# Data Analysis and Decision Making - II

**Prof. Raghu Nandan Sengupta**
**Department of Industrial & Management Engineering**
**Indian Institute of Technology, Kanpur**

## Lecture – 46
## Optimization

Welcome back my dear friends, a very good morning, good afternoon and good evening to all of you wherever you are in this part of the world. And this is the DADM-II which is Data Analysis and Decision Making 2 course under the NPTEL MOOC series and as you know this total course duration is for 12 which is 30 hours and each lecture is for half an hour so, in totality you have 60 lectures and you know that in each week we have 5 lectures, each again each being for half an hour as already mentioned and after each week we have assignments.

So, if we can see the slide number is the 46th lecture which we means that we are going to start the 10th week. And, my good name is Raghu Nandan Sengupta from IME department at IIT Kanpur. So if you remember, we were discussing about the concepts of reliability based optimization and I did mention at considering that the constraints are probabilistic, if the distribution for the constraints of the joint distribution for the decision variables are normal.

So, then converting those constraints considering normal distribution converting them into the; in the probabilistic sense, considering the standard normal distribution or the multi various standard normal distribution is very easy to find out the corresponding values of the access decision variables based on the fact, that what is the beta value; beta value, alpha value is the probability level.

Now, I also said using the diagram that in a two-dimensional case having the two normal distribution orthogonal to each other, you will basically have a circle in that two d space or the artesian coordinate. And in the higher dimension that is for the third dimension, fourth dimension and higher on, it will be respectively as sphere and atmosphere and so on.

Now, the point what we did mention it was that, that if it is a non-normal distribution then the overall area covered is definitely not a sphere or a atmosphere, it is a any shape on covered area in the two-dimensional case similarly, for the third dimension and higher dimension. Then trying to find out the centre of gravity of that particular shape based on which we can say that, what is the probability of that what is the solution of that probabilistic problem. It in many sense it becomes difficult if the distribution is non-normal because combining non-normal distribution to find out the multivariate case becomes difficult.

Now, again going back to the normal distribution, we have discussed all this things considering that the variants of both the x 1 and x 2, that is the decision variables we are taking in the two dimensional case. If the variances are same, then trying to combine the variances if they are same, then the overall covered area in the two-dimensional space depending on the beta value would be a circle. Similarly, the circle would increase or decrease in the radius value depending on the increase and decrease of the beta level value for the constraints or the level of reliability.

Now, the next question which will automatically come is that you have considered; so, you would be definitely be asking me that yes, you have considered the normal distribution with same variance. Then you on the other hand you said that, if the distributions are non-normal then trying to find out and combine them to trying to find out the multivariate distribution becomes difficult, which is also agreed upon. But, what happens if the variance is for the normal distributions are different and also subsequently if the variances of the non-normal distributions are unequal.

So let us consider the first case, if the variances of the normal distribution are unequal. Now, consider let us consider step by step; consider x 1 which you are measuring along the x axis, x 2 along the y axis, x 1 and x 2 are the decision variables and the constraints for already given. So, I am that this diagram I will again draw, again try to analyse it, but consider under these two cases, case 1 where the variance of x 1 is more than variance of x 2 and vice versa. So, how does the overall covered area looks like based on which you are going to utilize the concept of probabilistic constraints on or reliability based optimization concept. Such that you find out the centre of gravity of that area and then try to basically find out what is the reliable solution. So, let us first discuss that with a diagram and then; obviously, in the same flow off of discussion, we will consider the variance of x 2 being more than x 1; that means, x 2 being along the y axis.

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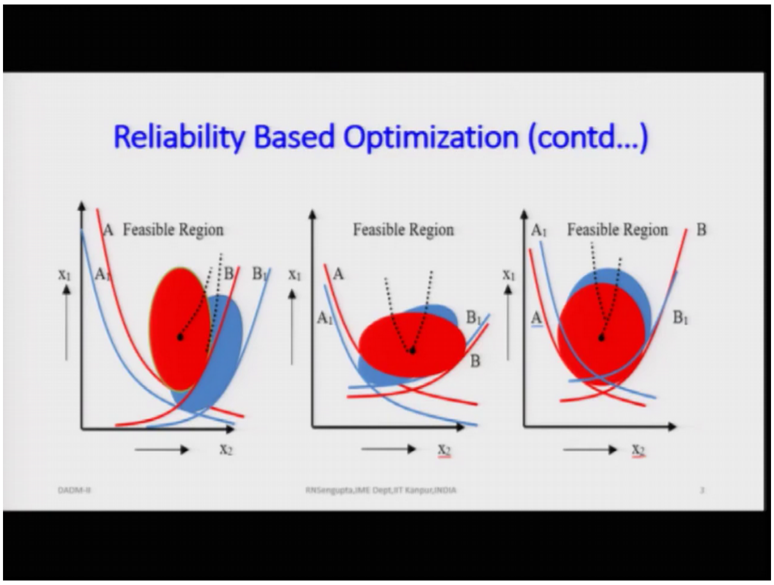


So, let us go from the last diagram. So, in this case what you have is the feasible region; so, let us consider the feasible region. If you concentrate on there are three diagrams frame A, B and C, let us consider frame C.

Now in this case, what you have the blue line? Which gives you the overall constraint areas and you have constraints being A 1 and a B 1 depending on the level of reliability. Then the overall boundary space for variance being equal would be actually depending on the reliability value would be the circle which is blue in colour, And now, we see the set of a gravity which is not possible to mark it out here with the other background is red so, that centre of gravity would give me the reliable solution based on the level of reliability.

Now, consider that the value of the reliability decreases or increases whichever direction. So, that would basically only shrink the space; that means, the feasible space shrinks because the feasible region is marked out as here. Now, if it basically shrinks the space or it increase the space then; obviously, you will have the 2 other boundaries, corresponding to A 1 and B 1 now would be A and B. Hence, the combined area considering that A and B is there would be the red circle and the centre of gravity or the centre of that circle will give you the actual solution based on the level of reliability which you have supposed for yourself for this problem.

So, as the level of reliability increases-decreases, the red lines would slowly move that is A and B would most move inside; that means, the feasible region would decrease, they would becoming A 1 B 1, then in the next case you will basically have A 2 B 2; A 2 B 2 I have not been drawn here. Slowly, the locai of the movement of the centre of gravity will move more towards inside the feasible region, and hence the level of reliability would dictate what is the actual solution.

Now in the deterministic case, if you remember I already discussed that depending on the on the constraints being deterministic the boundary space would be such that the point of intersection of the of the red lines which is A and B or A 1 B 1 whichever way you have been able to denote the deterministic constraints, that will give you the actual deterministic solutions based on which you can say that x 1 star and x 2 star are the optimum values of x 1 x 2 and the corresponding functional value of f x 1 star and f and x 2 star would give me the actual solution for the deterministic case.

So, as you keep changing the level of reliability, the overall feasible region as I already mentioned will decrease. Hence, the movement of the centre of gravity or at the locai would basically trace out the reliable solution based on the fact that what is the reliable solution or the reliability which you have place for your problem. Now, the other two discussion which I have mentioned before I showed you the diagram, one was basically the variability the areas of x 2 being greater than x 1 and in the other case it the third case was basically the variability of x 1 was greater than x 2.

So, let us consider the variability of x 1 being more than that of x 2 so; obviously, it mean the overall dispersion along the x x 1 axis by the way, here the axis have been taken in such a way that x 1 is along the y and x 2 is basically along the x axis. And, once you are able to draw that so, you will basically have the ellipsoid, all just like the rugby ball placed vertically up with the elongated axis being vertical or perpendicular to the ground and consider the ground is basically plain. So, you will basically have the baseball being placed in such a way, the dispersion along the vertical axis more, while the dispersion along the horizontal axis would be less because the variability of x 2 is less.

Now, and that corresponding overall feasible region is inside and the corresponding on the constraint boundaries would be A and B. And if they were know probabilistic part then; obviously, the part of intersection of A and B boundaries would give you the deterministic solution. Keep changing the level of reliability which would mean that the overall ellipsoid would technically move inside, more inside and you get the feasible region or move would basically start coming out; not coming out and go to in invisible in feasible region it will remain in the feasible region, but the feasible region technically is now expanded till the maximum boundary.

So, in that case the overall centre of gravity of that ellipsoid will trace the reliable solution based on the level of reliability. Now remember one thing, level of reliability or the beta value would basically expand and contract the circle area or the area of basically the ellipsoid which you are going to consider.

And ; obviously, I am repeating it again the way the centre of gravity or the centre of the ellipsoid or the circle traces the locai would basically give you the points based on which you can find out the x 1 star and x 2 star depending on the level of reliability. And hence, once you find out x 1 star x 2 star, you can find out the functional value of the objective function, based on the fact that x 1 takes the value x 1 star x 2 takes the value of x 2 star.

So, this was panel number A, let us move to panel number B which is the middle one. In this case, I am considering the reliability to be the same in all the three cases, but here the variability of x 2 is more than the variability x 1. So, hence the elongated part of the ellipsoid or the rugby ball would be placed horizontal to the ground hence, the major axis would be parallel to the ground while the minor axis would be vertical to the ground.

So, in that case again the locai of the centre of gravity as the reliability changes would trace out the probabilistic values. And if the probability values changes so, technically if the thus ellipsoid more inside hence the centre of gravity would move more towards the inside the feasible region.

So, based on that you can find out the reliability region, but in this normality distribution or the symmetric distribution on the normal distribution, the advantage is that you can use simple multivariate normal distribution or multivariate standard normal distribution to find out the values in their probabilistic sense and solve the problem and get the best solution, such that you will give you the objective function whatever it is maximization or minimization.

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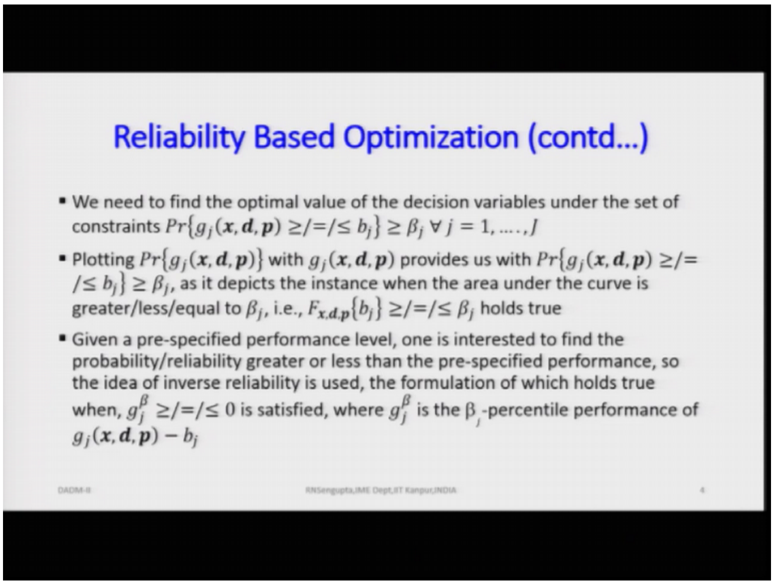


Now, how we are going to do it; so, this is the contractual framework based on which I discussed that how we can think or visualize, but technically you need to have a method. So, we need to basically find out the optimum values of the decision variables. Decision variables if you remember are the axes under the set of constraints that the probability of each and every constraints being greater than equal to or less than b b j's. Because we have considered j number of such constraints where the probable probabilistic concept is being considered and this probability would be greater than equal to beta j. So, beta j as I have already mentioned or alpha in the case if you have written it considering the level of confidence which you we all know in from DADM 1 or you must have studied in basic statistics.

So now, if you also remember that we considered in the last class which is the last class was basically the last class of the 9th week. We have considered the multivariate distribution and I said that finding out in the left the probability of z being less than equal to small z of or x being small less than equal to small x considering the normal distribution would be easily attainable or obtainable, once you know the standard normal table and have the concepts about that.

Now, if you want to plot the values with the corresponding values of the actual functional form which is g j. So, you want in one case you want to plot those values of the constraints and in other case you want to find out that, what is the probability that those constraints are greater than equal to some boundary value which you have set for yourself which is beta b j.

Now, once you are able to plot it so, it will basically defect the instances when the area under the curve is greater than; less than or equal to beta j depending on what values of beta j you have considered and what are the equality signs you have considered. It can be greater than beta j less than beta j and equal to beta j that is why I have written all the three constraints at one go in order to make you understand.

Now given a pre specified performance level one is interested to find on the probability reliability greater than or less than the pre specified performance. So the area, idea would be inverse probability or inverse reliability should be used; such that I can find out that these values of the constraints based on the fact that overall reliability is beta. So I you need to find out that, what is the functional form of g j to the power beta is greater than equal to or less than that value of 0 that will be satisfied.

So, what you are trying to do is that you have to given the probability, you will basically find out the overall value of that probability such that, you will simulate it and is going to come in this way. Say for example, you have the distribution considering is normal and you know the mean values; consider you know the mean values for g j's.

Now, the question would be how do you know the mean values, because if you remember when your formulator the problem whether deterministic case. Those deterministic values based on the fact that the mean values of those constraints or mean values of those parameters p or the mean values of the parameters x, based on which you will do the simulations are known with some with some certainty; I am not using the word certainty in a in a very probabilistic sense. You know that value due to some prior values which you have or the market conditions whatever it is practically.

Now, you will basically generate data, based on the mean value and considering the variances also known. So, you will generate say for say for example 10,000, So, once you generate 10,000, you will rank them from the lowest to the highest or the highest to the lowest whichever you do.

Now, if say for example, the beta value is 95 percent which technically means that you will follow the from the minimum to the maximum, keep adding the probabilities. The moment it the probable the total probabilities 95 percent or exactly little bit more, you will stop and report that value of the constraints which you have. Such that you know that those that that value of the constraints being greater than to beta value, beta value you have already known for yourself, will give you the probabilistic solution so, if it is known through the simulation.

You will basically generate such data depending on how you have rank them from the minimum to the maximum; if you do from the maximum to the minimum, then you will basically plot the values of 1 minus beta which we already know and we have discussed how we will do it in DADM 1. So, once you find out the probabilities, you can basically report and find out what is the value of g j to the power beta, such that it is greater than equal to or less than the value of 0 based on which you can find out, what is the constraint boundary and what is the value of the reliability; reliability solution which is the centre of gravity.

So, this in g j they this g which is the constraint, I am not mentioning the j because j would be the number of the constraint. So, g to the power beta is basically the beta j percentile performance or the value such that the constraint minus b b j which is on the right hand side is would be greater than equal to 0.

So, you are basically trying to simulate and find out at what values and for which values, the constraint minus that the right hand side would be exactly greater than equal to 0, such that you will report those values and do the calculation accordingly.

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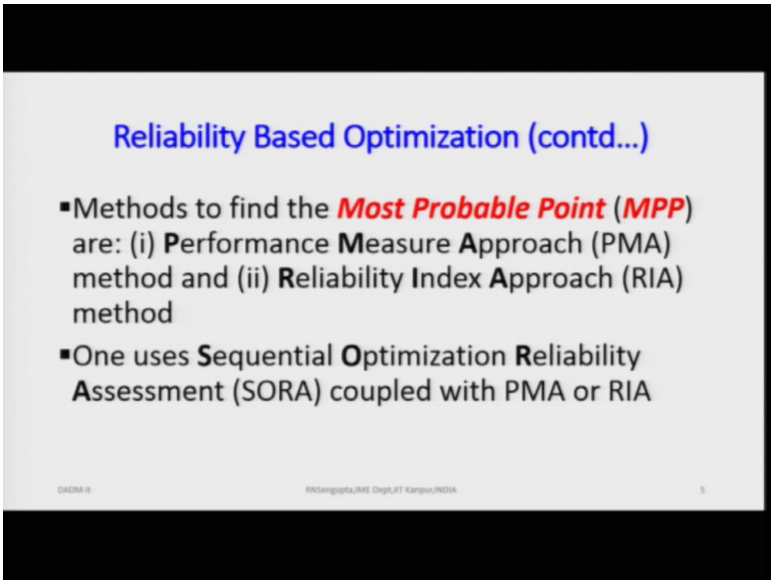


Now, to find this you have basically two ways or two methods. One is the most probable method and another is the performance measure approach; one is the most probable point approach, one is the performance measure approach. You need to find out the most probable point sorry for that. So, one is basically performance measure approach and one is the reliability index approach so, both intuitively at the same.

So, what you want to do is that? I will give you a background of or a little bit of a story about that. Considering there are two rooms and this two rooms, consider room number 1 and room number 2. So, in room number 2 you have those variables which are measured along the random variable which is x and consider there is n dimension. So, let us let us first go into the fact that consider there are two variables, x 1 and x 2. And in the other room you have the corresponding variables are u 1 and u 2 were x 1 is related to u 1 and x 2 is related to u 2, how it is let us proceed.

Now, consider both the variables x 1 and x 2 have a distribution; similarly, the random variables u 1 and u 2 have a distribution. Now, what are the properties of a of cumulative distribution or capital F of x. Now, we have already studied that actual value the overall property of CDF is basically, the sum of all the probabilities for minus infinity to plus infinity would always be 1. The sum of the probabilities which is true for any distribution point 1; number 2, if you find out if you are integrating it from minus infinity to plus in plus x whatever the x value is or u value whatever it is. If x tends to minus infinity, then the value which you are measuring the probability values would be 0. Similarly, this would also hold for the right hand room or the other room where the random variables are u, such that trying to basically find out the overall CDF some of the probabilities or the CDF values from minus infinity to sum u 1 value, such that u 1 value is tending to us minus infinity would also be 0.

Third case would be the in the integration or the overall sum of the probabilities from minus infinity to x as x tends to plus infinity in this room which is x space and in the other room when you integrate it from minus infinity to u 1. So now this x and u 1, I am using interchangeably; that means, x 1 x can be x 1 also or x 2 also there is a material; similarly, in other room you can be u 1 and u 2 also.

So, if you add up all the probabilities from minus infinity to u or u 1 or u 2 and 10 and let and you allow that u or u 1 or u 2. Basically tends to us, plus infinity then the area would be 1 which is very intuitive and also you will know that if there are 2 x values, then it is in a mono (Refer Time: 21:51) increasing function not decreasing because of probability values are either 0 or greater than 0 they are not less than 0. Such that if you add up any probabilities, it will be exactly equal to whatever the some of the probabilities, you have achieved or it will be basically a little bit greater than that. So, if you have this you will basically trying to utilize this concept accordingly.

So, you will use the performance measure approach and reliability based approach based on this fact, but the way you approach them is two different things so, how they are done I will discuss that with a diagram. So, generally we use the one of the methods is sequential optimization reliability approach and that is in tune with the PMA approach or the RIA approach. So, you use the sequential optimization procedure 1 step at a time you go and once you basically bring in the loop, the probabilistic part you use either the PMA or the r RIA not both of them, but you will basically use one at a time depending on the type of problem you are going to solve.

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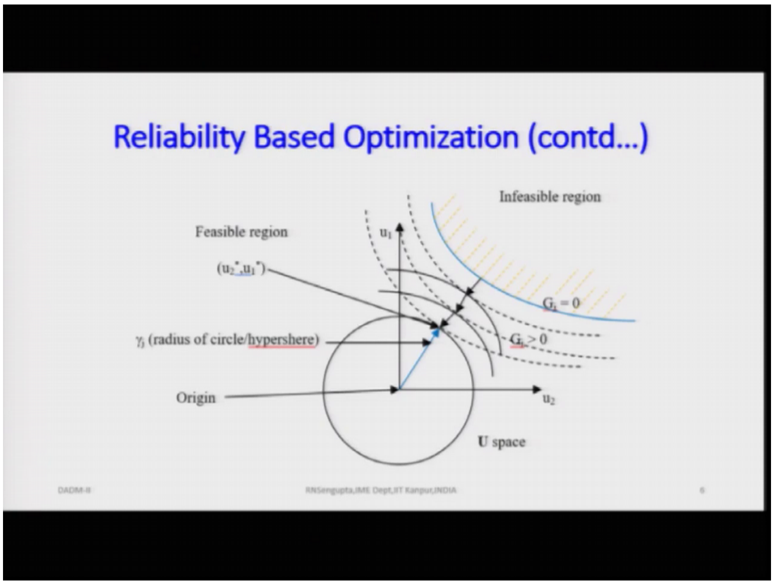


Now here is the background of the diagram, how the PMA and the ri RIA add up are solved. So, I am only considering the normal distribution because the normal distribution is easy for us to explain intuitively because graph wise as well as solution wise and this can be extended for the higher dimension also for the normal case, for the case when the variances are different and for the case where the distributions are not normal. So, let me come to the on the PMA and the RIA approach. So, both the concepts are subsumed in the same diagram.

So, consider the feasible region is given and you are considering that the feasible region is basically one of the considering which is G j and consider that is equal to beta B j or is equal to 0 because if you bring B on to the left side it the right hand side is 0.

Now, what you consider is that in the other room, you have basically map the constraints into G 1 and G 2 where the now the random variables are u 1 and u 2. And, you basically have a sphere; now the or a circle. Let me consider the circle in the two-dimensional space where the circle overall area depends on the level of reliability which you have set for yourself for that problem. So, if it is 90 percent or 99 percent or 65 percent or 60 percent, the overall the radius of the circle will depend on that and what you do is that? You basically consider, so once you start the iteration so, we know many of the iteration method like the Runge Kutta method, Newton Raphson method. So, basically you have one initial value and basically try to iterate about that point go step by step. And once when the change in the objective function is less than equal to some epsilon value which you have set for yourself depend on the accuracy will stop.

Now, consider that you want to start up the iteration. So, what is how would you do it? So, if you remember I have mentioned that x is a distribution which has a mean value, consider that as mu suffix x. So, x can be it can be a vector or a scale does not matter. Similarly, the probabilistic value p which you have considered for yourself which are the external factors which is effecting your problem also have a distribution consider that they have a mean value.

Now, you consider those mean values for x and p and see for example, based on that you solve the problem, once you solve the problem and you basically I would not use the word solve the problem is basically two steps. So, what you will do is that? You will consider this mu x and mu p and map that actual function value into the u space utilize. So, once you solve that in u space you will basically have the corresponding value of x in the u space or corresponding value of p in the u space which are the mean values in the u space and consider that is the origin of the circle.

Now, the radius of the circle will depend on the values of beta. So now consider the beta values are expanding or decreasing which is like a balloon or the overall radius of the circle expanding or decreasing. Now, if you remember if the reliability values in increasing the feasible regions technically starts decreasing; so, which means the centre of gravity of the circle is moving more inside the feasible region. So, this is exactly what you are trying to achieve.

So, as the circle radius increases at some point of time it touches the boundary level which is you have considered there is a division between the feasible region the infeasible region. So the area on to the right which is the hashed yellow one is the infeasible region and the region where we are basically trying to solve, it is the feasible region. The moment it becomes the tangent or it touches that that boundary of infeasible feasible region that value of beta would basically dictate what is the actual value based on which you are trying to solve the problem and that will give you the level of reliability and the solution. Solution means, the value based on which you are trying to solve the problem that iterate method.

Now, once you get those solution so, this is not exact yet. So, once you get that solution in the first iterative says state you will again map it back to the x space. So, this is the first iteration values of x and p, solve your problem using the methodology which you which you are which you are using in the very general sense, it can be linear programming, non-linear programming mixed (Refer Time: 27:45) program whatever it is, solve it and find out the second stage of values of x and p.

Again, map those x and p consider those are x 1 because that is that is the first step p 1 is the first say first step, you again map it back to the u space. Considering the pma, RIA approach whatever you have done, find out the solution there in the u space. Map it back to the x space and continue doing it till the difference between the objective function does not decrease less than equal to some epsilon value set for yourself.

Now, I did mention the PMA and the RIA are the same; if you look at the diagram in one case if you remember I said the circle increases that that disk increases, if it is basically sphere it is like a balloon expanding. Now, in the other case it can be you keep that the level of beta fixed; beta is now is being normalized to a value of 1, keep it fixed and then basically remove or push the infeasible region more towards the feasible region such that feasible region decreases.

The moment it is tangent that is the value based on which beta you will basically freeze your solution. So, in one case you are expanding the circle, in another case you are basically expanding the region based on which you will basically do the iteration back and forth till the level of efficiencies is reached. So, with this I will close the first class of the 10th week and continue discussing more about this in the next in the next class which would basically be the 40 45th; 46th 47th class. So, have a nice day and.

Thank you very much for your attention.