Total Quality Management - II
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Lecture – 15
Factorial Design – I

Welcome back my dear friends a very Good morning, Good afternoon, Good evening to all of you who are taking this TQM II course under the NPTEL MOOC. And I am Raghunandan Sengupta from the IME department IIT Kanpur. So, this is the fifteenth lecture for the TQM 2 lecture series of the course. And as you know that each week we have 5 classes and after each week we have one assignment. So, we are at the last lecture for the third week which is the 15th one.

Now if you remember that we were doing after going to details about the ANOVA concept that how you would use the concept of analysis of variance try to find out the sum of the squares for the error, sum of the squares for the treatment, I am using the word treatment in a very general sense it would basically be the experiment we are doing.

And then you will try to find out the total sum of the errors how we will try to differentiate using the concept of design experiments and, then you will consider the concept of hypothesis thing and if you remember we did go into details about the hypothesis testing concepts in the first 7 lectures, I would say almost till the end of the 7th lecture after which we started the concept of design of experiments, in a very preliminary way then we also considered under the ANOVA. So, called the models if the means are same if the tau values which is the difference in the average of the averages with respect to the treatment means, again I am using the word treatment for you for that example, if you remember the etching depending on the overall wattages or the energy levels of 160 180 220.

We also saw that how the degrees of freedom would change a minus 1, depending when you are considering inter or the intra level of sum of the squares. We also consider a very important concept was where the variances of the samples of the populations for each etching was same. And that means, variance was not changing not dependent on time that was a very important concept, I will be repeating it time and again please bear with me, then we also considered that if you are trying to compare. So, the compares would be for the balance model non balance model and the random factor models so and so, forth. And we took the example of the etching problem time and again to highlight all the different facets and different ways of trying to tackle the problem.

Then we went to into though not in that depth like the etching problem into the aluminum smelting problem and, also showed the different concepts of the models. And then we came into the concept of the of the factor models; factor models, if you remember in the fourteenth lecture which was the second last lecture of the third week, we considered that if there were two levels of inference, one was a plus A minus and with respect to that, you had another factor which was B plus me minus B plus B minus being the levels and, how those the relationship between the factors could be either increasing in a linear way, or they would be decreasing; that means, this inverse proportions and drew diagrams, or graphs to show that how they can be at least justified in a very nice manner. And though the relevance of the quantile paths was more discussed in the ANOVA models, but it will be utilized in the factor models also later on.

So, why this was important because if you remember the 4 distributions we considered the z distribution the t distribution which were only related to considering something with 2 with the mean, and z was being used when this all the informations about the standard deviation of the population was known T distribution with a certain degrees of freedom, which I did mention time and again, would be used for the case when the standard deviation the population was not known. Then we if you go into the second moment of the variances comparison, we took the chi square with change the degrees of freedom depending on what the problem formulation was then we went to the f distribution. Then we also consider the Bartlett test the Turkey’s capital T statistic and considered those examples.

Again for the same to detail example about the etching part and the aluminum smelting part. So, to again come back to the factorial models, we will continue discussing those and as I said that as you do the assignment small problems these things of the concept would become much more clearer to you and for all the queries, we as a group in from NPTEL the instructor which is me and my T as would be there to answer all your queries to the best of our knowledge and our ability.

And therefore, for any other queries; obviously, I would request time and again please read the book, I know it is a little bit more quantitative a little bit on the higher side, but please trust me that is one of the best books which you could have followed both in the TQM 1 the (Refer Time: 05:49) which was basically related to statistical quality control and, then this one basically being from the design of experience area. So, design of experience book is also there by Montgomery, but you can basically go through the initial framework, or the ideas from the statistical quality control and then switch over to the higher level book which is the design of experiments.

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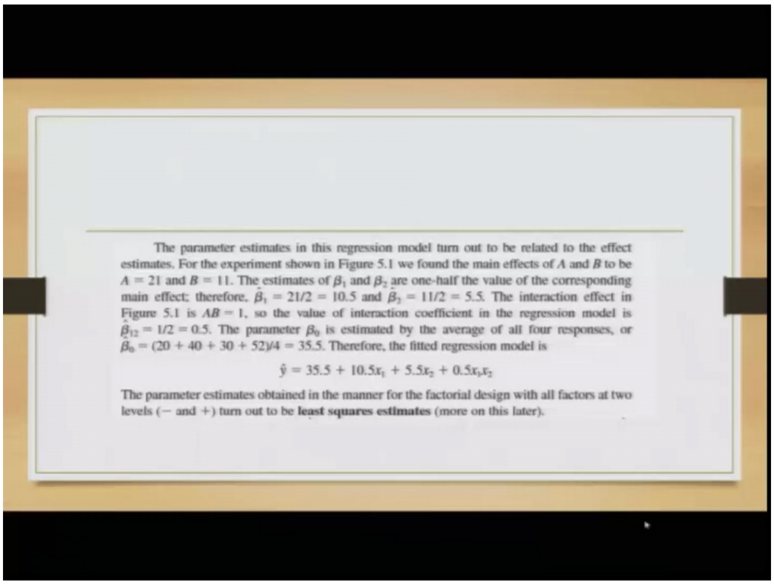
The parameter estimates in this regression model turn out to be related to the effect estimates. For the experiment shown in Figure 5.1 we found the main effects of A and B to be $A = 21$ and $B = 11$. The estimates of β_1 and β_2 are one-half the value of the corresponding main effect; therefore, $\hat{\beta}_1 = 21/2 = 10.5$ and $\hat{\beta}_2 = 11/2 = 5.5$. The interaction effect in Figure 5.1 is $AB = 1$, so the value of interaction coefficient in the regression model is $\hat{\beta}_{12} = 1/2 = 0.5$. The parameter β_0 is estimated by the average of all four responses, or $\hat{\beta}_0 = (20 + 40 + 30 + 52)/4 = 35.5$. Therefore, the fitted regression model is

$$\hat{y} = 35.5 + 10.5x_1 + 5.5x_2 + 0.5x_1x_2$$

The parameter estimates obtained in the manner for the factorial design with all factors at two levels (− and +) turn out to be **least squares estimates** (more on this later).

Now, continuing the discussion which we ended in the 14th lecture. So, the parameters we want to again find out. So, so if you go back. So, there was a basically a very simple linear model on the linear model would be f x would be coming on a standard basis from the factors and they would be so, called interrelationship of the factors. So, say for example, the factors as x 1 and x 2. So, the overall effect would be coming on a standalone basis from x 1 on a standalone basis from x 2 as well as the combination of x 1 and x 2 and you can formulate models accordingly. If they are 3 so; obviously, it will be x 1 x 2 x 3 and the combinations of 2 taken at a time for the 3 and then and the last one would be the combination of all the three taken at 1 go.

So, the parameter estimates in this regression model turn out to be related to the effect estimates. So, what are the effects and how you can find the estimates. For the experiment shown in the previous lecture which is the 14nth one, we found and the main effects of A and B, if you remember A and B are the factors. So, we mentioned them as a plus on the higher level of factor A minus and B plus B minus corresponding higher and lower level for the second factor. So, so continuing the reading as written in the slide we found the main effects of A and B to be A being 2 1, if you remember that was the so, called average effects and for B was 11.

Now, the estimates of beta 1 and beta 2; so if you remember beta 1 was the so called rate of change of consider in a very simple way the rate of change of the dependent factor, or dependent effect being dictated by 1 unit change of x 1 and beta 2 would be they basically the rate of change of the dependent factor per rate of change of x 2 which is the second factor and you can find out the factors rate correspondingly.

So, the estimates of beta 1 and beta 2 are 1 half of the value of the corresponding main effects so; obviously, you will try to find on the average. So, now, rather than beta 1 and beta 2 they would be mentioned as beta 1 hat and beta 2 hat corresponding to the estimates which you find out from the sample, which you will think is the best estimate of the best statistic which you are trying to utilize from the sample to find out the parameter values in the population. So, beta 1 hat comes out to be 10.5 or 10 and half and beta 2 comes out to be 5 and half. So, the interaction effect which is given in the figure 5.1. So, the figures which I referred would you one can basically simply refer to the book which I have been referring time and again.

So, a in the effect of A and B comes out to be 1. So, the value of the interaction of the coefficient the regression model, considering the joint effect comes out to be 0.5. So, the parameters of betas naught beta naught is basically the so, called intercept, if we consider the concept of very simple linear line which we have done in very simple algebra, very simple trigonometry, or very simple mathematics in class 10. So, we have y is equal to m x plus c. So, m is the rate of change of y with effect to 1 unit change in x and c was basically the intercept. So, that beta naught value is the intercept which is calculated as; obviously, it will beta hat. So I am, I may be mentioning beta hat, or may not be mentioning beta hat, but the crux of the discussion is that it is the best estimate which you are getting from the sample.

So, beta hat I am not is in the suffix comes out to be 35.5. Now, once this is solved when you write out the model this is what we have I will try to highlight it and again come back to whatever I said about few minutes back. So, this is the effect which I have so, it is not highlighting. So, I will just mark it here. So, this y hat now you will be thinking why it is y hat. Now, when you are trying to basically mention a model so, the model would basically have a part which is so, called deterministic part plus the error term. Now, if you remember the error term which we did consider in the ANOVA model and we will be considering time and angle in different type of models, we will consider it to have a mean value of 0 and a variance to be either one or sigma square, but the fact still remains that the sigma square value which is the error variance remains constant, such that the effect of the errors are not there to be consistent point, I will keep repeating it time and again.

So, therefore, the fitted regression model comes out to be now on the left hand side of the equality sine is y hat. So, you may be thinking why it is y hat because, y hat is basically the predicted value. So, that will be equal to beta naught which is beta naught hat is 35.5 plus 10.5 which is beta 1 hat into x 1. So, you plug in the value of x 1 you will get the second term that is 10.5 multiplied by x 1 value at that point of time where you want to find out the y hat. The beta 2 value would be 5.5 into x 2 hat and the 3rd term would basically be beta suffix 1 and 2, which is the joint effect multiplied by x 1 and x 2. So, again it is beta 1 2 suffix hat plus in the actual model there was an error. So, this error terms once you find out would basically be the difference of y actually at that point of time, minus the y hat value which you have found out.

So, if you basically have if you basically take a round amount way of trying to analyze the error terms. So, for each and every value different values of x 1 x 2 and x 1 x 1 and n 2 which is the suffix value so, all these thing combinations which you take for the first reading second reading, third reading, fourth trading so, and so forth you will have different values of y hats. So, it will be y hat suffix 1 y hat suffix 2 y have suffix 3, if we find out the differences of the corresponding y 1 minus y 1 hat you will have the error term 1, which is the first value of the error. If you find out the difference of y suffix 2 minus y suffix 2 hat, you will basically have the error 2 and so, on and so forth.

So, if you if you want to double check the actual assumption which we have mentioned time and again that the error term has an average value of 0. So, adding up all the error terms in the long term and, the law and for many readings would definitely come out to be 0 because the average value is 0. So, hence the value of error is not there because, you are trying to find out the estimated value the parameter estimate is obtained in the manner for the factorial design with all factors are two levels and, they turn out to be least square estimates in the sense that we are trying to find out conceptually; let me I am not going to digress, but I am going to go into a little bit different mode and, trying to give you a philosophical discussion is that when you find out the errors.

So, technically what you want to achieve let us step back and think, you want to achieve the errors such that the variance of the errors are minimized. So, if the variance of the errors are to be minimized which means technically that you are trying to basically find out the variances of the difference between the actual value and estimated value as low as possible so; obviously, cannot be made 0 it case to be as low as possible corresponding to the fact, that you find out the best estimates of beta naught hat beta 1 hat, beta 2 hat and beta 1 2 hat.

So, once you find out so, what you are trying to do is that you are trying to put those values in such a way that the error terms would have the minimum variances so; obviously, there would be other concepts or trying to find out the minimum variance also, but we would not consider those in our discussions for this TQM course, for the problem solving I will just give you an example as and when they and they come up in order to make you understand that there would be different ways, other ways also trying to analyze the error such that they may not give you the actual theoretical concept, but practically they would basically make things much clearer to you.

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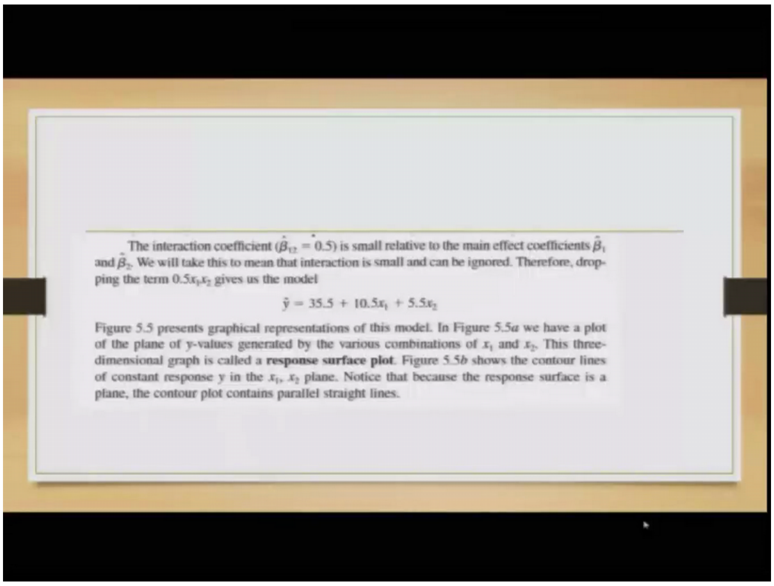

So, the interaction coefficient which is beta hat 1 2 is 0.5. So, is very small relative to the main effect so; obviously, you can drop it, in the sense that if you combine the effects of x 1 x 2 their rate of change or the effect on the value of y is only half of that value it is 0.5.

So, figure 5.1 again referring to the book represents the graphical representation of the model in figure 5.5 a we have a plot of the value of y value generated by the various combinations of x 1 x 2. So, you take different combinations of x 1 and x 2 and try to plot them, this three dimensional graph is called the response service graph. Now, you may be thinking why it is three combinations because, you have y y and you want to basically find out the effect of y on 3 combinations number 1 on x 1, number 2 on x 2 and number 3 is x 1 suffix 1 and 2; that means, combined effect of them. So, the figure 5 point B shows the contour lines of the constants response of y in the x 1 x 2 plane because x 1 and x 2 any draw the plane, they would combine the combinations of x 1 and x 2 also which is x suffix 2. To notice that because the response surface is a plane the counter plots contains parallel straight lines.

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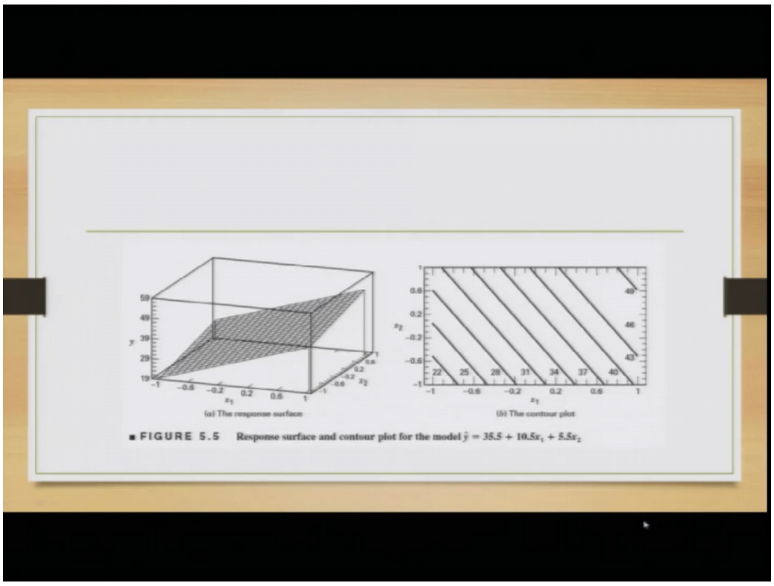

So, now, is the diagram which I am did mention I will try to come to that that discussions once again. So, if you consider figure 5.5 a, you have if you notice on the y axis which is where I am pointing my left hand this is the y value and on the x axis and the z axis because, there is no z. So, what we are trying to do is that plot x 2 and plot x 1. So, as is the linear line hence the counter the contour of the surface would basically be a plane. So, in the two dimension case if there was only x 1 and no x 2, or x suffix 1 and 2 was there, it would be a straight line in two dimension in the higher dimension it would be the a plane and, in the and higher still higher dimensions it would be hyper plane, which we did basically utilize that in the concept of theoretical concepts to be clear.

Now, if when you are looking and the plots of x 1 and x 2 the relationship between them; obviously, you are if you can notice figure 5.5 a and if you eliminate y what you are looking is that you are looking from the top like if just to digress. So, in mechanical design or civil engineering design, we use the trying to draw on any object we use the plan the front view and the side view so, in the plan and the front view and the side view when you consider want them to be constant in the sense that you are looking parallelly along the diagram. So, there is no variations in the values of y, if you plot them they would basically be slanted lines with a tan value of often of negative values. So, which means that if you plot them those lines as parallel as I am I pointing my this pointer. So, these are the values which gives you a relationship between x 1 and x 2 depending on what are the cons on the values of x 1 and x 2 correspondingly.

So, if you plot say for example, x 1 consider this value as 1 and if you if you go along this diagonal line. So, you will basically be able to plot for different values of x 2 and x 1 what are the combinations values you will take amongst them, such that you get the overall effect in the y value so, y value is not being shown here, because they are just a plane which you have not considered because, you are considering this diagram here in the two dimensional plane.

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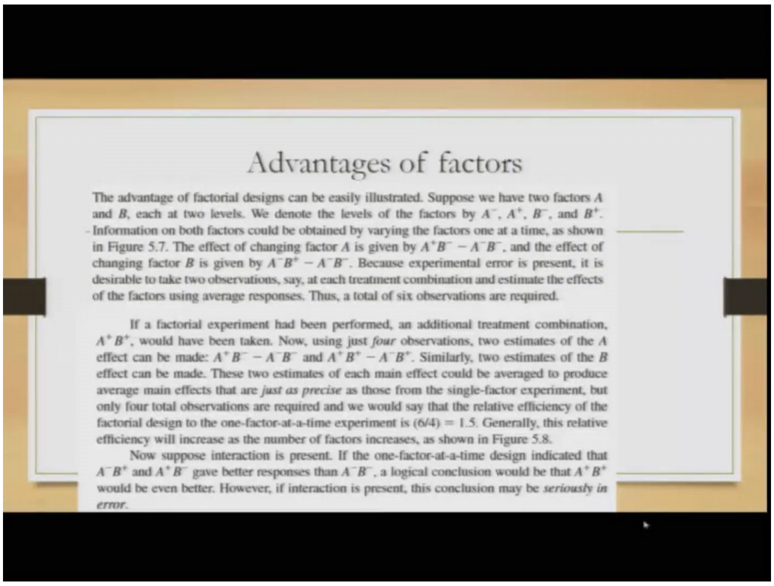

And; obviously, in a three dimensional plane and higher up, or you can basically visualize how they are basically being plotted.

Now, another thing in the model we have considered that y is equal to beta naught plus beta 1 into x 1 plus beta 2 x 2 plus beta 1 2 into x 1 2 plus the error term. So, for the timing considering the error term is not there. So, what we get is basically the y hat value now, if you extend that model to a higher dimension say for example, x 1 x 2 x 3 and you have basically trying to find out the relationship of y based on the effect of on an individual basis on x 1 on x 2 on x 3. And, again as I mentioned the combinations of 2 taken at a time which is x 1 into a and x 2, x 1 and x 3 and x 2 and x 3 and the last 1 basically being the combinations of x 1 x 2 x 3 taken all together at a time.

So, if you plot that it will be hyper plane, but in the case as I mentioned, but in case if it is a non-linear one. So, obviously, the plane which had do or which was shown on in 5.5 a would no longer may be a plane, but in with curved surface and you can find it accordingly how the relationship varies. This is just for a thought process and you can basically visualize how would the diagrams would look like.

The advantage of factorial design can be easily illustrated suppose that we have two factors A and B each I had two levels. So, the levels are A minus A plus as I mentioned and B minus B plus. Information on both factors can be obtained by varying the factors 1 at a time keeping the other fixed as shown in figure 5.7 in if you refer to the book, the effect of changing the factors a is given by so, there would be basically combinations where you considered A plus in 2 and B minus and A minus and B minus. So obviously, you are you are taking keeping a fixed as the highest higher level and, trying to find out the combinations of a plus with all the combinations of the other factors considering there are only two factors.

So, that is why it is given as A plus B minus and the other combination is A minus and B minus. And if you find out the difference of them and the effect of changing the factors B would basically, now be given as now you keep a minus at this a minus level, or you can consider m A at this a plus level and the values of B has been changed from B plus to be minus or B minus 2 B plus whichever way you considered because, the experiment error is present it is desirable to take 2 observations, say at each treatment combinations that what treatment again is being utilized in a very general sense depending on the experiment.

And estimate the effects of the factors using average responses such that you are able to find out the average, or the best fit for those combinations thus they would basically be total 6 combinations, 6 combinations depending on the 2 combinations of each which you have if a factorial experiment has been performed and additional treatment condition conditions of A plus B bar plus which is the higher levels for both a plus and A and B are there, would have been taken. Now, using just 4 combinations two estimates of can be found out which are one is A plus B minus minus A minus B minus, which is basically you keeping B as fixed at a lower level and changing a from 2 from A plus to A minus or A minus to a plus and then find out the difference. And the other combinations would be A plus B plus minus A minus B plus which means that you have kept B at a higher level B plus fixed and basically trying to come combine the changes accordingly.

Now, you may be thinking why I am taking two stages of combinations only the reason is very simple because, we considered that a would be at a higher level and lower level. So, what if it becomes that a has three levels A plus or consider like A 1 A 2 and A 3 where A 1 is the higher level A 2 is the middle level and A 3 is in the lower level. Similarly you can have combinations of B as B 1, B 2, B 3 where B 1 is the higher level B 2 is there is the middle level and B 3 is an lower level and; obviously, you will have combinations like this. Listen to me carefully it will be come in combinations of A 1 with respect to B 1 B 2 B 3 then, A 2 with the combinations of B 1, B 2, B 3 and the last one would basically be A 3 with the combinations of B 1, B 2, B 3.

Now, if you look from the other side it will be B 1 with the combinations of A 1, A 2, A 3, B 2 with the combinations with A 1, A 2 a three and the last one would basically be B 3 with the combination of A 1, A 2, A 3 and you can basically expand it and basically have a had a very simple notion that how these things can be done. So, if you are able to draw it pictorially and find out the calculation the component the model can be made much more better in order to do your this calculations, in a much more robust way robust again I am using in a much nicer way.

Similarly, two estimates of B can affect can affect can be made there these two estimates; that means, for this example we had two effects because keeping A at two levels you found out two as keeping B fixed at two levels you found out Bs. So, that is why it is said as two levels these two estimates of each main effect could be averaged to produce average main effects that are just as precise as those from the single factor experiment.

But only four total observations are required to do this and, we would say that the relative efficiency of the factorial design to A 1 factor model would be 6 by 4, because in the in the two factor model, if we found out there were 6 combinations in the one factor model we found out the basically they were 4 combinations. So, the efficiency would be about or the so, called ratio differences of the ratio relationship between them would be 6 by 4, which is 1.5. And obviously, it will change depending on what levels of combinations which we take, generally this relative efficiency will increase as the numbers of factors are increased.

So, now it will make sense that when I was mentioning A 1 with combinations with B 1 B 2 B 3, or A 2 with the combinations of B 1, B 2, B 3 or A 3 with the combinations of A

1, A 3 A B 1, B 2, B 3 and similarly the other way around like B keep fixed find it out with respect to A 1, A 2, A 3, then go to B to find out the combination of A 1 A 2 A 3 and basically go to the last level of B B which is B 3, then find the combinations of A 1 A 2 a three so; obviously, the efficiency would increase because the ratios would increase.

Now, suppose interaction is present. So, in this case this interaction is present. So, if what the one factor at a time design is indicated then; obviously, you will have the combinations of A minus B plus and A plus B minus because, now they are basically affecting each other and the combined effect. So, they would give a better response than trying to find out with only A minus B minus or, A plus and B plus a logical conclusion would be that A plus B plus would be even better because, if you combine them in that case the combinations or interrelationship would be much better. However, if interaction is present this conclusion may be seriously an error because; in that case you are only taking A minus B minus and A plus B minus.

Hence, you are eliminating the cross relationship which we have, cause relationship I am using the word in this sense A minus with an B plus and another is B minus and A plus, again only two levels combinations.

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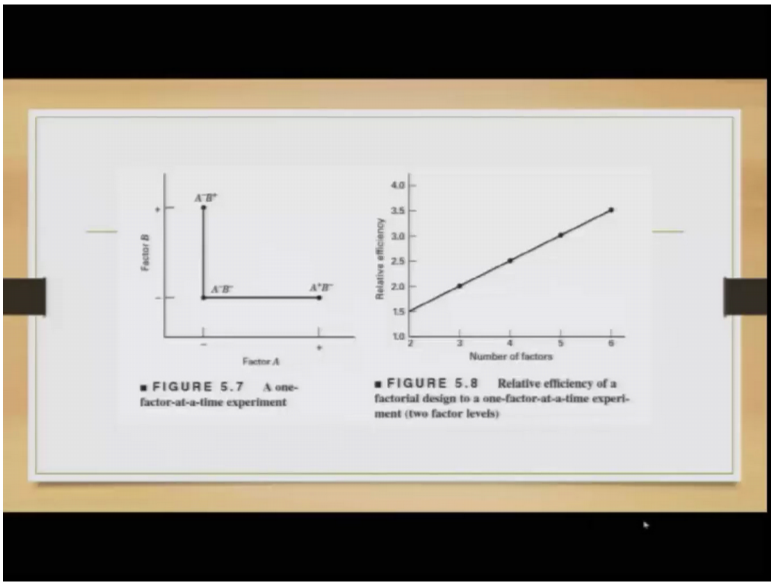

So, if I basically have 1 at a time factor model and a time experiment. So, the combinations are if you see the graph, along the x axis we have your plotting factory with two levels of importance which is minus and plus which is A minus and A plus for the B 1, you have basically along the y axis which is B minus and B plus, hence the combinations which you have would be A minus B plus the combination on the top most point and, the other one on the bottom most right would be basically a plus and B minus and other combinations would be A minus and be B minus.

So, now if we find out the relative efficiency of the factorial design, or 2 A 1 factor at a time experiment; obviously, we find out the efficiency would be 1.5 and so, on and so forth. So, these values are given. So, if you find out for the combinations of a number of factors being 2, then the efficiency is 1.5 which is 6 by 4, if the combination is 3, now the relationship what the factors are 2. If the combinations are 4, then the efficiency is 2.5 with the combinations is 5 the efficiency is 3 for 6 combinations it is 3.5 and so, on and so it will basically go linear me up. So, this is for the linear model I am not considering the non-linear models.

So, with this we will end the 15th lecture and, continue the discussions for the factor models in more details and slowly expand our ambit of discussions for the ANOVA and ANOVA model, then in the designer experiments and continue this in more details.

Thank you very much for your attention, have a nice day. Bye.