Microeconomics: Theory & Applications
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Lecture – 9
Optimization Theory and Techniques Part-2

Welcome back to the lecture series on Microeconomics. Let us continue with the discussion on Optimization Techniques. Previously, we have seen a class of optimization problems which are known as unconstrained optimization. It means, that the decision variables or the choice variables we are dealing with can take any value, there is no constraint on them. But, in reality we find in many maximization or minimization type problems the decision variables are not allowed to take on just any value so; that means, that decision variables the choice variables have to satisfy some kind of side relation, and that is why they are constrained in nature.

So, if we are dealing with such constraints some side relations which we need to satisfy while we are optimizing we have the class of optimization problem which is known as constraint optimization problem.

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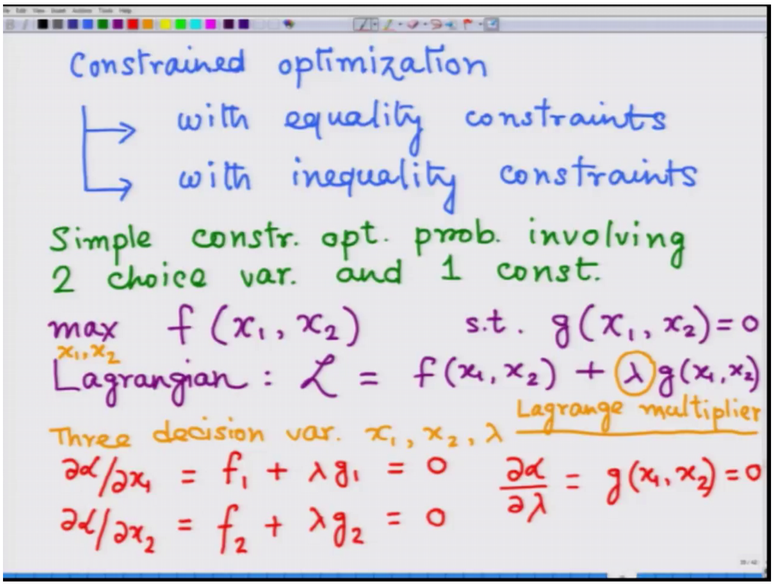

And now we are going to discuss this class of problems. Constrained optimization type problems can be classified in 2 groups. One is with equality constraints and the second one is with inequality constraints.

Now, we are going to discuss the first type which is the optimization problem with equality constraints. And now we are going to learn a technique which is called Lagrangian constrained optimization technique. And this technique is named after an 18th century Italian mathematician named Joseph Louie Lagrange. So, we are going to start with a very simple constrained optimization problem involving 2 choice variables and 1 constant. Needless to say that we can easily generalize this, but let us start with a simple problem to see what is the matter.

So, let us assume we are dealing with a maximization problem. There is a continuous and differentiable function defined over x 1 and x 2. And we need to maximize this function subject to this constraint. Now, the Lagrangian technique that we are going to study evolves around a function which is known as Lagrangian. And that Lagrangian function is to be defined as follows.

So, here the decision variables will be x 1 and x 2. And we introduced this new entity or the concept called Lagrange multiplier, which is also unknown. So, in total we have 3 decision variables x 1, x 2 and lambda. Later we will see that this concept Lagrange multiplier has very interesting interpretation both in mathematical terms and also in economics. So, if we need to maximize this Lagrangian function lambda, then what to do? We will take first order derivatives of this function with respect to the decision variables that is the way to go. So, we are talking about del lambda, del x 1 and we get f 1 the first partial derivative of function f of the decision variable x 1, plus lambda g 1, we need to set that equal to 0.

Similarly, we now need to differentiate the Lagrangian function with respect to the second decision variable x 2. And this will give this partial derivatives f 2 and g 2. And, as we all know we need to equate this first order condition equal to 0. And do not forget lambda. So, that is also you know a decision variable. So, we need to also differentiate the Lagrangian function with respect to this lambda. And that will give us the third first order condition. So, we get 3 unknowns in 3 equations. So, of course, this system of equations is solvable.

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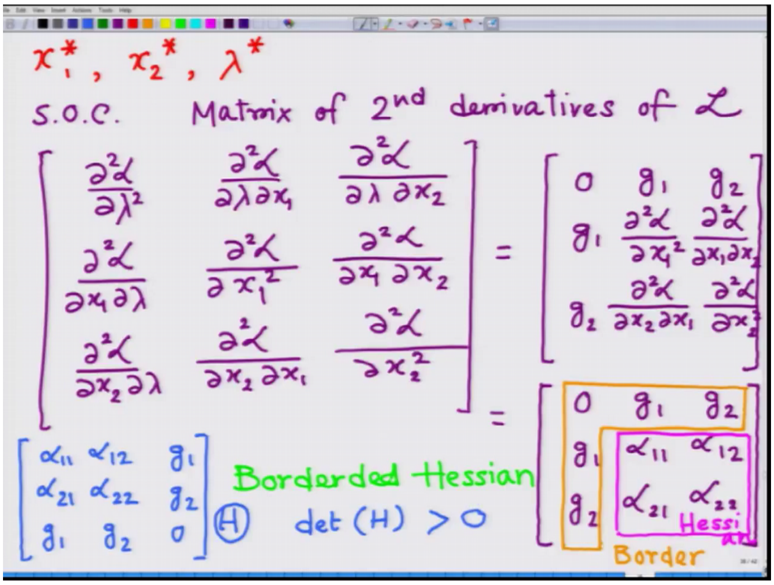

Now, after solving we will get the optimized values x 1 star, x 2 star and lambda star. So now, let us consider the second order condition of this maximally constrained maximization problem. So, we know we have to consider a matrix of second derivatives of the function that we are maximizing. In this case, that is this function Lagrangian. So, we construct this particular matrix. So, we first concentrate on the third first order condition, which is basically can be obtained by taking derivative of the Lagrangian function with respect to the Lagrange multiplier.

So, we first focus on that constraint, and then we will differentiate with respect to lambda x 1 and x 2. Then after, we are done with that, we have to now differentiate the first and the second first order condition with respect to all 3 decision variables. And, that is how we are going to get our components of this matrix. So, this is the matrix that we create from the second derivatives of the Lagrangian function. Now of course, given the Lagrangian function it is easy to derive for certain components and if we do so we get the following.

So, this g 1 and g 2 are the first order partial derivatives of the constraint with respect to the decision variables 1 and 2. Now, we can simplify this further to a form which has easy to deal with notations. This is the matrix that we are going to consider when we are going to discuss our second order conditions. Now note that there is a very interesting stuff going on here. So, this is look at the square matrix here. This is looking more or less like the Hessian that we have talked about previously. And now there is this border component coming, these border components are coming from the partials of the constraint.

So, this is the border that is generated, and it is bordering this Hessian and that is why it is called bordered Hessian matrix. So, note the difference, in the case of an unconstrained optimization problem we get imple Hessian matrix in the case of a constrained optimization problem we get a bordered Hessian matrix. Now you may ask why the border should be like this. Now, there is an alternative representation possible and if I go for that alternative representation, then bordered Hessian matrix can also be written like this.

So, this is also allowed. Now once these bordered Hessian matrix is constructed, let us see the second order condition how can we state the second order condition. For maximization problem Hessian has to be negative definite matrix subject to the constraint. Thus, if we name this matrix say H then determinant of this H matrix has to be positive and that is the second order condition for maximization.

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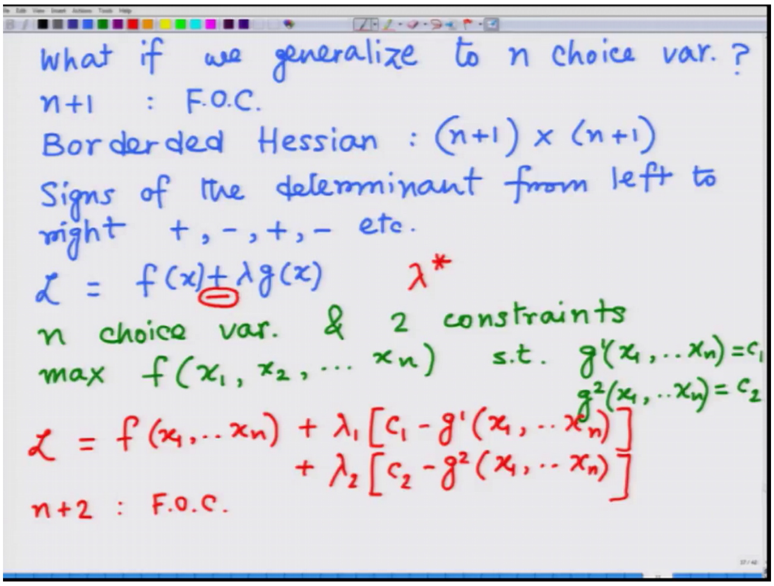

Now, we have discussed a 2 variable one constraint case. What if we generalize to n choice variables. Then how my first order condition and second order condition are going to change? First order conditions will not change. You know, you have to take the first order partial derivatives set them equal to 0, and you know if there are n number of choice variables there will be n number of first order conditions. And of course, you need to differentiate with respect to the Lagrange multiplier lambda.

So, if there are n number of choice variables you get n plus 1 number of first order condition equations. So, in the case of n choice variables and one constraint model, you know the bordered Hessian matrix is going to be n plus 1 by n plus 1 matrix. And the signs of the determinants from left to right, we will change in sign we will alternate in sign. And the order should be plus minus plus minus etcetera.

Now, note one thing, we have defined our Lagrangian function as f x plus lambda g x. One may ask what if I replace this plus and you know I replace this plus with a minus. No change, only the sign of the Lagrange multiplier value the optimized value of lambda star we will have a different sign. So, only that much difference will happen if you replace plus with a minus here. Now, let us move on to an extension. And this extensions we will now have n number of choice variables, and 2 constraints.

So, let us write down a fresh maximization problem. Maximize f of x 1 x 2 x n and subject to g of x 1 x n equal to c 1, this is function g 1. There will be another function g 2. So, in this case, how can we write the Lagrangian function? Now, the Lagrangian function we will take the following shape. So, of course, as there are n number of decision variables, and there are 2 Lagrange multipliers, there will be n plus 2 number of decision variables, and you are expecting n plus 2 number of first order conditions. And you know of course, you know following the way I have shown previously, you can construct the bordered Hessian (Refer Time: 21:56) all. But, now let us concentrate on the interpretation of this lambda 1 and lambda 2. What are these variables? So, the issue at hand is to get the interpretation of Lagrange multiplier.

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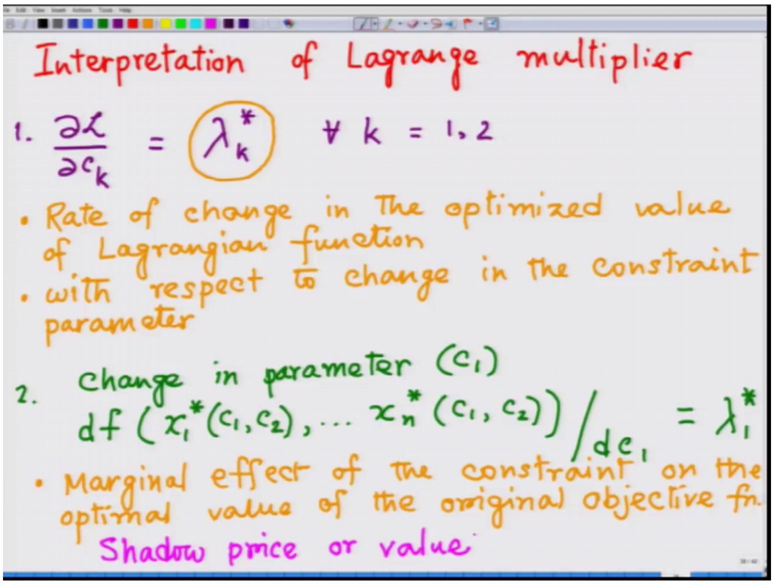

Now, we can approach this interpretation in 2 different way. So, in a general case where you know we are dealing with k number of constraints, we can write and this is the first way of looking at it. And of course, in the case that we were dealing with k takes 2 values, because in we were dealing with 2 constraints. So, this c are the parameters, c 1 and c 2. So, this Lagrange multiplier is the rate of change in the optimized value of Lagrangian function. And, this is with respect to change in the constraint parameter.

Now, let us look at the second approach to interpret the Lagrange multiplier and for that you know let us remember the discussion that we had earlier that the optimized values of the decision variables in this case x 1 star and x 2 star are functions of the parameters in the model. So, if you change in the parameter value then it will have an impact on the optimized values of the choice variables. That is what we have seen under the heading of comparative statics. Now, we are going to follow that approach to interpret the Lagrange multiplier.

So, suppose we are changing the parameter 1 and that is c 1. So, we can write Lagrange multiplier value Lagrange multiplier as the following expression. So, this expression is the Lagrange multiplier 1. So, if we follow this line of thought, then this can be interpreted as marginal effect of the constraint on the optimal value of the original objective function. And in economics, this particular interpretation is very popular and this Lagrange multiplier has another name and this is also known as shadow price or value. Now, we are going to discuss an example of this Lagrangian method. So, this is it for right now. We will continue with these discussion in the next lecture.