Microeconomics: Theory & Applications

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Lecture – 8
Optimization Theory and Techniques Part-1

Welcome back to the lecture series on Microeconomics. Today we are going to have a discussion on Optimization Theory and some of the associated Techniques. Whatever we have discussed so far it is quite evident that all economic agents are optimizers.

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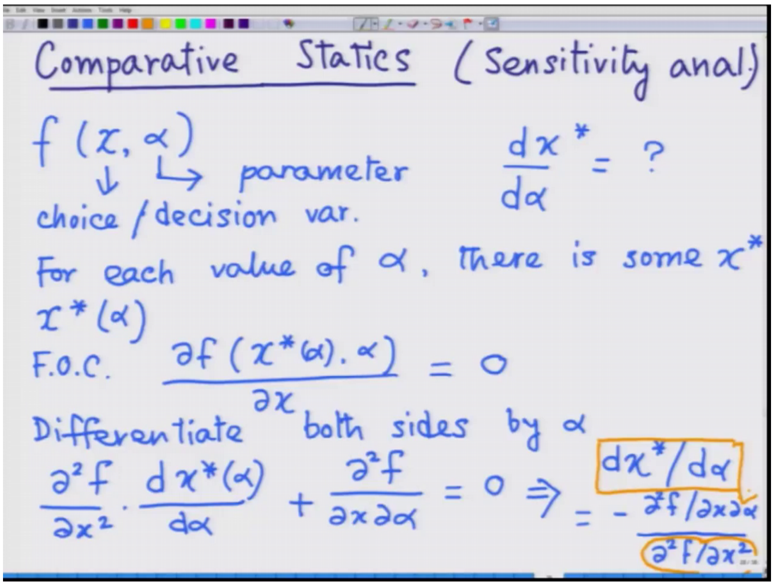

Now, we are going to move on to the case of 2 variable optimization.

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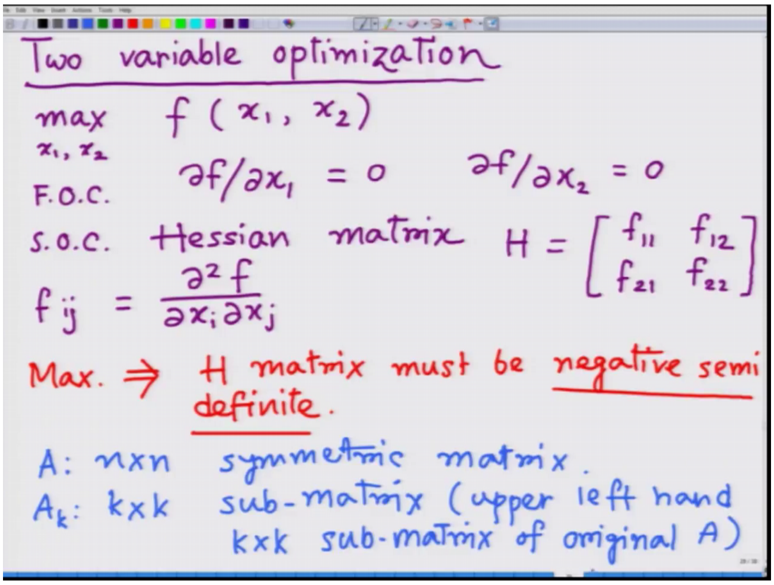

Of course, this can be generalized to n dimension, you know, n number of variables, but let us not get into the complications. This is not a you know advanced level microeconomic theory course for this course, if we know up to this that would be good enough.

So, in the case of 2 variable optimization, let us start with this maximization problem a function defined over 2 independent variables x 1 and x 2. These are the decision variables so, my maximization will be with respect to x 1 and x 2. So, what will be the first order condition? In this case first order condition will be the partial derivative of the objective function with respect to the decision variables in we need to set them to 0.

Now, the second order condition is a bit complicated compared to the first case. Here we have to introduce a concept called Hessian matrix. The second order condition is represented through a Hessian matrix. Now the Hessian matrix is defined as H f 1 1, f 1 2, f 2 1 and f 2 2. So, these are all partial derivatives right. So, in general one can write this f I j is this partial derivative. Now, again there can be 2 types of problems one maximization and one minimizations. So, we have started with the maximization problem. So, let us look at the second order condition for the maximization case. For maximization problem the Hessian matrix must be negative semi definite.

Now, what do I mean by this new term that is introduced here negative semi definite. Those who are doing college level matrix algebra may be aware of this concept. Those who are not for them let me define this very quickly. So, let us assume A to be n cross n symmetric matrix. And A k to be the k cross k sub matrix. So, this is basically the upper left hand k cross k sub matrix of original matrix A.

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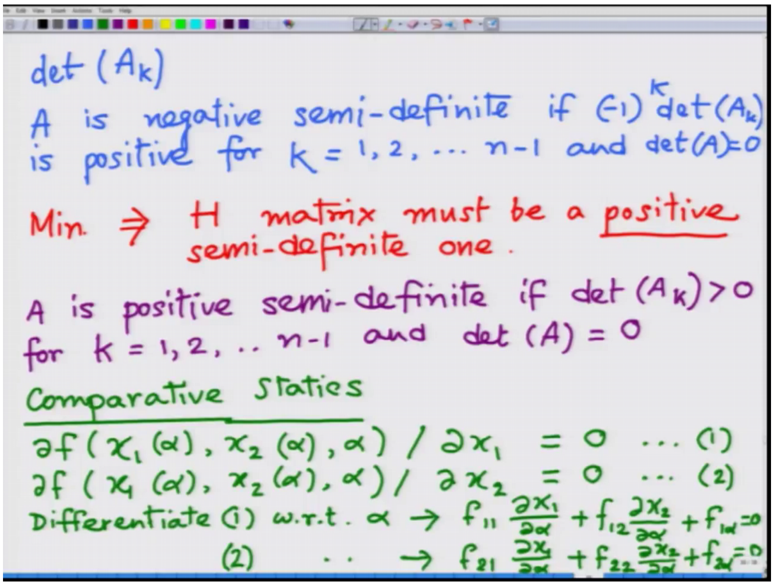

Then let us also define determinant A k the definition of determent should be known to you, then A is negative semi definite if is positive for k taking values 1 2 up to n minus 1 and determinant A should be 0.

Now, if we are dealing with a minimization problem, then what will be the shape of the second order condition? Again we have to construct the hessian of course, and then this is the result that I am going to state H matrix must be a positive semi definite one. What do I mean by positive semi definite? So, let us stick to that same definition of A, and then A is positive semi definite if determinant A k is positive for k equal to 1, 2, 2 to up to n minus 1, and determinant A should be equal to 0.

Now, in this case, let us look at comparative static analysis. So now, we are going to discuss a 2 variable maximization problem. So, you will start with the first order conditions, and then you know we are going to see how from there we can conduct our comparative statics analysis. So, we will start from the first order condition directly. I said that I can assume that the decision variables are basically the functions of the parameters. So, I am writing explicitly as I did earlier also. So, this is say equation number 1, then I have del f. So, I am differentiating the same function, this time with respect to my decision second decision variable which is x 2, and I am setting that equal to 0, and these 1 2 equations they will give me the first order conditions.

Now, for a comparative analysis differentiate 1 with respect to the alpha the parameter. And that will give you f 1 1 del x 1 del alpha plus f 1 2 del x 2 del alpha plus f 1 alpha and that would be equal to 0. Now, differentiate 2 with respect to the parameter alpha which is changing, and now we will get f 2 1 del x 1 del alpha. And you need to set that equal to 0 as well. So, you get 2 equations, and you can name them number 3 number 4, what are the you know variables in this equations? We are actually interested in the comparative static components which are namely del x 1 del alpha and del x 2 del alpha.

So, these are the variables of interest. We have been discussing the role of optimization techniques in economic analysis. Now we are going to continue with what we have learned so far. We are going to see an example of optimization exercise in microeconomics.

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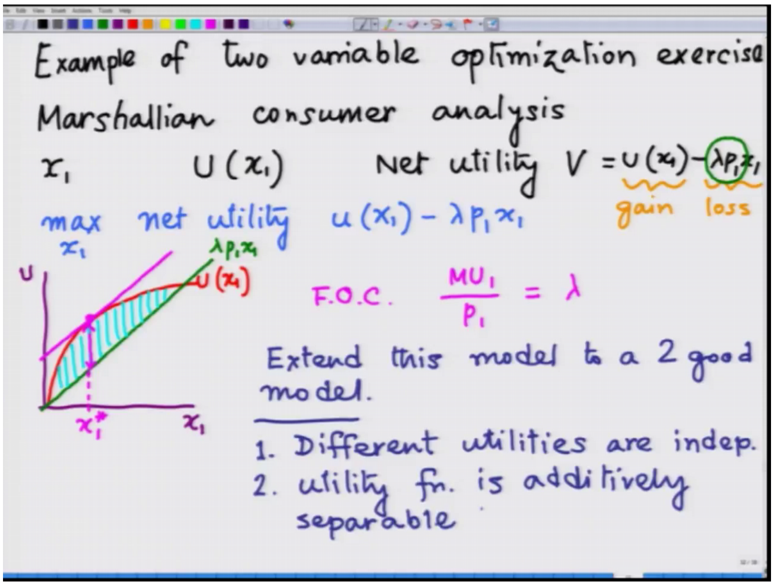

We are going to go back to the discussion on the Marshallian consumer analysis. We are we have already seen how in a Marshallian world consumer optimizes. Now we are going to start from there and we are going to extend that to a 2 good model. In Marshallian consumer analysis, we have made couple of assumptions. I suggest you go back to the previous lectures and you know have a look at them.

So, you know I am not repeating these assumptions. So, what we have done there we have just worked with one consumption good, namely x 1. And then we said that the consumer derives utility, which is a function of the x 1 consumption only. And there is a concept called net utility; say, it is denoted by V and that is basically u of x 1 minus lambda p 1 x 1. So, if you remember the discussion we had earlier. So, in the one variable model this is the gain of the consumer and these in terms of the utility, and this is the loss of the consumer in terms of utility as well.

So, a consumer maximizes net utility u of x 1 minus lambda p 1 x 1. We have looked at the first order condition and all. And of course, the decision variable has been x 1 there. Now, we do not want to get into the first order condition. Let us look at the graph so, that we can get some insight into it. So, I am plotting u along the vertical axis, and quantities or units consumed of commodity 1 along the horizontal axis. Now utility is a concave to origin function, because of the law of diminishing marginal utility. And let us see how the consumer actually optimizes in this given setup. So, as the consumer consumes 0 unit then of course, there is no loss, but as the consumer started consuming positive units, the positive units will be multiplied by the price of the unit and the marginal utility of money. And of course, you know this lambda 1 lambda p 1 that you see here will be a constant.

So, we are talking about basically a positively sloping upward sloping straight line. And let us draw such a straight line and this can be lambda p 1 x 1. So, you can see the area that the consumer that that the area which gives the consumers net utility is given by this dashed area. And the net utility is maximized when you have a straight line drawn as parallel to this lambda p 1 x 1 straight line and the point of tangency of that line to the utility function. So, that tangency point at the tangency point the gap between this straight line lambda p 1 x 1 which is the loss function. And u of x 1 which is the gain function is highest. And that is why at this point the utility is maximized and we can say that we have obtained the optimized value of x 1 star.

So, this is where we have stopped. No, not really because we have then talked about the first order condition also. And then we know we have derived the first order condition to be marginal utility of commodity 1 divided by the price of commodity 1 and that was equal to the marginal utility of money lambda. Now, up to here we have started. Now, let us think about an extension of the model to do that we have to have some more assumptions. Number 1, we assume different utilities are independent, by that we mean that when the consumer derives utility from consumption of x 1, it does not depend upon the quantity of consumption of commodity 2. We also assume the utility function is additively separable. What do I mean by that? I am going to show you right now.

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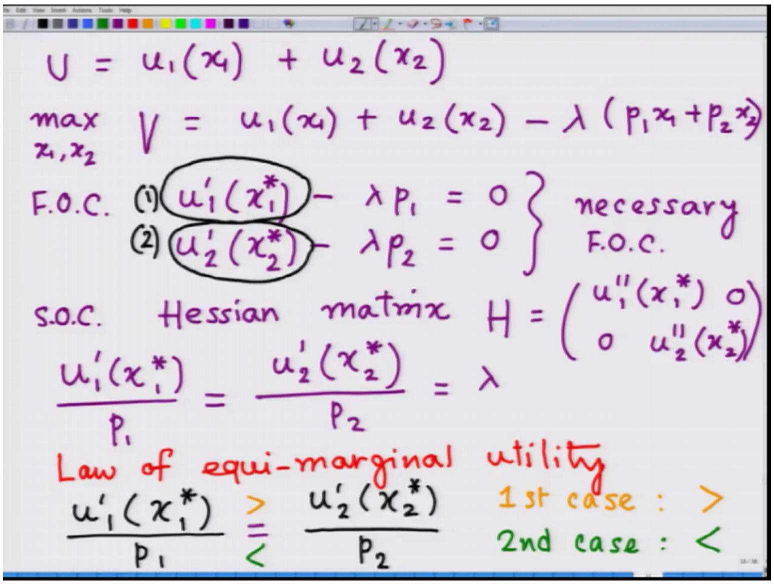

So now, we are going to start with the new utility function U. This has 2 components u of x 1 plus u of x 2. So, you see the utility consumed from consumption good 1 and utility consumed from consumption good 2, they are independent. Now, if we start with that utility function how would the consumers maximization problem would look like? So, we can talk about a net utility function V. Remember we have to extract the loss also. So, here comes the loss. This is the expenditure on community 1, this is the expenditure on commodity 2, and you multiply that with the marginal utility of money.

So, this gives you the loss and of course, the decision variables are x 1 and x 2 in this case. So, in this case you can very well see it is very difficult to draw a graph, as you know we have 2 different variables consumption goods here. So, we will you know try to solve this optimization exercise using calculus, and to start with we have to write down the first order conditions for maximization. So, we have to partially differentiate with respect to the good 1. That will give us 1 first order condition and you know at optimal this is going to be equal to 0. We have to differentiate with respect to the community 2 and at optimum this is going to hold. So, these are basically my necessary first order conditions.

Now, what is the second order condition. For second order condition you remember that we have to focus on Hessian matrix. And we define hessian matrix H in this case as; now let us try to interpret something more out of these 2 first order conditions, because we have 2 goods now. Note we can write from these first order conditions the following stuff. And this is known as law of equi marginal utility.

So, if you have n number of commodities of course, you will we have we have you will be having more components. Now how to interpret this law of equi marginal utility? So, if you have some amount of money, you are thinking to allocate this amount of money between 2 goods x 1 and x 2. How to split the money into consumption of x 1 and x 2 such that your utility is highest possible? The rule is the following. So, if you are spending say 1 rupee on x 1 it should give you the same level of utility if you decide to spend the amount of same amount of money on good 2. So, that is basically the law of equal marginal utility.

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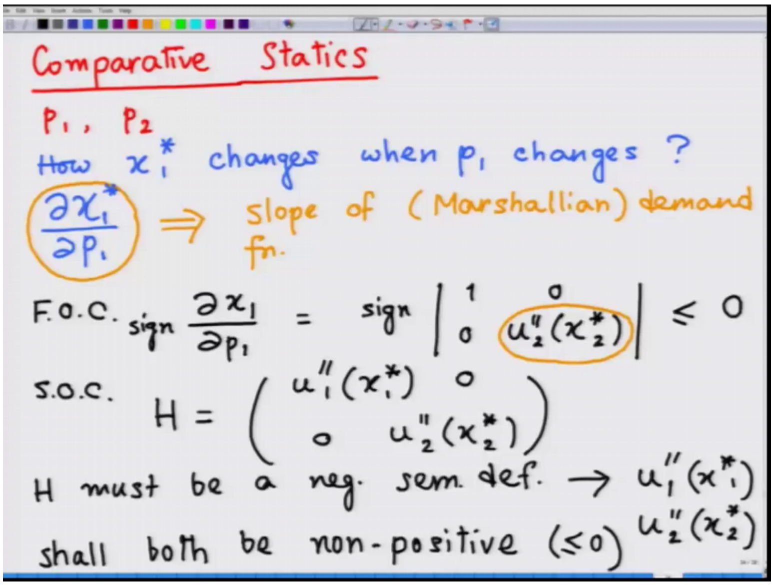

Now, let us look at the following comparative static analysis in this case. In this model the parameters are p 1 and p 2. So, these are the market prices of this consumption goods 1 and 2 respectively. So, of course, as p 1 and p 2 changes, the demand for x 1 x 2 shall change. And this is what we are going to capture through the comparative static analysis. Let us talk about this exercise how x 1 star changes when p 1 changes.

So, in symbolic sense we are looking for; remember, this component or this derivative can you identify this derivative with what we have already studied. This is the slope of what we call a Marshallian demand function. So now, you can realize that one of the beautiful example of usefulness of comparative static analysis is to find out the slope of the demand function. Now, let us start from the first order condition itself.

So, we have 2 equations there. And then the sign of this thing which is important to us can be given by 1 0 0. How we can get this done? You know, you have 2 different equations there, and you have to differentiate these 2 equations with respect to the parameter which is changing. Here it is p 1 and then you know you will get a set of equations. And then from that set of equations you have to apply the Cramer's rule to get you know the expression of you know del x 1 del p 1. And that expression the sine of that expression actually is the sine of this particular determinant, and we can say that this is going to be less than or equal to 0, how come? Because, of an assumption that we made regarding this component.

So, we have assumed the utility function to be concave. And that is due to the law of diminishing marginal utility. Hence as the second order derivative is negative the sine is definitely less than or equal to 0.

Now, let us look at the second order condition of this maximization exercise. So, we have to write down the hessian and that is going to be u double prime 1 x star 1 0 0. So, for the maximization H must be a negative semi definite matrix, right. And that means, u double prime 1 x 1 star and u double prime 2 x 2 star shall both be non positive. It implies less than equal to 0. So, you can see the assumptions that we make in the case of Marshallian utility analysis or Marshallian demand analysis. How that is playing a very critical role to establish, that we have a downward sloping demand function. And we have the maximized utility achieved for a consumer.

Now, let us go back to the first order condition. Say, let us call this to be equation 1 this to be equation 2. Now, this u prime component here that is the marginal utility of commodity 1. And this component here is the margin utility of commodity 2. Now note we got u prime 1 say I denote these to be the ith good know as the generalized case. It can take value 1, 2. We can have greater than and we can also have the case of less than. Let me start with the inequality. So, this is the first expression for first group good. And then we have this expression for good 2. If equality is not made, then what will happen? First case would be the case of this.

So, u prime one divided by p 1 is greater than u prime 2 divided by p 2. So, in that case we can very well see that the marginal, you know utility from commodity one is much higher compared to the utility gained from the commodity 2. So, in that case, the consumption of good 1 should increase and consumption of good 2 shall decrease. If the reverse is assumed and that is basically the second case, then what will happen? Then we can see that as the utility from commodity 2 is higher compared to the utility coming from community 1. In that case, the reallocation should be such that the consumption of x 2 shall increase and consumption of x 1 shall decrease.

Now, this is in a nutshell all about Marshallian consumer demand analysis. But you can probably see that there are some lapses or some limitations in Marshallian consumer demand analysis. There are some we know very strict assumptions. Take the case of assumption that different utilities are independent. So, you know this very assumption of additively separable utility function. That is absurd think about you know the consumption of you know a cup of tea. You have a cup of tea, then you have you know some sugar, you have some milk and you know there is an optimum proportion of tea sugar and milk will give you the highest you know utility.

So, you know it is very difficult to assume that utility coming out of one consumption good does not depend on the level of the consumption of the other good. The another strict you know assumption is the constant marginal utility of money. In reality these assumptions do not take place. That is why you know in the 20th century Marshallian demand analysis was heavily challenged, and we get a new stream of theory known as hicks aligned indifference curve theory. This is the modern approach to consumer behavior in this course we are going to study that and you know we are going to continue with this discussion.