Microeconomics: Theory & Applications
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Lecture – 29
Cobb-Douglas Production Function

So, we have already discussed the concepts of production function, isoquant, slope of the isoquant etcetera. Now, with these concepts already defined, let us look at some elasticity measures and this elasticities are very critical concepts in the theory of firm. First, we are going to study the concept of elasticity of substitution. Now, for that let us look at a diagram first.

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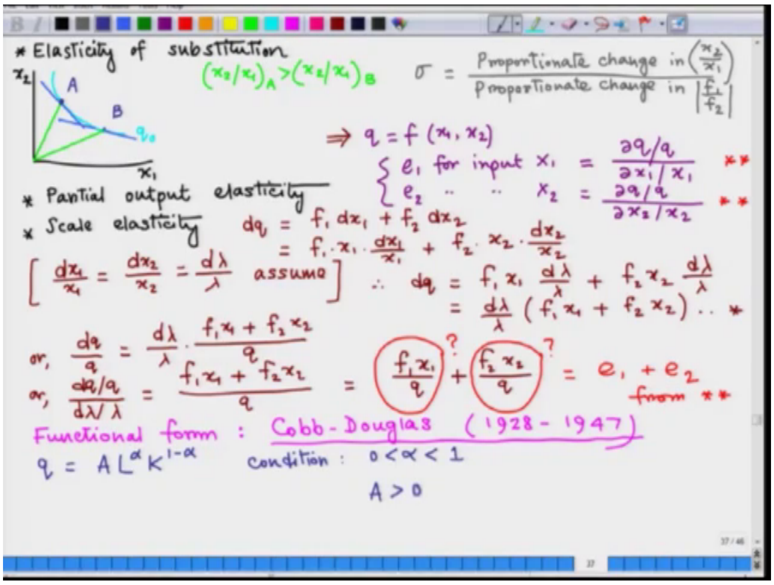

So, let us consider one isoquant displaying quantity output level q naught and let us take two points on it say point A and point B. Now note that, as we move down along the isoquant from A to point B, not only the slope gets changed like this, the absolute value of the slope actually falls right, but it is not only that but note that input ratio is also changing.

So, if you join origin to this input bundle A the input ratio will be x 2 over x 1 A say and if you join origin with point B or input bundle B then we get another input ratio x 2 over x 1 at point B and we can see that, this input ratio of two inputs is higher at input bonded A than at B. So, elasticity of substitution is defined as sigma; and so sigma is going to be proportionate change in x 1 over x 2 or x 2 over x 1 whatever divided by proportionate change in the slope of the isoquant which is given by f 1 over f 2 right, the ratio of marginal products right.

And we know that we are talking about MRTS. So, it has to be the absolute value of the slope of the isoquant right. The next elasticity concept that we are going to study is partial output elasticity. So, if we assume there are say multiple inputs then if we bring in change to one particular input that will lead to some change in the output. If we keep all other inputs fixed then, we can express these percentage change in output due to one percent change in one single input in terms of partial output elasticity. Let us express that mathematically, through a formula.

So, suppose we take one production function defined over only two inputs, but definitely it can be done on more. So, in that case there will be two partial output elasticityies, one for the input 1 and that would be defined as del q divided by q del x 1 x 1 right; and of course, needless to say, that q will be basically this production function right. So, similarly we can have for input X 2 and then it will be third we are going to look at one interesting result, we are going to derive scale elasticity and we are going to show that the scale elasticity is sum of the partial output elasticityies. Again we are going to work with a two variable input production function, but definitely this result can be generalized to n number of inputs.

So, let us start with that expression or production function. So, we can totally differentiate to get first partial with respect to input 1 times change in input 1 then, this is the marginal product of input 2 times the change in input 2 and we can rewrite this as, f 1 times x 1 times dx 1 by x 1 plus f 2 times x 2 times dx 2 divided by x 2 right. So, now, what do we mean by scale change? By scale change we mean all inputs are changed in the same proportion.

So, if that is the case then we can write suppose, lambda is the rate of change in inputs. So, we can assume this right. So, if we assume then we can express dq accordingly, f 1 x 1 times d lambda over lambda plus f 2 x 2 d lambda over lambda. We are basically replacing the proportionate change in input through this expression d lambda over lambda, which is the constant proportional change.

So, definitely we can now write or we can write dq divided q equals d lambda divided by lambda times f 1 x 1 plus f 2 x 2 divided by q. We are just dividing both sides of this expression denoted by star by q. And then we can write dq q d lambda lambda and that will be what that will be right simple. Now, we can again write this as f 1 x 1 divided by q plus f 2 x 2 divided by q right.

Now, note at these expressions what are these expressions. So, these are basically my partial output elasticities. So, I can now, write this as e 1 plus e 2 as I have derive these things earlier say in star star. So, scale elasticity can be directly computed from the production function iself. So, far we have discussed production function isoquant and this elasticity formulas through a general functional form we have not assumed a specific functional form an equation.

So, now we are going to look at one particular functional form which is very popular in applied economic research and also in theoretical economic research. The functional form is popularly known as Cobb-Douglas production function and it is named after economist Paul Douglas and statistician Charles Cobb. Now, we have to go back to 1920's where Cobb and Douglas shaked hand and then they embarked on this journey to find out a functional for production function. Now, they are not the first one's who tried to find out a functional form for a production function.

Actually, there were some European economists who were also trying to find out functional forms and they have been successful as well, one can mention the attempts made by Professor Newt Rixel much before, Cobb and Douglas to find out a functional form and it was quite interesting to note that almost 30 years before Cobb and Douglas found out this functional form nue tweak cell actually had discovered it. But, when Cobb and Douglas embarked their journey to find out a functional form they did it again independently.

So, now what did Cobb and Douglas? Cobb and Douglas do they gathered data for almost 30 years on American manufacturing industry and later on the macroeconomic variables of aggregate output and aggregate input and then they used least squares regression method to find out a functional form and estimate the parameters of that functional form. So, now, let us look at the functional form that Cobb and Douglas prescribed, based on the statistical evidence that they have obtained from American data.

It is important to note that, it took almost 20 years to establish the simple functional form that they have proposed in the academia. So, now, we are going to look at a very simple functional form or a mathematical equation for production function which Cobb and Douglas proposed this was very simple, but you know it was initially criticized by scholars a lot. They successfully defended their functional form in a span of 2 3 decades of research and later on Nobel laureate economists like Paul Samuelson and Robert Solow have adopted this functional form in their research to make it widely popular.

Let us now, focus on the shape of the Cobb Douglas production function in it is initial form. So, it was prescribed as. So, here q is of course, output L and K are basically the variable inputs labor and capital respectively and alpha and 1 minus alpha are basically two exponents right. And there is a condition that Cobb and Douglas made when they proposed this functional form that is alpha lies between 0 and 1. Now, we are going to look at the properties of a Cobb-Douglas Production Function. We are going to look at 4 or 5 of them one by one.

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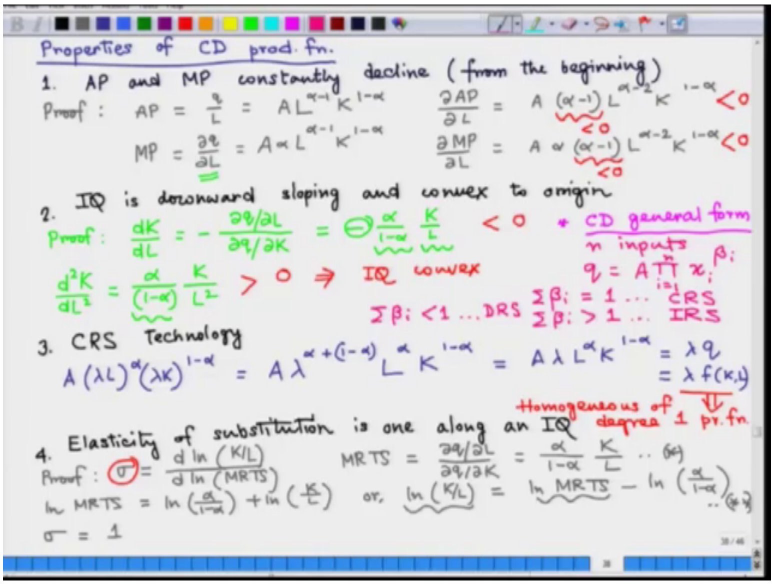

I am abbreviating Cobb-Douglas by CD. So, the first property is going to be on the concepts that we have already seen and this concepts are average product and marginal product. So, for a Cobb Douglas production function, average product and marginal product constantly decline; that means, that the decline from the beginning ok. Note that here, when I am going to derive the average product and marginal product I am going to get these derivations done with respect to the labour input, but it can also be done for the capital input without losing any generality.

So, now, let us look at the proof of this property. So, in the case of Cobb Douglas average product is going to be defined as q divided by L and then it is going to be A times L to the power alpha minus 1 and K to the power 1 minus alpha right. So, next we need to compute the slope derive the slope. So, we are going to write A alpha minus 1 L to the power alpha minus 2 and then K 1 minus alpha. Now, can we comment on the sign of these derivative? Yes we can. Because note at this entity. This is certainly negative because alpha is going to lie between 0 and 1 by assumption. So, overall we can assume that del AP del L is negative so; that means, the slope is negative from the outset.

Now, let us look at the case of the marginal product then. So, in this case marginal product would be del q divided by del L and this is going to be A times alpha L to the power alpha minus 1 and K to the power 1 minus alpha. So, the next would be to derive the slope of the marginal product. And we get A alpha alpha minus 1 L to the power alpha minus 2 and K 1 minus alpha right. So, again we observe this entity here, which is negative raised elements are positive. So, safely we can assign negative sign to this derivative showing that marginal product curve also slopes downward from the beginning.

So, the next property would be on the curvature of the isoquants. So, isoquant is downward sloping and convex to origin. So, now, let us look at the proof of that. So, we have already derived dQ dL right. So, to complete the expression for the slope we need to also compute or derive del Q del K which I leave it to you as an exercise now if, I assume that you have done that then the slope can be given as del Q del L divided by del Q del C del K ok. Now, this is positive this is also positive. So, we have a negative sign in front so, no wonder we can assign negative sign for the slope.

Now, let us look at the convexity. So, the for curvature we know that we have to look at the second order derivative. So, d square K dL square and this would be I am giving you the final expression to save time right. Now, by assumption KL are positive and alpha lies between 0 and 1. So, this is also positive. So, we can safely assume these to be positive. So, that means, that isoquant is convex.

Now, we move on to the third property and this is very interesting and important property. So, here we are going to assume or here we are going to find the scale of the technology displayed by Cobb-Douglas production function. And here goes the result a Cobb Douglas production function in it is original form displays constant returns to scale technology; now, we are going to prove that. So, to be consistent with our definition with scale elasticity let us now multiply the inputs L and K by a scalar lambda. So, if we do that then we get what? And we get A times lambda to the power alpha and then of course, there will be 1 minus alpha I am skipping steps here L to the power alpha and K to the power 1 minus alpha.

So, we get A lambda L to the power alpha K to the power one minus alpha. So, that is what that is basically lambda times my q now original production function that I have assumed right. So, this is basically, lambda times function of K and L. So, we see that if there is a 1 percent change in the inputs or if there is x percent change in both the inputs that is exactly x percent change in the output as well. Hence, we can say that the technology is constant returns to scale. Now, as this is CRS we can also infer that the original formulation of Cobb-Douglas production function actually, is a homogeneous of degree 1 production function.

Now, let us move to the elasticities. And here goes the property elasticity of substitution is one along an isoquant of Cobb-Douglas production function. So, now, we are going to look at the proof for it ok. So, we are going to now start with the formula first right. So, let us write like this now, MRTS is basically what? The absolute value of the slope of the isoquant and we have already derived these partial derivatives now I leave it to you to figure out that the final expression for these would be this.

Now, as the formula for sigma involves logarithm let us take natural logarithm of this expression denoted by this star. So, we can write after taking natural logarithm and we can rewrite. So, now, we can differentiate this expression given by double star. So, we can now differentiate this with respect to this and we can get sigma equal to 1 ok.

So, we have completed with four properties of Cobb-Douglas production function. Now, as a digression let us quickly look at the interpretation of the exponents of the Cobb-Douglas production function. Now, there are two exponents alpha and 1 minus alpha, what are there? So, it can be shown that alpha is partial output elasticity of the labor input and 1 minus alpha is partial output elasticity of the capital input. To find an interpretation let us go back to the statistical study done by Cobb and Douglas back in 1930's.

So, they have analyzed data to find out the value of alpha to be almost equal to 0.75. So, what does that mean? So, if there is a 1 percent change in labour use then output is going to change by 0.75 percentage given the capital input remains same. So, the next thing, we are going to discuss is the general form of a Cobb-Douglas production function. Do not be under impression that Cobb-Douglas production functions always show constant return to scale. Later on economists have generalized this functional form to accommodate other returns to scale as well and it has been extended to multiple inputs also

So, now let us look at a modern form of a Cobb-Douglas production function it can also be called a general form of Cobb-Douglas production function. So, here we are going to assume an input production function and they will be x 1, x 2 to x n and let us write the functional form as A is basically, the technology parameter or efficiency parameter as we have defined it before. So, now here in this case we do not assume whether beta will strictly lie between 0 or 1, it can take any value.

So, basically in this type of production function if, summation beta i the sum of the partial output elasticitys basically. If it shows 1 then we say that, we have a CRS or traditional Cobb-Douglas. If summation of b i takes value greater than 1 we say that, we have increasing returns to scale and of course, if it is if the sum is less than 1 then, we say that we have a decreasing returns to scale production function.

We will continue with this discussion in the next lecture.