Microeconomics: Theory & Applications
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Lecture – 57
Linear Programming (Part-4)

So, we have found the shadow price or the duel price of the input or resource number 1. Now, let us also say that this value remains constant for a particular range of the input 1's capacity or the availability. So, for that let us revisit the graph to understand what is happening there.

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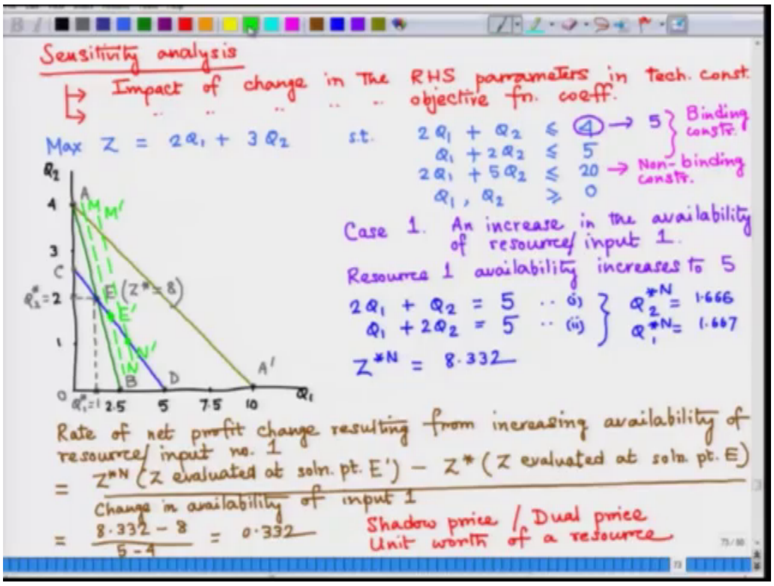

So, here as I have already told you that if my input or resource constraint line AB shifts to MN and then M prime N prime then basically I am relaxing my constraint and there can be another case, where I can also increase the or strain then the resource constraint by reducing the availability of the input. In that case, this line will shift backward parallelly to line M double prime N double primes.

So, basically what I am trying to say here is that that these movements of this initial resource constraint for input or resource number 1 which is given by line AB is basically movements along this input or resource constraint line CD. This is basically the resource constraint line or boundary line for input 2. So, this is basically defining the range of input 1's usage within which the shadow price or the dual price of input 1 will remain constant. So, for that let us have a look at a numerical analysis ok.

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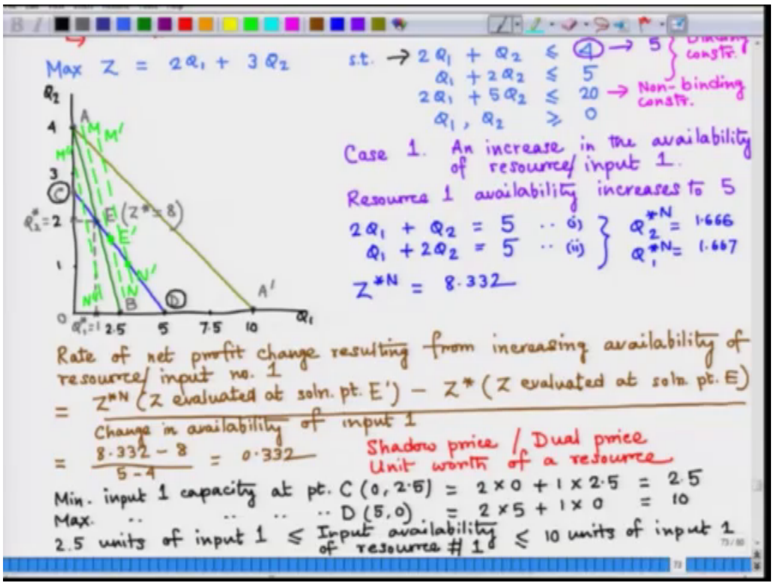

So, one can find out the minimum input 1 capacity at point C which is basically one corner point in the feasible solution space given by coordinates 0 and 2.5 ok. So, now, we have to look at the input constraint 1, use the technology coefficients to write 2 times Q 1's value is 0 here plus 1 times Q 2's value is 2.5 here at this point C. So, finally, we get 2.5 units of input 1's use right.

So, we are done with one extreme point C let us circle it. Now, we will move to the other extreme point of the range in which these movements parallel movements of AB line is permissible. So, for that we can now talk about maximum input 1 capacity at point D So, point D is basically given by the coordinate 5, 0. So, now, again we look at the input constraint 1. So, 2 times the value of Q 1 which is 5 at point D plus 1 times 0 which is the value of Q 2 at point D. So, we are talking about 10 right ok.

So, that means, that we are talking about some range which is given by 2.5 units of input 1 and 10 units of input 1 ok. So, if my input availability is within 2.5 and 10, then basically in that range the shadow price or the dual price will remain constraint and that is equal to 0.332.

Now, let us look at case number 2 which talks about an increase in the availability of resource number input number 2. So, let us also assume that here you here also the availability of resource 2 is increased by only 1 unit. So, that will have some impact on the technical constraints. So, let us write down the technical changed technical constraints in this case.

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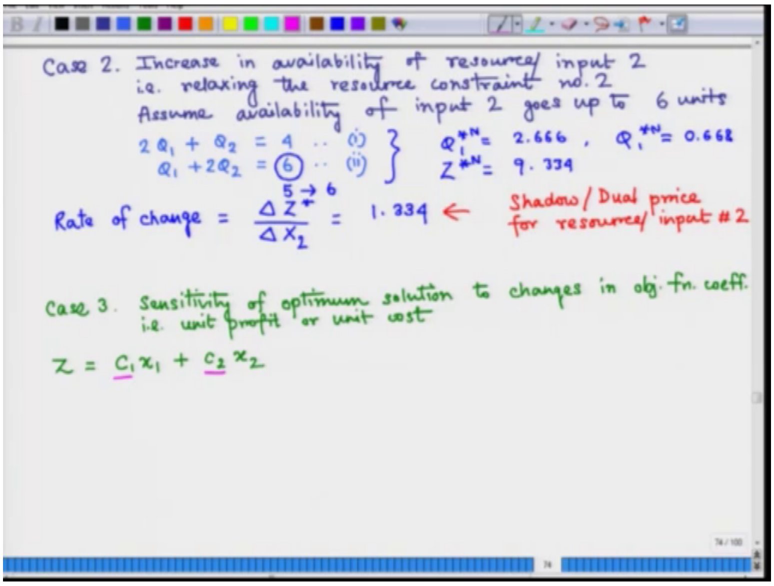

So, here we will have the technical constraints written as note that there is no change in the first technical constraint, but there is a change in the second technical constraint. Of course, I am assuming that slag variable values are 0 here at the optimal solution.

So, note the change here. So, here earlier we had number 5. So, now, it has gone up from 5 to 6. So, let me quickly jot down the solution. I do not want to spend a lot of time to solve this. We have already seen how to solve. So, I can quickly write down the solutions. Let us denote this by N to compare with the old 1 ok. So, now, basically we can talk about that rate of change as change in the Z star value divided by the change in the availability of input 2 and here in this case that is basically equal to 1.334.

So, here this is the shadow or dual price for resource or input number 2 ok. So, let us now move on to the case number 3 of sensitivity analysis and here we are going to study the sensitivity of optimum solution to a linear programming problem to changes in the objective function coefficient. It implies unit profit in the case of the primal problem or unit cost in the dual problem.

So, basically we have to now talk about the; suppose, we have this maximization type objective function where we have this linear functional form, so, here in the sensitivity analysis, we can also change the values of C 1 and C 2 to see what impact will it have on the solution of the LP problem. Now, to study this case in greater detail, let us take help of an illustration that will help us to understand this case in greater detail.

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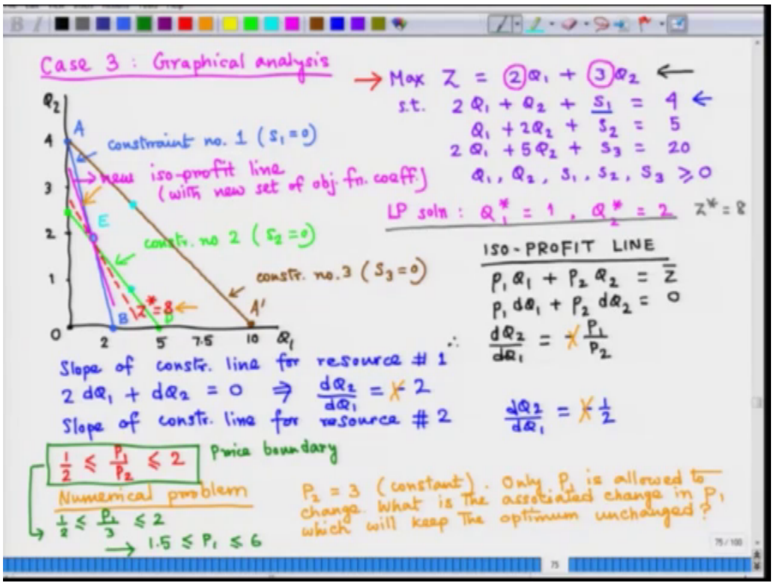

So, here is the old graph that we have used before and here is the old maximization type problem that we have solved before ok. So, here we have already found that this point E which is the intersection point of the constraint for input 1 and 2 or the boundary lines for input 1 and 2, intersect each other and we have found the optimum solution to be obtained at point A and the optimum solution was Q star 1 equal to 1 and Q star 2 equal to 2 right. So, this is basically the LP solution to this problem.

Now, we are going to study the sensitivity analysis. What if I say that my net unit profit is now going to change, ok. So, let us start our analysis with the objective function itself ok. So, we have the objective function which is given here. Now, let me introduce a concept called iso-profit line ok. So, when we talk about iso-profit line, let us write the profit expression in general terms ok so, let me write C 1. So, the way we are writing our iso-profit line expression, we do not assume any specific value of net unit profit P 1 and P 2.

So, how to compute the slope of this iso-profit line? So, now, let us see how this is going to be useful ok. So, let us now go back to the graph. So, you can see that the iso-profit line in this case will actually pass through the intersection point E right and that will give the highest possible iso-profit line that can be obtained by the firm. So, let me assume that we have iso-profit line that represents the original profit expression. So, this one I am talking about. So, although you know I have not calculated the exact slope of this iso-profit line, I just you know approximated it through this red broken line.

So, but note one thing that iso-profit line; so, let me assume this particular iso-profit line takes value Z star equal to 8. If you remember that was the initial solution right ok. So, now, you observe one thing that this iso-profit line can actually tilt little bit around this red broken line and it will still pass through the same point E. So, let me draw one such line. So, I am drawing this pink line which is also an iso-profit line which passes through the point E and note that although the slope of these new iso-profit line; so, this is my new iso-profit line; new iso-profit line means that with new set of objective function coefficients; hence, the slope change.

So, this new iso-profit line although it has a different slope, but still it passes through the same point E and hence, the LP solution does not change. So, we get the same LP solution as we have obtained before. So, which is precisely this and we also get back the same Z star value which is 8 ok, but note that there is a limit to this rotation. So, this rotation of the original iso-profit line which is given by the red broken line is limited by this green constraint line and the blue constraint line. So, these are basically the constraint lines corresponding to input 1 and input 2, right.

So, basically the change in slope is limited and the upper limit and the lower limit of this range is basically dictated by the slope of these 2 binding constraints line. So, to have some comparison between the slopes, let us now work with the absolute values of the slopes right. So, I will now just take the mod value of all this. So, I will ignore this negative sign and that negative sign and negative sign here ok.

So, basically if from the diagram I see that my original iso-profit line which is basically this red broken line and the changed iso-profit line which is basically this pink line has to lie between this input constraint lines, the green line and the blue line; then that means, that this relation is valid. They have to satisfy this condition right ok.

So, basically what does this mean? It means that the P 1, the ratio of P 1 and P 2 can lie between half and 2 such that any small change in P 1 and P 2 which lies within the range will have no impact on the optimal LP solution. It will not change the value of Q 1 star and Q 2 star and finally, Z star ok. So, let us talk about a simple numerical example to understand ok. So, suppose unit profit of a commodity 2 is fixed at its current level right.

So, basically now I declare that my P 2 is 3 and that is not going to change ok. So, only P 1 is going to change. So, a question can be asked that how far P 1 can change? So, we can write ok. So, how to approach these problem? Easy; so, we have to first look at that boundary condition ok. So, we have to use this price boundary condition in our problem. So, here we know for sure that P 2 will be 3. So, let us plug the value of P 2 here.

So, if we do so, then we get this range for P 1. So, if unit profit for commodity 1 lies between 1.5 and 6, then the optimum solution to this LP problem will not change ok. So, I hope that this clarifies the sensitivity analysis case number 3 which is change in objective function coefficient ok. Now, we are going to give another numerical exercise linear programming problem from its primal counterpart pause. [FL]

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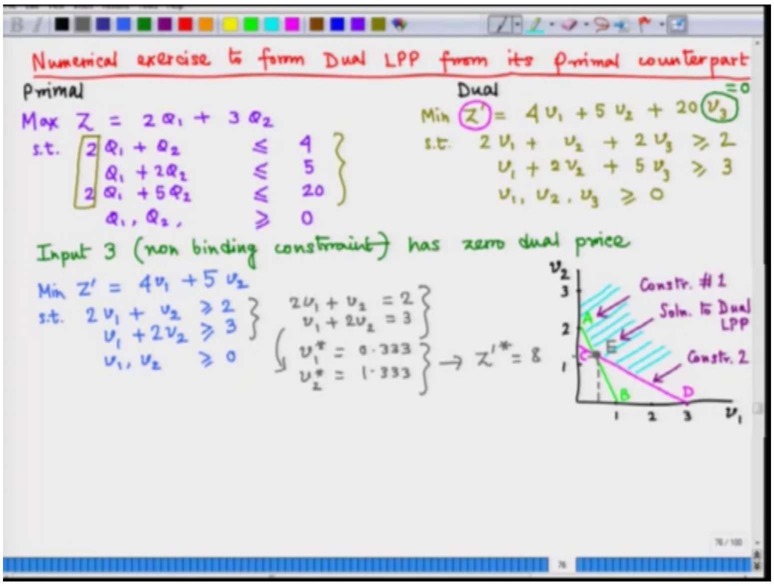

So, let us revisit the same old primal problem that we had, the primal problem of a profit maximization exercise of a farm and now let us construct the dual LP problem from this primal problem. So, this is primal. Now, let us have the dual of this. So, dual will be a minimization exercise and Z prime is basically a new linear objective function which we need to minimize now.

So, remember that the number of constraints in the primal problem will be exactly equal to the number of objective coefficients and variables in the dual problem. So, here we have 3 constraints. So, they are in the dual problem we expect 3 decision variables and 3 objective function coefficients right. So, for each of these inputs, now we have a dual price which will be the decision variable and the parameters in the right hand side of the constraint which are basically 4, 5 and 20, will now become the objective function coefficients right.

So, we write 4 v 1 plus 5 v 2 plus 20 v 3 as the objective function. Now, let us write down the constraints. So, as we know that the column vector that we see here, will now become the row vector right. So, we write 2 v 1 plus v 2 plus 2 v 3 right and this sign of the inequality will change from less than equal to 2 greater than equal to and this will be greater than equal to the objective function coefficient for the first decision variable. So, that is basically 2 here in this case.

Similarly, we can construct the second constraint for the dual and this time it will be v 1 plus 2 v 2 plus 5 v 3 greater than equal to 3 right and of course, needless to say we have to have non negativity constraints for my decision variables which are basically the dual prices. Now, remember that in the previous solution, we found that our input number 3 is somewhat special in the sense that it has a non binding constraint. And if you remember our discussion initial discussion on the dual price, we have said that any input which has a non binding constraint has 0 dual price; right. Why? Because this input is not fully optimized, so, there is some surplus left ok.

So, if that is the case then we know that even before we attempt to solve the dual LP problem, we know a prior the v 3 is going to be equal to 0, right. So, we can now simplify our dual LP problem and rewrite it as and I know I am not writing the ok, well let me write. So, this will complete our dual LP problem. Now, we will see how graphically we can solve this dual LP problem. So, of course, we need to first plot a 2 dimensional solution space, where now the input prices or the dual prices are going to be measured along the axis.

So, let me measure v 1. The shadow price or the dual price for input 1 along the horizontal axis and v 2 along the vertical axis ok. So, now, we need to plot the constraints right. So, here first, so now, let us first plot the boundary line for the constraint number 1. This will be given by this green colored straight line. Let me give it a name 2. So, AB ok. Now, let me also plot the boundary line for constraint number 2 and now let me have this pink line, depicting that let me also name it. Let me name it C and D; let me denote by C D fine.

So, now what will be the feasible region? It is a minimization problem right. So, basically the firm will try to get down to iso-cost line which is as minimum as possible. So, by iso-cost line basically, I indicate to this expression Z prime this linear function because this is basically, this can be given a cost interpretation ok. So, here in this case, the feasible solution space will be given by this shaded area ok. And note that there is an intersection point given by say point E and that basically satisfies both the constraints and this is the point where an iso-cost line could be perceived which will have the minimum possible value ok.

So, let me see at this point E, what would be the values of v 1 and v 2. So, in other words what will be the solution to this dual LP problem, fine? So, let me point out. So, this is basically my constraint number 1 and this is my constraint number 2 and this point E is basically my solution to dual linear programming problem ok. So, basically, what we can do, I mean how to solve them numerically? Well, I have already ignored the third constraint. Now, I have 2 constraints here. Now, I need to use the linear form for these constraints.

So, basically I will have now, 2 v 1 plus v 2 equal to 2 and v 1 plus 2 v 2 equal to 3. So, a system of equations I get and you know that can be easily solved. So, if we solve then we get the values v 1 star equal to 0.333 approximately and v 2 star with has a value 1.333 approximately. So, now, if I plug this v 1 star and v 2 star in my dual objective function, then I get the optimized value of this objective function Z prime star which is 8. So, I get the same solution. So, this is basically the solution to the dual problem.