Microeconomics: Theory & Applications
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Lecture – 56
Linear Programming (Part-3)

Hi, welcome back to the lecture series on Microeconomics. Last time we have started our discussion on a numerical exercise to solve a Primal LPP. So, now, we are going to see how graphical method could be used to find the optimal solution for the Linear Programming problem that we have stated earlier.

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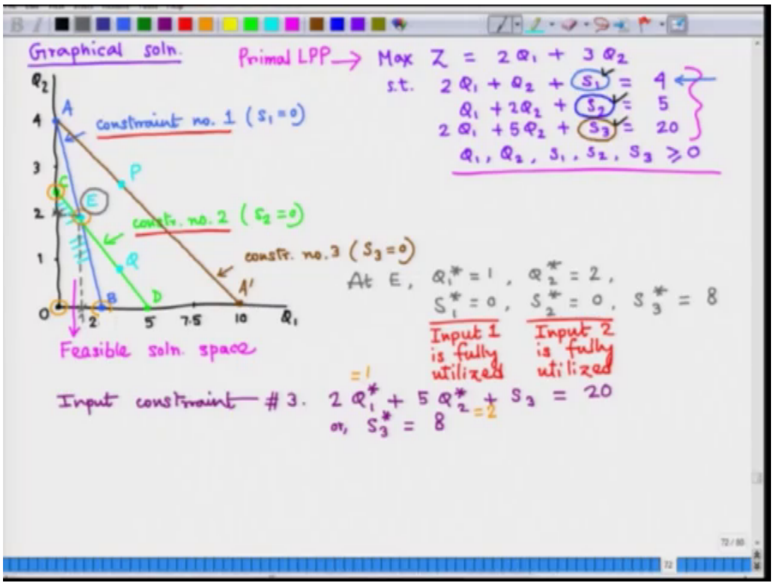

So, let us look at the objective function that is a linear function to start with and here we have the technical constraints in the slack form and here we have the non negativity constraints right. So, in the graphical method, we have to first plot the technical constraints as linear functions or straight lines. So, to do that we have to assume the slack variables take 0 values. So, if we assume so, then let me show you how we can draw 3 constraint lines in this case.

Let us first start with the input constraint number 1 and here we have to assume that the slack variable is one takes 0 value and if that is the case, then we can draw this blue line and this is basically my constraint number 1 right and that assumes S 1 is 0; that means, that the entire amount of input 1 is consumed in production of both the commodities.

So, now let us move on to the second input constraint. So, if we want to plot linear input constraint, then we have to assume the slack variable S 2, value is 0 and if we do so, then we can plot this constraint for input number 2 as this green straight line ok. Let me name this straight line. So, that later we do not get confused.

Let us now move to the constraint number 3. So, in the case of constraint number 3, we have to follow the same approach and we can assume the 0 value for the slack variable S 3. So, that we can plot and if we do so, then we get a straight line which is given by these points A and A prime right. So, that is basically my constraint number 3 ok. So, the theory of linear programming problem says that solution of a linear programming problem actually see situated at one of the corner points of the feasible solution space. So, in this graph let us now mark the feasible solution space.

So, let us now look at point on the A A prime line. So, let me name this point as P. So, here at point P the constraint number 3 is satisfied, but the constraints number 1 and 2 are violated right ok. So, let us now talk about a point on the line CD and let me denote this point by Q. So, at point Q, you see that the constraint number 3 is made constraint number 2 is also made, but constraint number 1 is violated. So, basically what do we see?

We see that there is an area in this solution space which satisfies all 3 constraints right and that is given by this shaded area and there is an intersection point here, ok. So, let me name that intersection point as E ok. So, basically this shaded area is known as feasible solution space. So, basically what does the LP theory says? LP theory says that if I now denote my origin as say O, so, there are 4 corner points right here.

So, one is basically my origin this point, then this is another corner point of the feasible solution space, the point B and then this intersection point of input constraints lines for one input, 1 and input 2 and finally, this point C. So, these are basically the corner points of the feasible solution space. So, the LP theory says that the final solution will be one of these four.

So, now I am going to show you the value of the objective function evaluated at this intersection point E and I am going to show that, we will get back that value 8 that we have derived earlier by using algebra, but now I leave it up to you to find the objective function value evaluated at the other corner points. If you evaluate the objective function at other 3 corner points, then you will see that the value of the objective function is less than what is obtained at the intersection point E. So, the intersection point or the corner point E gives the solution to this profit maximizing LPP.

So, now let us look at the graph again to see the optimal point which gives us the profit maximizing solution to this LP problem. So, here you see this point E is basically giving me the optimum solution to this profit maximizing primal LPP. Now, let us evaluate the objective function Z at this point E right. So, basically we see that here at the intersection point E, we get Q 1 star equal to 1 and Q 2 star equal to 2 and then we have; So, what about the slack variables? So, these are the optimized values of the slack variables ok.

So, now let us wait for some time. I will come back to this a star 3 value, but now let us first look at the case this and this. So, what do they mean basically? So, an optimized value of the slack variable if it becomes 0, then that means, that particular resource constraint is satisfied with equality. So, that means, my input 1 in this case is fully utilized. Well, how do I know?

I know it for sure because the point E lies on the constraint number 1 which is given by the blue line. Now, let us move to the A star 2 value which is also equal to 0. So, this signifies my input 2 is also fully utilized that is why the slack variable takes 0 value. Well, how do I know? I know it for sure because the point E also lies on the constraint line for input 2 which is given by the green line, CD right.

So, now let us look at the interesting case of input 3. now, here at the point E, actually lies within the area which is covered by the axis and the boundary line for input 3 which is given by this brown line A A prime. So, what does that mean? So, the point E when it is lying inside the area below the input boundary line for input 3; that means, that my input 3 is not fully utilized. So, that means, that there will be some value for the slack variable some positive value for the slack variable. How to obtain? Let us see that.

So, now we will move to the input constraint number 3. I am calling this input constraint, you can also call it resource constraint does not matter ok. So, now, you plug these values of optimized the optimal values of Q 1 star plus Q 2 star. So, basically or we can write S 3 equal to 8. So, the final solution will have a positive slack variable value for the input constraint 3 ok. So, we have seen how to solve LPP primal type profit maximization problem through graphs.

So, now we are going to look at this interesting and practical concept of sensitivity analysis and see its usage in practical managerial decision making problems. So, what do we mean by sensitivity analysis? By sensitivity analysis we mean, if there is a change in the parameter values in LPP, then how it is going to impact the objective function value. So, here we have assumed several values as parameters in solving the LPP right. But how we can be so sure about these values?

So, if there is some small change in the parameter value, it may or may not have an impact on the objective function. So, sensitivity analysis actually helps us to study that fact.

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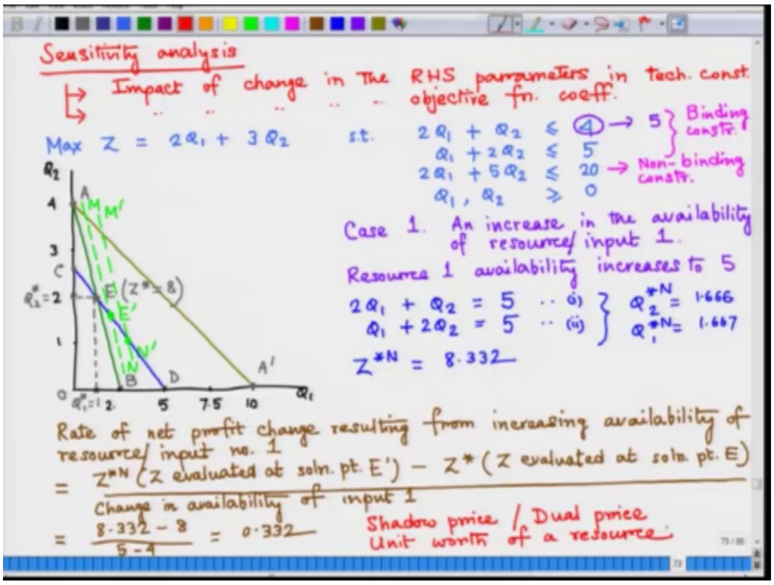

Now, sensitivity analysis could be of 2 types. One is studying the impact of change in the RHS parameters, in technical constraints and the second one is basically impact of change in the objective function coefficient. First, we are going to look at a case through numerical exercise again to understand the first case of sensitivity analysis which is through a small change in the parameter value in the RHS of the technical constraint and that is basically talking about a small change in the input availability.

So, now in this context of LP solutions, I would like to introduce the concepts of binding and nonbinding constraints. So, you have probably noticed that these constraint number 3 which is basically for input 3 or resource 3 is not intersecting with the other constraint lines, it is just you know intersecting with the input constraint line of input 1 and that is also along the Q 2 axis right.

So, if there is a small change in the parameter value for the input constraint number 3 that will have no impact on the optimal solution of this particular LP problem which is obtained at the point E which is intersection point between the input constraint lines for input 1 and 2. But that is not the case with the input 1 and input 2. If there is a small change in the right hand side value of these constraints, then that will basically lead to a different intersection point and of course, that will have an impact on the optimal solution of the problem the Q 1 star, Q 2 star values and finally, Z star value.

So, basically the letter constraints where a small change has a direct impact on the optimal solution of the linear programming problem is called a binding constraint. But the first constraint that we have discussed the constraint for input or resource number 3, that is basically a nonbinding constraint because a small change in that constraint we will not have any impact on the solution of the linear programming problem ok.

Let us now move to the same numerical exercise and the graph that we have seen before and let us now conduct some sensitivity analysis ok. So, now, we are going to study 2 different cases. So, first we are going to talk about a change which is basically an increase in the availability of the source or input 1 ok. So, what does that mean? That means that now I am relaxing the linear constraint which was there for my resource or input number 1.

So, now let us assume that input 1 or resource 1 availability increases to 5 units ok. So, earlier we had 4 units. Now, this is increasing up to 5 ok. Let us see how my linear programming problem is going to react to that, then rewrite my technical constraints. So, we will assume that the slack variables will take 0 value at the optimum, ok. I also assume that this nonbinding constraint plays no role. So, ok.

So, now I am going to assume that nonbinding constraint does not play a role in my LP solution. So, I can safely ignore that and I assume that at the point E, the optimum point at the optimal solution the slack variables take 0 values S 1 star and S 2 star are basically 0's. So, then in that case we can rewrite the first 2 technical constraints as; so, there is a change now because the RHS parameter value has changed from 4 to 5, but there is no change in the second resource constraint ok. So, earlier the RHS parameter value was 5, now also it is 5 ok.

So, now let me give these equations some number ok. So, now, I can plug equation number 2 in equation number 1 to in order to solve for Q 1 and Q 2 and I am skipping some steps, but you can check that the solution will be given by these values of Q 1 star and Q 2 star ok.

So, now, if that is the case, we can plug these Q 1 star and Q 2 star values in the objective function and we obtain a new Z star value right. So, let me call this Z star new ok, let me also write new here, so that you can differentiate this from the earlier solution. So, the Z star n value will be 8.332 ok. So, now let us see what is happening through the graph ok.

So, as you relax the constraint then basically what happens for input 1? There is a parallel upward shift in the line a B right. So, I am talking about us parallel shift like this. So, we can call this something like M n and now note that there will be a new intersection point here, let me call this E prime and of course, you can see that the Q 2 star value will be little less than 2 and Q 1 star value will be little less than 1 and this is what we have obtained.

So, gradually we can move from 5 to 6 and 6 to 7 and then we can see that there will be series of parallel shifts like this and as the constraint for input 1 is a binding constraint, it will have an impact on the objective function value right and now we know let us study the sensitivity analysis. So, as we have seen there is a change in the right hand side parameter value in the technical constraint for input or resource number 1. There is a small change in the objective function value as well.

So, now we can express these change in the objective function value in terms of concept, which is a concept of rate change which we have already seen and let us represent that. So, in this case one can write rate of need profit change resulting from increasing availability of resource or input number 1 and that is basically Z star n which is basically Z evaluated solution point E prime minus Z star, which is basically Z the linear function evaluated at the earlier solution point E right and that is divided by the change in availability of input 1 ok. So, here I get 8.332 minus 8 divided by 5 minus 4 and that is basically 0.332 ok.

So, we have evaluated or enumerated the rate of change in the objective function due to a change in the right hand side parameter value of the technical constraint. Now, this has a theoretical name and the theoretical name is shadow price or dual price of the input or the resource it is basically the unit worth of a resource.

So, why it is called the shadow price or the unit worth of resource? It is simple. So, if the firm gets lucky to use one more unit of the input 1 or source 1 in this case, the net profit of the firm increases by 0.332 rupees or dollar whatever you know monetary unit we are measuring net profit in net profit. And so, that means, that that the actual value or the true value of 1 unit of input 1 or resource 1 is basically this amount of money. Now, that is why it is called the true worth of or unit worth of the resource.

But why it is called the shadow price or dual price? Because we will look at the duality problem later on for the same primal problem, you will see that these are the prices which are exactly solved in the dual linear programming problem. This is called shadow price of the input because actually the input is bought and sold in the market at some price if we assume that then this price is not that particular price. So, the price that here we have derived is basically from the mathematical exercise. So, this is known as the shadow price because you know there is no link between the market price of this particular input if it exists at all.

We started the discussion saying that there can be many inputs in short run whose availability are fixed and you know some inputs can be of that sort where the price is not readily available in the market. So, we can actually you know use the linear programming problem, the dual problem to you know get some proxy input prices for those type of inputs.

So, here you know that is why this is known as the shadow price. So, it does not say that if the firm visits input market to purchase 1 unit of this resource 1, then the firm actually is paying 0.332 rupees or dollar per unit of the input. It is saying that if it employs 1 unit of resource 1 or input 1, then its net profit is going to go up by 0.332 dollars or rupees. So, this is the maximum amount that the firm will be willing to pay to purchase 1 unit of the resource. So, we are going to continue our discussion on sensitivity analysis in the linear programming problem in the next lecture as well.