assign variables, to assign variables to the decision of whether to travel; whether to travel from one city to another city ok.

So, here what you are trying to do is, you are trying to make the decision, your decision is once you reach Bombay ok, so the idea is that from Kanpur, where do you want to go, you have three options in front of you. You can go to Bombay, Delhi or Madras ok. From here once you decide, I want to go to Delhi let us say for example, say you want to go to Madras, then you have only two options in front of you, that is either you can go to Delhi or go to Bombay ok.

So, once you finish decide that ok, I want to go to from Madras, I want to go to Bombay, then there is only one option in front of you go to Delhi. And from Delhi then you have go back to Kanpur in that regard ok. So, depending upon which you choose your path or your the order in which you visit the cities will change. So, what we are trying to do is whether to travel from one city to another city whether we want to do that option is what we are actually trying to model that is what the variable that we are trying to do.

So, if there is a variable, if there is a variable X 1 2 which implies student traveling to city 2 from city 1 ok. And in this case city 1 is Kanpur city 2 is, city 2 in our case will be Delhi.

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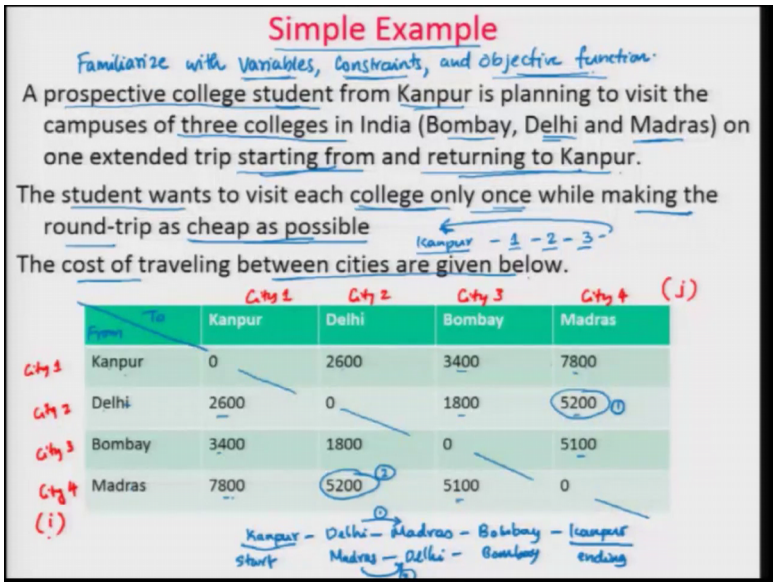

How did I get this? The answer to that question is this is our city 1 Kanpur then or if you think about a city 1 ok, city 2, city 3 and city 4 ok, same way city 2, city 3, city 4. So, this is the, so in this case I can think about this as the cost of the travel ok. And X 1 2 is pretty much a travel whether the student will travel from city 1 to 2 in this regard.

So, if student travels, then X 1 2 is equal to 1 which means the student will travel from city 1 to city 2. If student does not travel, then X 1 2 is equal to 0. So, what we are saying here is that the student does not travel from city 1 to 2, it actually went to some other city. So, the value of this travel is that the student will this basically says that this tells you whether the student can travel from city 1 to city 2. Then there is nothing important about city 1 city 2.

If you think about this and I am going to let say use these indexing go through this as with indexers j and I am going to use this as index of i so then I can make this variable as X i j ok. X i j means the value of i can vary between 1 2 3 and 4 ok; and j can also vary between 1 2 3 and 4. So, this will give me by changing the indexes of i and j, I can model any travel from any city to another.

But then there is also one thing you should notice that note that ok, there is no such thing there is no such thing as tripped trip to the city itself. So, when I am saying is that if I say X 1 1 which means travel from Kanpur to Kanpur this is always should be equal to 0;

same way X 2 2 should always be equal to 0; X 3 3 should always be equal to 0 ok, and same way X 4 4 should also always be equal to 0.

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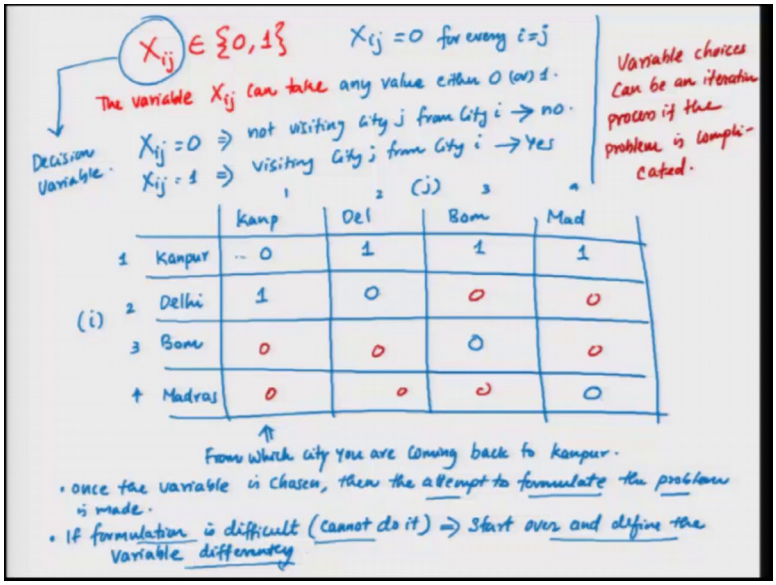

So, if I have to say for all these things or if I have to write about this then the next major thing that I have to do is I have to say X i j is the variable is an element of the set 0 or 1 which means the variable X i j can take any value either 0 or 1 ok; 0 means you are not visiting city i. So, X i j equal to 0 means do not or the student not visiting not visiting city j from city i. X i j equal to 1 implies visiting city j from city i. So, if it is equal to 1, then you are visiting city j from city i; this is the yes, and this is the no alright.

So, if I have to write this down, then if you can think about it that is why for other way is having a matrix the previous matrix that we talked about, I can think about a scenario like this. So, we have Kanpur ok, Delhi, Bombay, and Madras and same thing here Kanpur, Delhi, Bombay, and Madras.

So, if I we said also that X i j equal to 0 for every i equal to j whenever i and j are the same, then they are one on the same, then these values are equal to 0. This is only possible when so this is your index i and this is your index j ok. So, variable is 1, 2, 3 and 4; this is 1, 2, 3 and 4. So, when i equal to j, this can happen is at this place this is X 1 1, this point is X 1 1. So, this will be 0. This is X 2 2 – 0. This will be 0; this will be 0, because this is Delhi to Delhi, Bombay to Bombay, Madras to Madras. So, in all these

cases, where the index I and j have the same value the variable will take 0 which means the tour within this city is not allowed.

And then if I write, if I do an example, if I do this like right this ok, then what does that mean; that means, I am visiting from Kanpur I am visiting to Delhi. So, that is the idea ok. If I say something like this, then I am visiting Kanpur to Bombay; if I write something like this, then it is Kanpur to Madras ok. If I do something like this, then it says I am visiting and coming back to Kanpur from Delhi. So, this column tells you from which city you are coming back to Kanpur. So, that is idea in this case alright.

So, the idea is as I said earlier so this X i j this variable that we talked about, this is most of the time what we call as the decision variable. We need to decide whether to visit city j from city i, yes or no, that decision is that is why it is called as a decision variable. So, the other thing is once the variable is chosen, once the variable is chosen, then the attempt to formulate, attempt to formulate the problem is made, problem is made ok.

So, what do you try to do with this, you have chosen the variable. So, then once you have chosen the variable, if you try to for we attempt to formulate the problem ok. If formulation if formulation is difficult or cannot do it ok, then you want to do is start over and define the variable, defined the variable differently. So, if it is impossible with the current variable definition, and you find it is extremely difficult for you to formulate the problem, then it is better to start over, and define a new set of variables.

And usually what happens is many a times ok, variable choices is an iterative process. So, one of the rules that you need to remember in this regard is this. Variable choice or choices can be an iterative process if the problem is complicated ok. So, in this case, what we are trying to do is, if you think about this problem instead of looking into this cost structure, what we are doing is we are actually converting this into a 0 1 problem ok.

Like this if you can say all of these are 0s. And you are saying that this is the order in which you are visiting. Whether this visit is or not that is a different story, but yes this actually means that you are traveling from Kanpur to Delhi, Kanpur to Bombay, Kanpur to Madras, and returning back to Kanpur from Delhi kind of a thing. This may not be a feasible thing or may not be something that make any sense, but for the time being this is one option because you have no other thing that prevents you from choosing a tool like this. I can actually make it tool like this ok.

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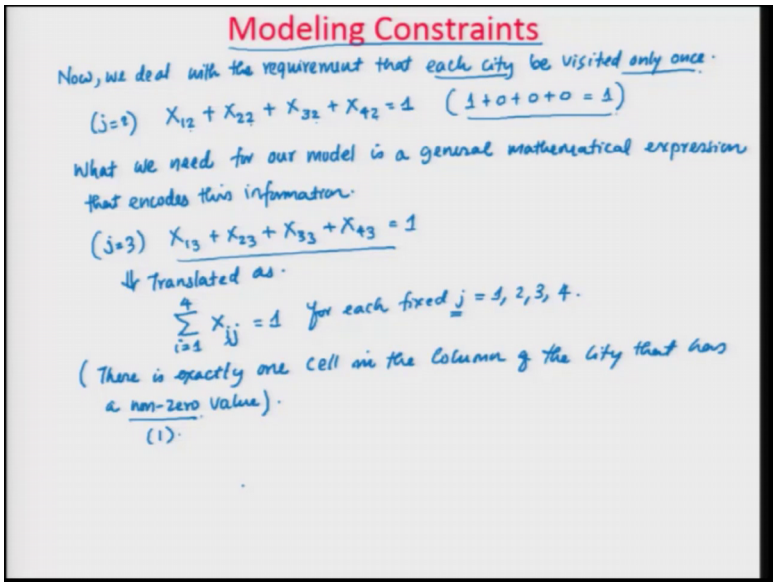

So, then the second part of it is now the next most important part of this problem is all about the how we model the constraints or the modeling of the constraints ok. Now, we deal with the condition with the requirement with the requirement that each city each city be visited only once. We want to visit each city only once ok, you do not want to visit more than that.

So, how would you do that ok. So, let us take for example, this case of the city 2 which is Delhi ok, think about this then the simplest way to think about it is if you want to ensure that the city Delhi is visited only once, that should be only 1 in this column right, only 1 in this column. Or another way to say is if I say something like this X 1 2 plus X 2 2 plus X 3 2 plus X 4 2 is equal to 1. If I say this, what am I saying here this is here 2, the column number 2 j,. So, here j is 2 in this case.

So, in this case you can see that this j is equal to 2, in this case, it remains 2 2 2, the jth position always remains in 2. And the ith position varies from 1 2 3 4. So, each one of this X 1 2, 2 2, 3 2, 4 2, if I sum all of them, we are actually summing 1 plus 0 plus 0 plus 0 which should be equal to 1. If we can create write a mathematical constraint like this, then what we are actually saying is no matter what each city is visited only exactly only once.

So, what we require what we need for our model our model is a general mathematical expression, mathematical expression, mathematical expression that encodes this

information ok. So, I can say that for the city of let us say for that matter city of Bombay which is to be 3, I have can say that when j equal to 3, I can always say that X 1 3 plus X 2 3 plus X 3 3 plus X 4 3 ok, they should be equal to 1.

So, since you can approach this, you can apply the same logic to any column, you can write this as this purely mathematical expression translated as translated as sigma summation of I is equal to 1 t 4 X i j, i comma j equal to 1 for each fixed j equal to 1, 2, 3, 4. So, what you are saying that for each fixed j the value of j remains fixed from 1, 2, 3 and 4. The value of I can change from 1 to 4 for each value of them the summation the sum of them should always be equal to 1 ok.

Or what it says is there is exactly 1. So, in a in another word you are saying is that ok, there is exactly one cell in the column of the city that has a non-zero value non-zero value; non-zero value in this case is 1 ok; non-zero value is 1 in this case ok. I hope you guys understand what am talking about, so each column if you do this, so we think about it yes in this case it is this whole column sums to 1, this sums to 1, this also sums to 1 this also sums to 1 this also sums to 1. So, ideally speaking the solution says that yes you are visiting already each city once fine. So, this is the first part of the approach ok.

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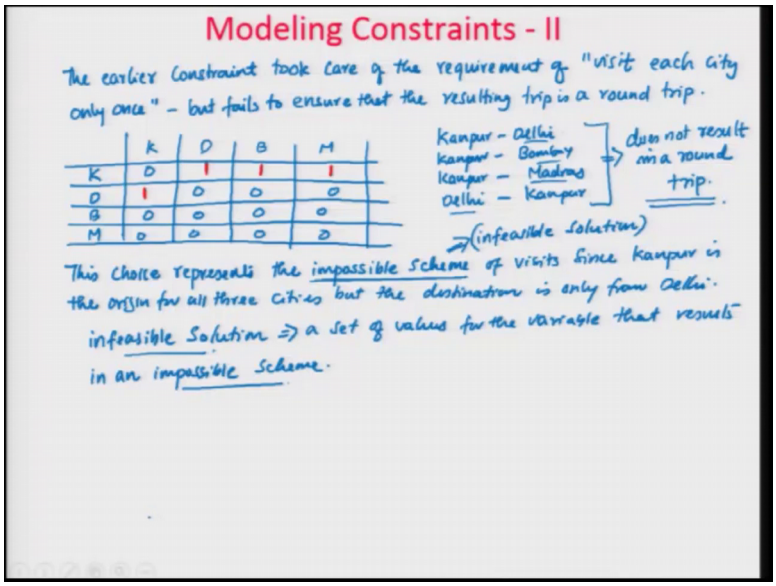

So, in our case, the earlier constraint took care of the, took care of the requirement of visit each city only once ok. The constraint took care of the requirement of visiting each city only once ok, but fails to ensure that the resulting trip is a round-trip.

So, for example, what I can say here is the previous example that we draw, I am just going to draw that small in a small fashion right here not a large one, we will draw a small matrix. So, this is your Kanpur, Delhi, Bombay, and Madras, and we start already started abbreviating and I am lazy in this regard so that is what we are doing Kanpur, Delhi, Bombay, and Madras. So, what I did earlier was this ok, everything else was 0s ok. In this case the ones are here 1, 1, 1 here four 1s in this regard and the rest of them are 0s right.

So, while this ensures that the city is visited once, the option that you are having here is Kanpur-Delhi ok, then you have these Kanpur-Bombay ok, and then you have is Kanpur-Madras, and then Delhi-Kanpur. So, by this option there is no way you can ensure a round-trip you are never moving around too because, you want to visit only the city once. You have visited city once, but the point is that, this does not result in a round-trip ok, even though you can visit the city, city exactly once, but it does not guarantee a round-trip ok.

So, what are we saying is that somewhat in words if you translate this means this assignment or this choice represents it represents this choice represents the impossible scheme, the impossible scheme of visits since a Kanpur is the origin for all three cities, but the destination is only from Delhi. So, you can see that in words in English, you are the origin of each 3 city visitors from Kanpur that the destination belt to Kanpur is only from Delhi.

So, this gives you results in what you call as an impossible scheme or this is also what we call as some in feasible solution in the optimization terms. When somebody says in feasible solution, what we are talking about is an impossible scheme or a set of values. So, an in feasible solution implies ok, a set of values for the variable that results that results in an impossible scheme ok. So, the impossible scheme is also known as in feasible solution in this regard ok. So, what the solution that we draw here is an feasible solution or an impossible solution in that regard. So, how can you address this?

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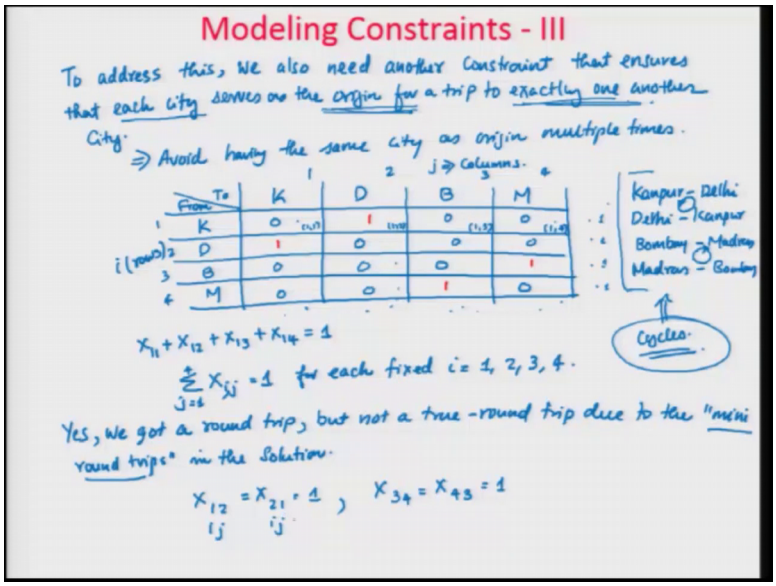

So, to address to address this to address the end to and just to ensure that infeasible solutions are not happen, we also need another constraint. We also need another constraint we also require another constraint that ensures that ensures what does this constraint should ensure that the constraints would ensure that each city each city serves as the origin for a trip to exactly one another city, one another city.

So, you cannot have one city. So, this implies avoid having the same city as origin multiple times ok. So, you are basically saying that exactly one. So, your cities are, any city, each city serves as the origin for a trip exactly once, exactly once to another city. So, if you want to do that, then the way to do it is, if you draw that diagram that we drew earlier which is something like this I will try to draw nicely today.

So, this is your from and to ok, from, to. This is Kanpur, Delhi, Bombay, Madras I believe, Kanpur, Delhi, Bombay, Madras. And the what we were saying is that all these diagonal elements are 0, because the tour within the city is not allowed, and then let me put the 0s in that color actually not 0s the 1s in the red color. So, this was the scheme that we came up with, and then the rest of it was all 0s. This was the impossible scheme, where you are these results in a violation of this condition, origin for each trip is exactly one another.

So, what we have done is this is your j which is the columns ok, and this is your i which is the rows. So, we ensured that the sum of the columns should be individually one. So,

this should be summed to 1; this should be summed to 1; this should summed to 1. Now, to ensure that this 1s, you how to also ensure that the rows are also summed to 1; so, to mathematically write that, so I have I can say it as if I have to sum in this row, then I have to say it as X 1 1 plus X 1 2 plus X 1 3 plus X 1 4 is equal to 1.

So, 1 1 is this column; this is 1 1 ok; this is 1 comma 2, this is 1 comma 3 this is 1 comma 4; it should sum to a maximum value of 1. In this case, if you think about it, it actually sums to 3, which is a valuation of this. So, then what will happen in our case is we will have to change this. So, one way to do it is I am just going to change this portion only, so that you have an idea ok. I am going to change these two 0s and 0s. And then what I am going to do is I am going to just change these two red color 1 and 1.

So, now if you think about it I have a scenario where each column has ones summed to this fashion ok. You have 1 summed to this fashion; you have 1 summed to this fashion; 1 summed to this fashion; 1 summed to this fashion; same way you have a 1 summed in this fashion as well. So, if I write this constraint, then I can say it as same way as we wrote earlier ok, sigma j equal to 1 to 4. This time the change will happen our j it will go over the columns, it will, but the row will remain the same of X i j, i comma j is equal to 1 for each fixed i equal to 1, 2, 3, 4 ok.

So, we are basically we are mathematically saying that each row should also summed to 1, so that exactly you will have something called as say each city should serve as an origin for a trip of exactly one another city. So, if you think about this, then the trip actually eh is something like this at this point. We can see it as you know it is a you can say it is Kanpur to Delhi ok, and Delhi to Kanpur ok, then you have Bombay to Madras, and Madras to Bombay yeah.

So, if you think about it yes you have visited each city only exactly once ok, but then you did not visit them in a complete tour you are not completed. So, from Kanpur to Delhi, even from Delhi to Kanpur came back, then somehow you reach the Bombay, and then from Bombay to Madras, and Madras to Bombay. So, you violated. So, this is not equivalent to the tour where we thought told earlier. So, this has actually two there is 1 cycle here, and then there is another cycle here.

So, what we have here is yes we got a round-trip, we got a round-trip, but not true round-trip, due to the mini round-trips inside, mini round-trips in the solution. So, what you are

saying here is that because of these mini round-trips, you are not being able to you are not being able to reach up to true round-trip. Such kind of behavior these kind of things are called as cycles in the graph ok.

So, when you have these kind of cycles where your solution, yes, it will satisfy these constraints, the current constraints of ensuring that these cities only visited exactly once, but you might not get a true round-trip yet ok. So, to ensure that you have you can get a true round-trip you need to you know enforce a constraint. So now, one other way to think about it is how do you enforce this constraint.

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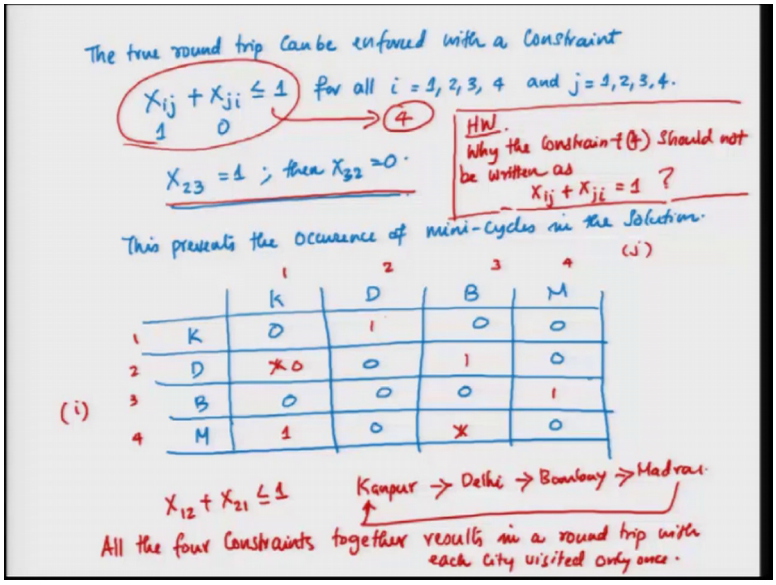

Enforcing the constraint is the true round-trip, the true round-trip can be enforced with the constraint enforced with the constraint with the constraint. So, like for example, if you think about here, it is this is 1, 2, 3, and 4; this is 1, 2, 3, and 4. So, the values you have here is X 1 2 equals X 2 1 equals 1. The other one is X 3 4 equals X 4 3 is equal to 1. So, these are the two 1s that we have ok.

So, what we are trying to do is if you try to create a constraint saying that, so this is your i and j i and j ok. So, you think about it that if the i j and j are if you change these guys ok, then you are reading the same stuff. So, if I say that X i j plus X j i ok, if I say that it is equal to 1, that means, I am really instead of let us not talk about equal to 1. Let say it is less than or equal to 1 if I say this, that means, the maximum value I can take this is 1.

So, if any case, if this case is 1, then that means it should be 0. So, that will ensure that the cycle will come not happen ok.

So, this is for all i equal to 1, 2, 3, 4; and j equal to 1, 2, 3, 4. So, all possible options, if you say X 2 3 is equal to 1, then X 3 2 should be 0 that is what we are actually saying. This prevents the occurrence of occurrence of mini-cycles in the solution. So, as we do earlier the graph or the table ok, we will draw once again here ok. This is your Kanpur, Delhi, Bombay, Madras; here is your Kanpur, Delhi, Bombay and Madras ok. And the solution that we had at this point was this and we had it was this right and everything else was 0s ok. So, let me do this one, so that makes life easy to understand, 1 and 1, 1 and 1 ok.

So, when you enforce this constraint, when you put this constraint into picture then when you say this is 1, 2, 3, and 4, this is 1, 2, 3 and 4, this is your i and this is your j ok. So, when you say X when you have this is 1, 2 ok, so when you say X 1 2 plus X 2 1 is less than or equal to 1. If I say this when you look into this X 1 2 and X 2 1 should be less than or equal to me 1 means 1 of this has to be 0. So, I have you know I can say either I choose this, then they should become as 0 ok. If that happens then what I will have to do is I will have to make something else as 0. So, I might have to maybe do this. If I do that, then I will say fine that like this violate the other constraint.

So, I might end up making this and moving this to what we call as a 1 in the process I will probably get a tour out of this. So, you think about this assignment then what would happen is it is from Kanpur, Kanpur to Delhi; and from Delhi, we are going to Bombay; and from Bombay, we are going to Madras; and from Madras, we are coming back to Kanpur. So, this actually gives you a two tour where we are visiting each city only once. So, all these things put together ok. So, this is your one constraint. Then in the previous slide we have another constraint right here right; this is another constraint.

Then if you think about the previous one, we had a constraint which is right here ok. Then in the previous slide we also had a constraint which is this ok. Therefore, elements of 0, 1 and the other 1 is not equal to 0 ok. So, these constraints put together, if I call this as constraint 1 ok, and this as constraint 2 ok, and this is constraint 3 ok, and this is constraint 4 ok. Then with these 4 constraints put together, all the 4 constraints together results in a round-trip with each city visited only once fine.

So, there is one homework question for you guys ok. So, this is a homework question you have to think about it this homework ok. Why the constraint 4 should not be written as should not be written as should not be written as X i j plus X j i is equal to 1 ok? Why are you writing this is not the right thing to do ok? So, think over it, and try to answer this question this is your homework question alright.

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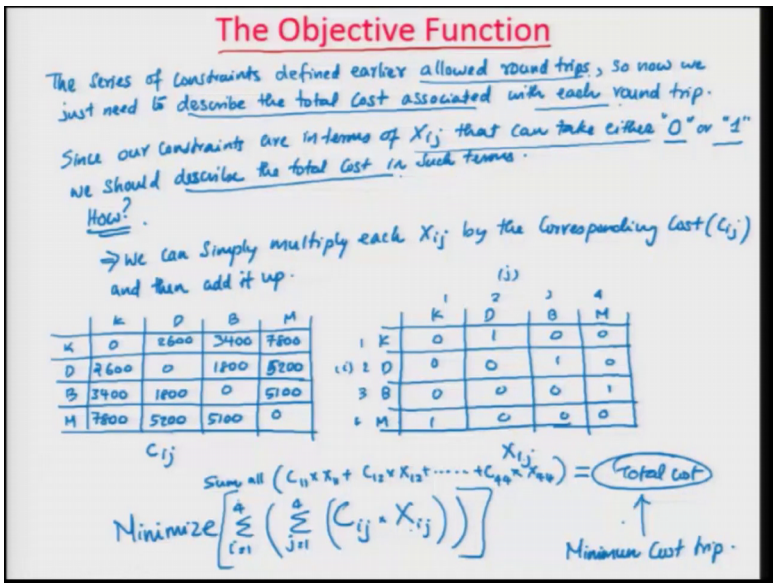

With that now we get into the last part which is called as the objective function ok. So, the objective function, so we are going through now the variables, we are gone through the object the constraints, and now we are in the objective function. So, the series of constraints of constraints series of constraints defined earlier. So, the constraints that we defined earlier allowed a round-trips ok. So, now we just need to, so what do we now need to do, need to describe the total cost total cost associated with each round-trip ok.

So, what we are now doing is, now the constraints allowed us to model the round-trip. Now, all we need to do is describe the total cost associated with the each round-trip ok. Since our constraints are in terms of X i j that can take either 0 or 1 ok, we should describe the total cost in such terms ok. So, how will you do that that is the big question? You want to describe the total cost in terms of X i j which can take all the values of 0 and 1.

So, how can you do that? The answer to this is we can simply multiply, we can simply multiply a each variable each X i j by the corresponding cost let me call it as C i j ok,

corresponding cost C i j, and then add it up ok. So, what we are saying here is in a way if you think about it you had your I am drawing it in together, so that you have a nice idea of what we are talking about ok.

We had this initial thing where we had Kanpur, Delhi, Bombay, Madras, Kanpur, Delhi, Bombay and Madras ok. And we had the values in this regard as kept as the cost numbers, we gave it as these were 0 0 0 0s. And we had 2600, 3400, then 7000; we had 1800, 5200 and 5100. And this was 2600 3400 then 7800, 1800, 5200, and 5100 and this was our costs matrix. So, this was C i j, C i comma j.

And you had another matrix my bad let me draw this properly 2, 3, 4 ok. This much say 1, 2, 3 and 4 ok. So, this is the same Kanpur, Delhi, Bombay and Madras; Kanpur, Delhi, Bombay and Madras. And what we had was a bunch of 0s and 1s ok. So, the previous one we had a 1, 1, 1 and 1. So, we had 1 here 0, 0 had a here 1 here, 0, 1 here ok, and 1 here. So, this was the solution we had. So, this is our X i j (Refer Time: 45:04).

So, what we are actually saying is take the X i j individual values of X i j and multiply the values of this. So, you would actually think about it as you would think about it as sum all C 1 1 times X 1 1 plus C 1 2 times X 1 2 plus all the way like this to see this will be 4 4 times X 4 4 ok. If you think I will give you sum all these values and put it out that will be the total cost of the round-trip ok. So, we write this mathematically, this whole thing if you want to write it mathematically, I can write it as C i j multiplied by X i j.

And what do you need to do is you need to sum to sum it to two times sigma j is equal to 1 to 4. And same thing sigma i is equal to 1 to 4. So, for all values of i varying from i is here which is 1, 2, 3 and 4; and j is here which is 1, 2, 3 and 4. So, all the values of i, j varying from 1 to 4; and i varying from 1 to 4 multiply the cost C i j with the X i j which are 0s and 1s and sum them up that will give you the total cost of the particular option that you have.

And when you if you can evaluate all these total costs like this for all the round-trips, then the minimal the minimum cost trip is what you should take, so that puts the criteria here as minimize. So, when you are saying that minimize this whole thing, that means, minimize the total cost of travel for a given roundtrip that is, so that they becomes this becomes the objective function of your problem alright.

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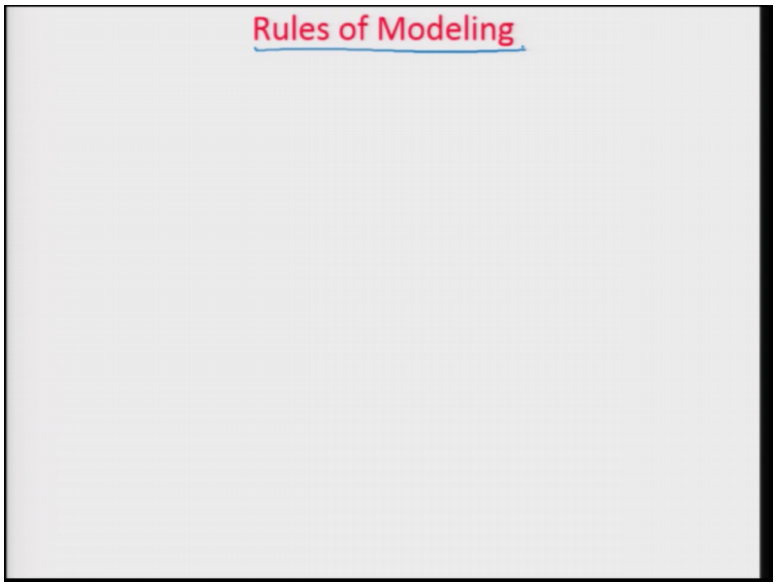

So, then there are additional aspects to this, the rules of modeling. But what we will do is because I would like you guys to read through this whole content and work through these examples. And once you understand this example, what we will do is, we will take the next class and we will take it forward the rules of modeling in the next class.

Thank you.