Quantitative Finance

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Module – 06

Lecture - 34

So, continue our discussion about Ito's Lemma, venna process, generalized venna process. Based on that, we proceeded and found out that stock prices can be simply modeled using very simple concept of geometric prominent motion and continuing that if we find out. So, generally the geometric prominent motion would consist of two terms; one is left; one on the right hand side; would basically, be the mean drift rate which will be given by mu and the second term would be the volatility which is the external noise. So, if we expand the equation in the discrete form, this is what we have.

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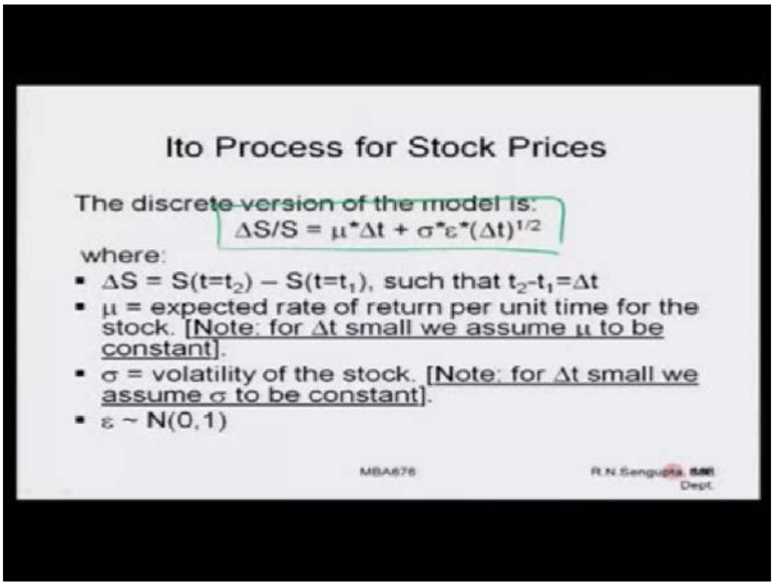

Ds by s or delta S by s would basically, be equal to the drift rate plus the volatility, multiplied by the venna process, where delta S is basically spot price or stock price difference between t 2 to t 1, t 1 to t 0, t 3, t 2, t 1 so on and so forth. Here mu is the expected rate of return per unit time for the stock. Note for delta small t, we assume it to be constant. For other cases, it would be dependent on t, but in general we consider mu is not dependent on time. This is a very simplistic assumption. If we consider the generalized venna process, we saw x was dependent on x and t, which is basically mu would have been dependent on S and t; that means, the mean value would depend from where the spot stock price is as of now and how far you will go.

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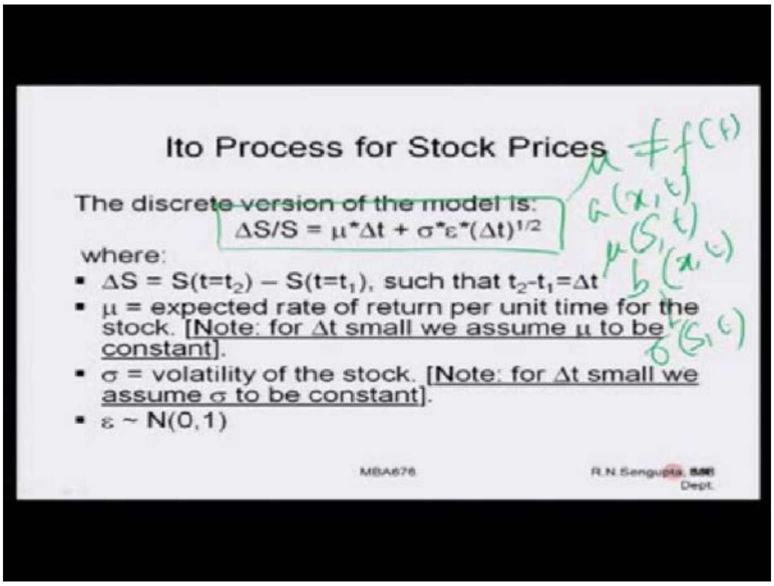

Sigma is the volatility of the stock. Note for again, for small delta time t volatility is independent of time. Again if you go back, we did consider b was dependent on x and t. If you consider this, sigma would be dependent on s and t and also finally, we did mention and this we will stick throughout the assumption, is that the epsilon would basically have a normal distribution with 0 mean time on standard deviation.

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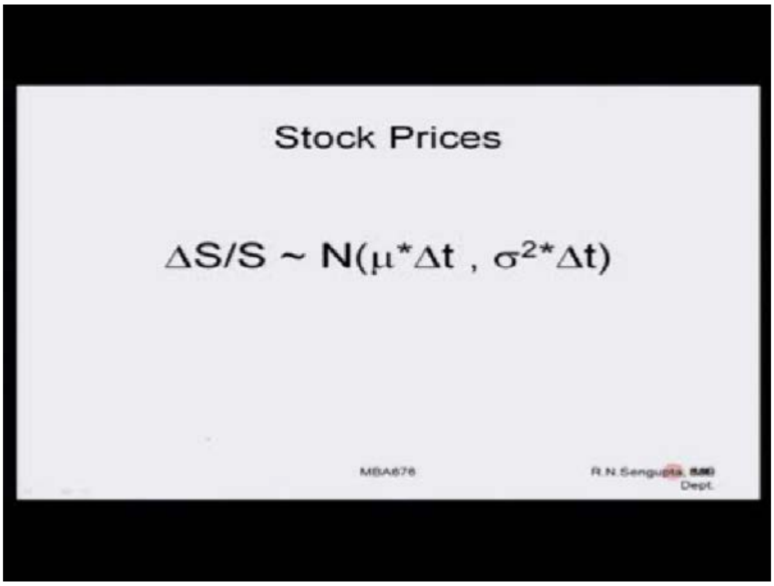

So, now finally, once we are able to derive we will see that the rate of change of the stock price with respect to its present value would be a normal distribution. Here is what is the interesting part; this normal distribution mean and variances are not constant; they are changing with respect to time. So, the first term which is the mu value would be, the normal value would be, for the normal distribution the expected value is mu into delta t, which is the drift rate multiplied by the rate of change of time which you are considering, and in the other case, if you consider you will basically have the variability of the volatility given by sigma square into delta t, which means that if I am able to draw the functional form of delta s into s, the mean value would change.

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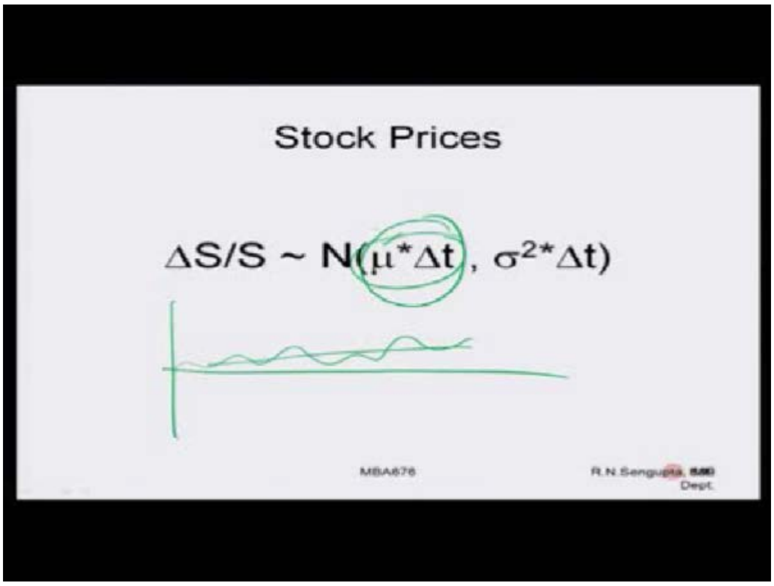

So, the rate of change per unit time is this, and if I basically consider delta t time t this is the total rate of change; total value change in the drift. If I consider sigma square is the white noise then for the time difference of delta t, from where we are and where you want to measure the volatility; it would be given by sigma square into delta t. Now pause; let us pause here and note. If you remember I did I have been mentioning in the last class time and again few times that the volatility is dependent on time either to a scale of delta t or to a scale of square root of delta t. Now, this is what is; if it is variance, it is dependent on delta t, hence variance could be additive. If it is basically standard deviation, it would be dependent on that function by square root of delta t, hence it is not additive.

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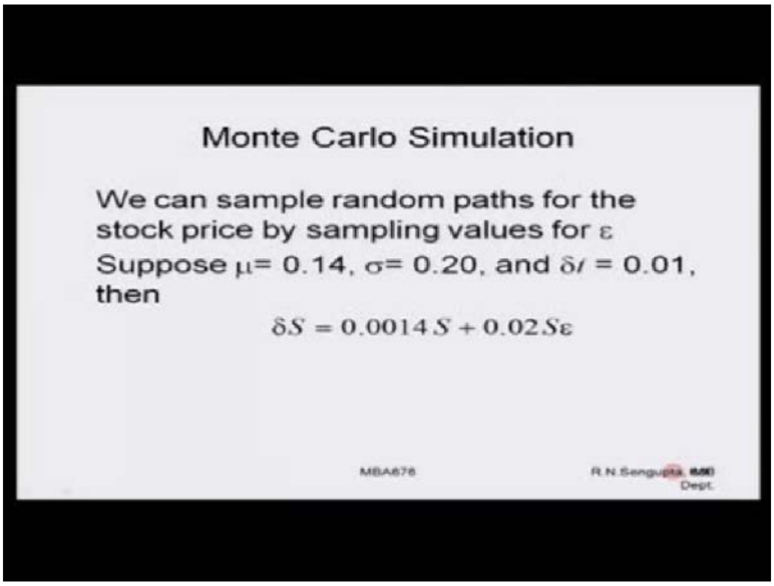

We can sample random parts for the stock market as I was discussing, considering that the values of mu is 0.14, sigma is 0.20 and delta t is given by 0.01, whatever unit of time. So, if we find out; if you take S down so, obviously, S would not be there, S would not be there. So, mu into delta t which is, this comes out to be 0.0014 and the second term was sigma into the epsilon into the square root of delta t. So, this you have multiplied by square root and it would be given by the case, multiplied by standard deviation of the epsilon.

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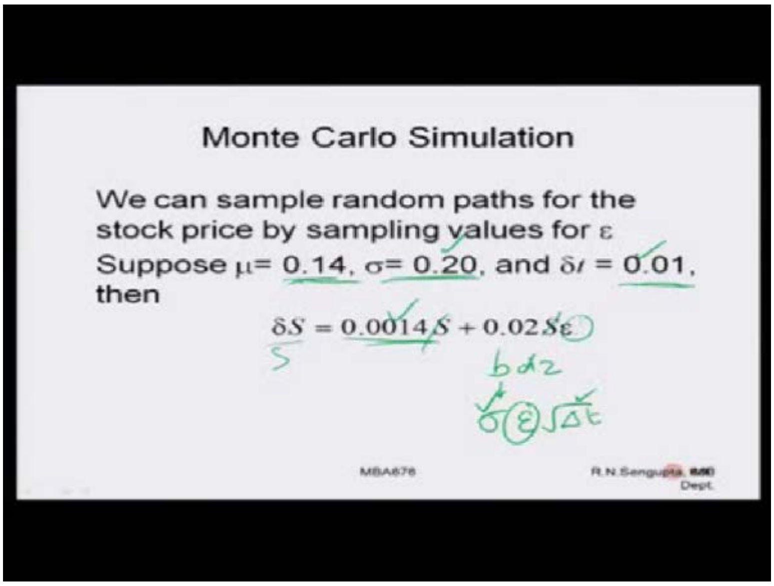

So, what you are doing is b into z; so, b is basically what you have is sigma into the value of epsilon into square root of t; this you know; this you know. This is generated and you can find out.

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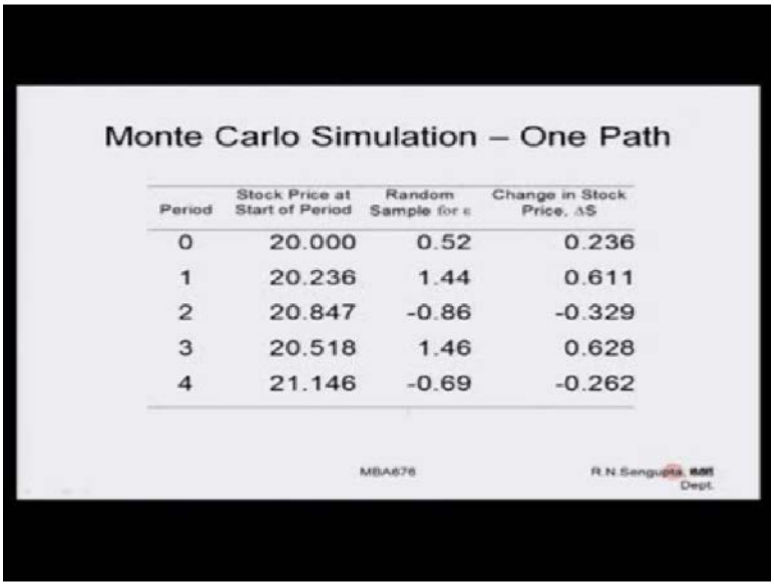

So, we have a taken a sample shot at time t is equal to 0; the stock price is 20. So, I randomly sample from epsilon 0 and I take a time, whatever the time zero zero 1.11 whatever, and change in the stock prices given. So, what I do is that given this price using this formula of mu into dt plus sigma into epsilon into square root of delta t; this you consider as constant with respect to time; this you consider constant with respect to time; find it out from the historical values. Delta t can be whatever the time difference is. Here in this case, is 1 minus 0, 2 minus 1, is just a theoretical example. So, once you are able to do that you generate, and generate from a standard normal deviate. Once you find out, you find out the change of the price is, because on the right hand side what we actually are finding out.

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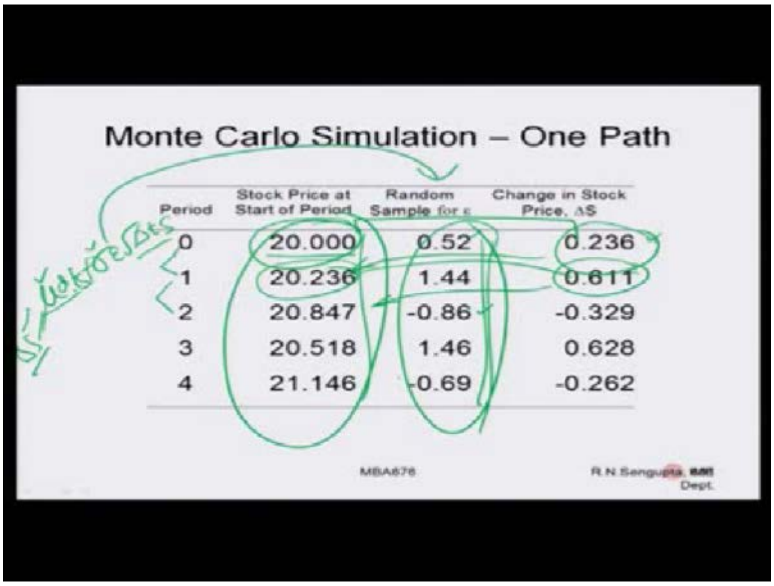

So, this would be multiplied by s; this would be multiplied by s; this is delta s. So, once delta s is found out, this plus this would give you this. Then again generate, you would get this value. This plus this would give you this and you basically, bootstrap by side. That is 20.000 plus 0.236 would give you 20.236. Then again 20.236 plus 0.611 would give you 20.847. Again 20.847, now there is not a plus, because the random number generated epsilon is not now negative. So, 20 plus 20.847 minus 0.329 would now give you a new value of the stock point price, which is 20.158 and so on and so forth. So, these values which generate are dependent on the random sample, which you have generated from the epsilon. So, if you are able to generate another set; obviously, these values would change. So, what I meant was that if you do it a long time, infinite number of times, then the expected value of the change of x or delta x, t minus delta x 0 or x t and x 0 would be 0, because whatever fluctuation you have they would always add up to 0 in the long run.

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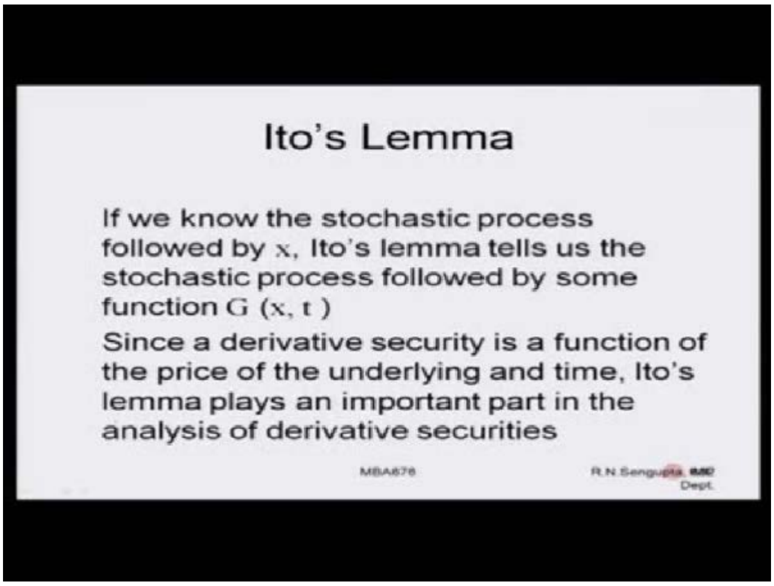

So, Ito's Lemma, if you know the stochastic process followed by x and which is the random variable, this Ito's Lemma tells us that the stochastic process would follow, will be followed by function, by some functional form of g x, t. Since a derivative security is a function of the price of the underlying security and the time, Ito's Lemma plays a vital role and an important role in trying to analyze the price of this derivative, which means that given the concept that they are Brownian motion and given the Ito's Lemma, it gives you of two information that how the rate of change of that function is happening with respect to the underlying the stock. It will give you a lot of information that how Ito's Lemma can be formulated in a very simple way to find out the price of that derivative, given the spot prices of the stocks are available to you.

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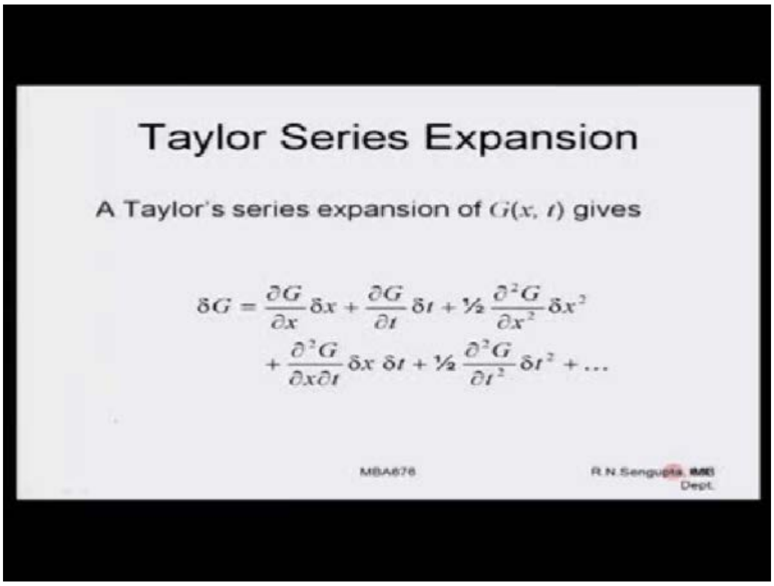

Now, consider a Taylor series expansion, in a very simple one of G x, t, where x is the place where you are and t is the time for that particular random variable. So, if we expand this, which is very simple. It is basically two variables. So, basically what I have; this is the first differentiate with respect to the first variable dx; second differentiate, the first differential with respect to the second variable. So, if I go to the second derivative, second derivative would have basically three terms. The second derivative with respect to x twice; second derivative with respect to t twice and second derivative with respect to x, then t and then second derivative of respect to t and then x. So, what we would be doing is this.

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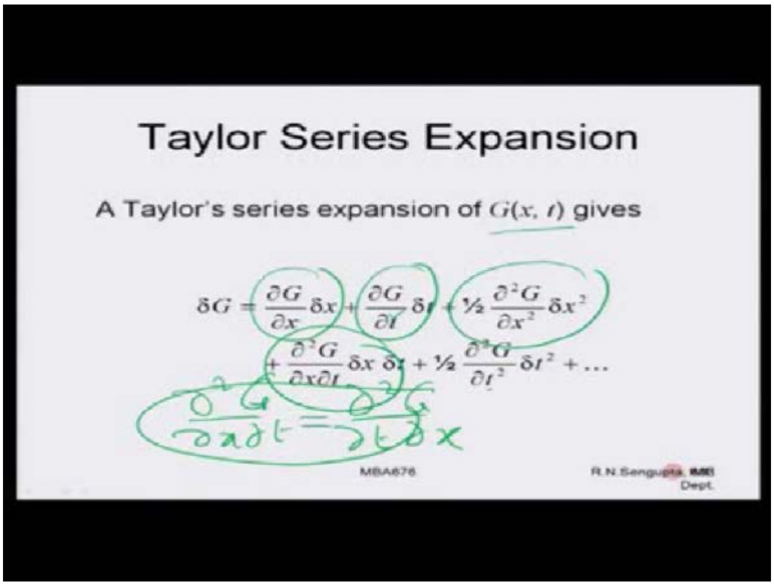

So, if you consider the function to be symmetric, they are equal. If they are equal, they are multiplied twice, hence there is no half or two terms here. So, this is the term and then if you consider then in the cubic form it will basically expand. So, cubic form what do we have? You have delta 2 g delta z cube plus delta 2 g delta t cube and you will take the values of x and t in this proportions. You will differentiate with x. Then differentiate twice with t or else you differentiate with t and then differentiate twice with x and basically have a combination of that and proceed. So, this is basically a simple multivariate Taylor series expansion.

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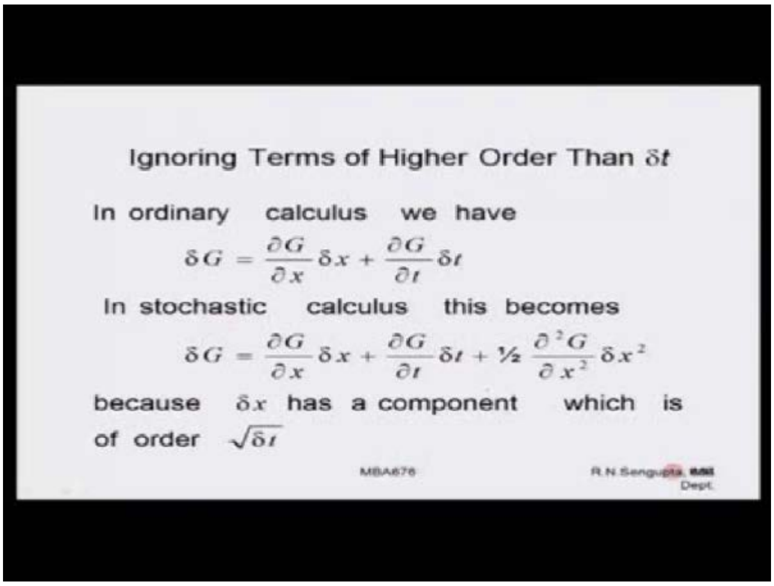

So, in ordinary calculus we have, if we are able to find out considering that we are able to ignore the third and higher terms. So, this is the first term. First term, because they are only first differentiate; obviously, there would be second term, second term differentiate. So, now if we bring into the picture and considering that the terms of delta t as t tends to 0 can be ignored. If you go back to the slides it will make sense.

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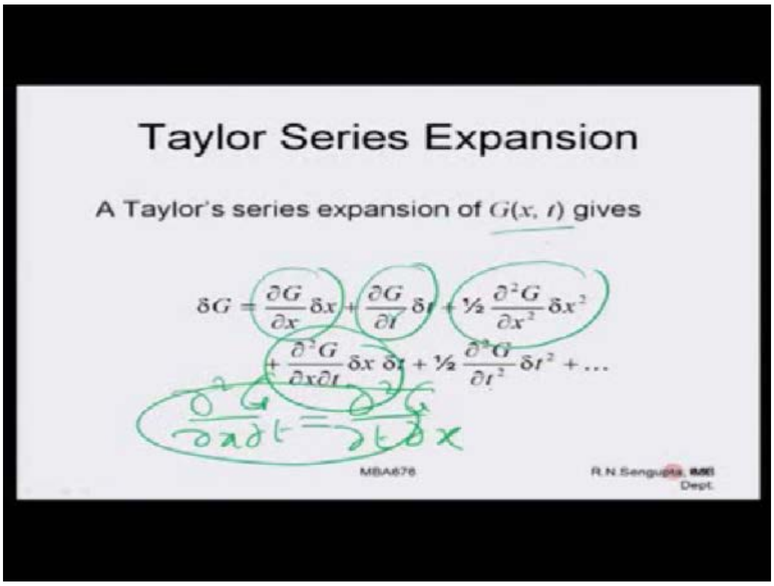

This cannot be ignored. This cannot be ignored, because delta t has some value, but now consider this term. What you are doing? You are trying to find out the differential of that particular function, twice with respect to t and then multiplying by a very small term, multiplied twice, which is here. So, obviously, it can be safely assumed to be 0 and also the interesting fact is that we would be differentiating the function with respect to two times. One with respect to delta x which is not that small and again, if you differentiate with respect to t, the smaller value; it will now tend towards 0, plus the fact it is being multiplied by delta t. So, technically even though theoretically, it may not be right, but practically, we will assume these values are almost tending to 0; that means, this last term and the second last term. So, if I highlight it, this is the values. These two terms are basically ignored.

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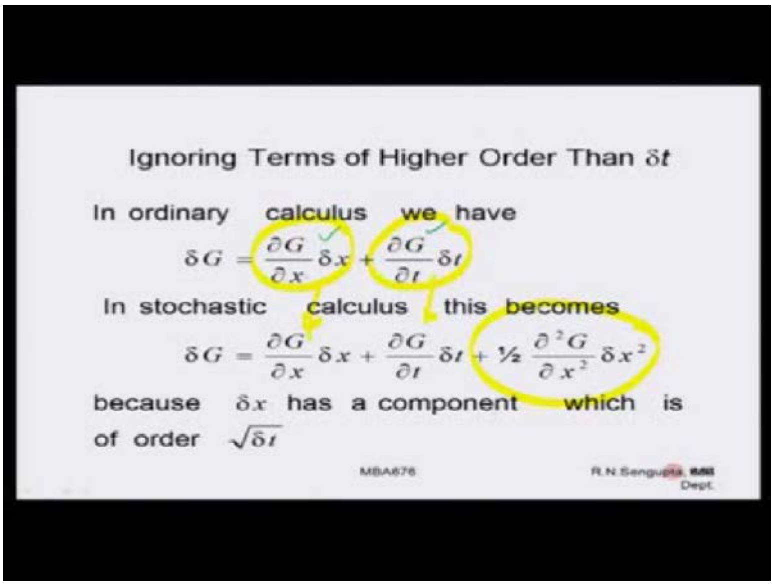

So, if you do that you will find out the question, is the first terms which are here and the second term is basically, coming from the second differentiate of the x term; nothing to do with t, because if you note; delta x is a component which is at order of square root of delta t, hence it can be ignored.

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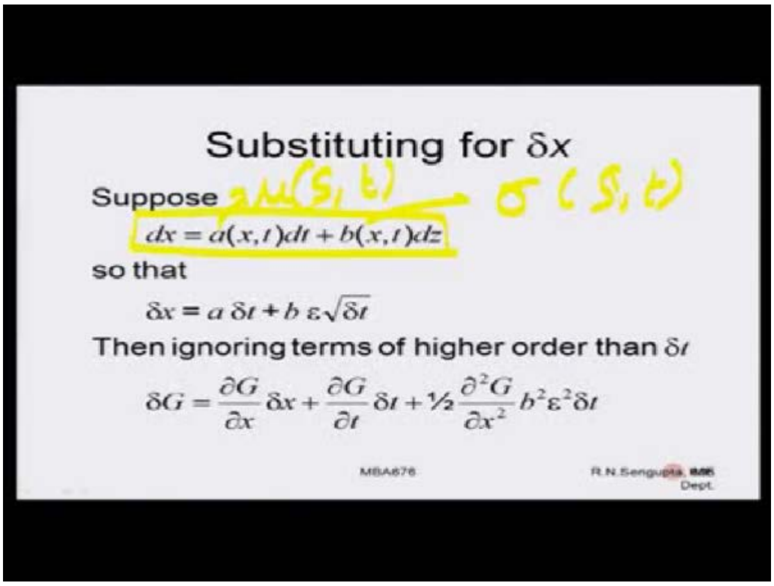

Now, suppose the rates which is mu, which we did consider that it is constant with respect to time if delta t is almost tending to 0 or it also said that sigma is a constant, if you consider rate with respect to t, where delta t is tending to 0; then if you are able to expand that, then we had the simple equation where factor of time and x are not going to come, but if you consider the factors of time and space are going to come; equation is this. If you are, if we bring here this as mu S,t and this is sigma S, t, then they finding the equation finding the Taylor series expansion may be tricky, even though it may give you better results, but that would not be what the overall effort you are going to spend.

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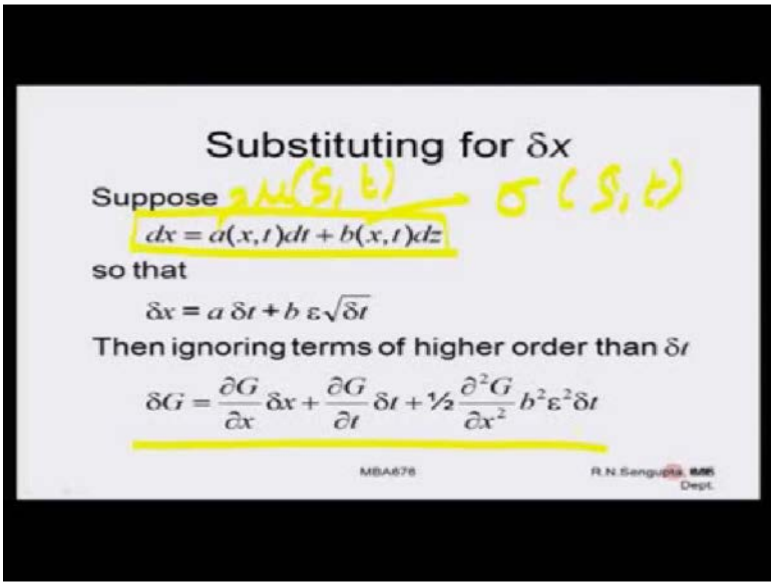

So, now, the final equation which we have considering that a or mu, b or sigma, are not dependent on where you are and not dependent on time, the final equation, ignoring any second terms of delta t, square root or delta t, square root squares; actual equation is this, which means the partial differentiate equation, using the Taylor series expansion is given by this term.

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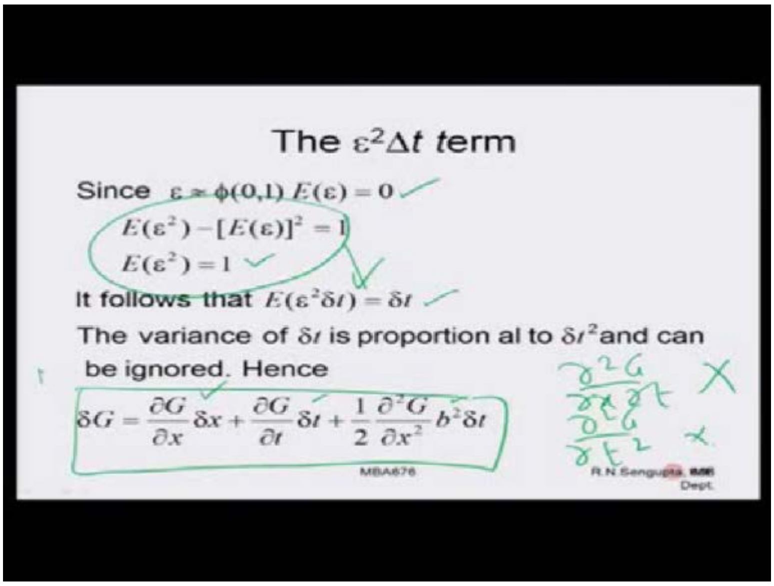

So, now, as epsilon we have considered is basically, standard normal deviate with 0 mean and one standard deviation, hence the mean value is 0, and if we consider the variance would also come up to 1 as you already considered. So, it follows that if we find out the expected value of epsilon square delta t, it would come out to delta t, because the respect of these two assumptions, which you have already noted down. The variance of delta t is proportional to delta t square and can be ignored, hence the final equation as we mentioned is this. This is with respect to x first differential. This is with respect to t first differential and this is respect to x, but the fact that higher term of this can be ignored and higher terms of this can be ignored. This is the case where you differentiate between x and then t or t to x and here, you are differentiating twice with t.

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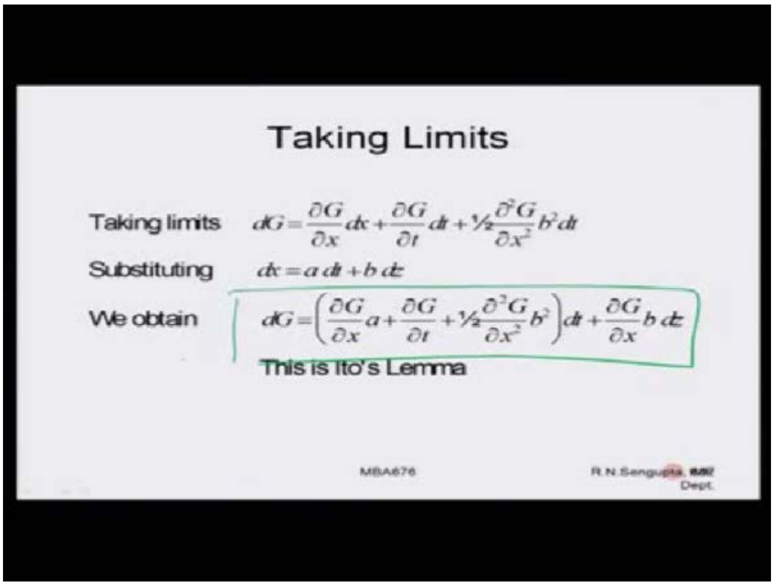

If you take the limits, we obtain the final formula which is this, which is basically, the Ito’s Lemma. So, now, what we need to do is that given the spot price is s, you are going to find out a derivative which is a functional spot price, depending on the values of mu and sigma we can find out what is the actual price of that derivative.

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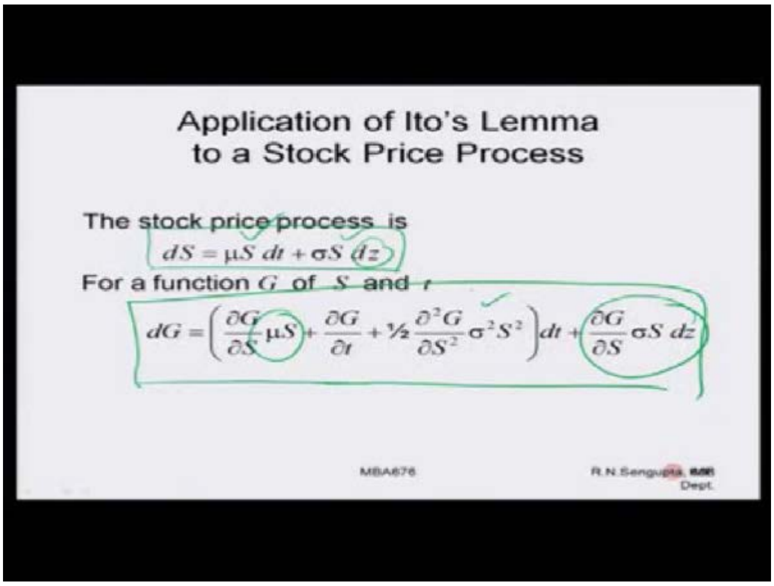

So, applications of Ito’s Lemma towards stock price process stocks, price and its process; the stock price, as we already know is given by this drift term, volatility term. This is the vennas process; this s is the spot price. Now, if you bring it into picture for a functional z, which is basically of s and t; not x and t; s is the spot price. The actual equation basically becomes this. So, now, you will basically have the drift rate. You will have the volatility rate here and you will basically, have the squared terms of the volatility depending on what is the stock price, but remember one thing; the fact that we have assumed a and b are independent of x and t or mu and sigma are independent of s and t, even though are logical to serve and solve; in practical sense, they may not be true in the sense, that the assumptions we are taking as the mu on sigma to be independent are not true.

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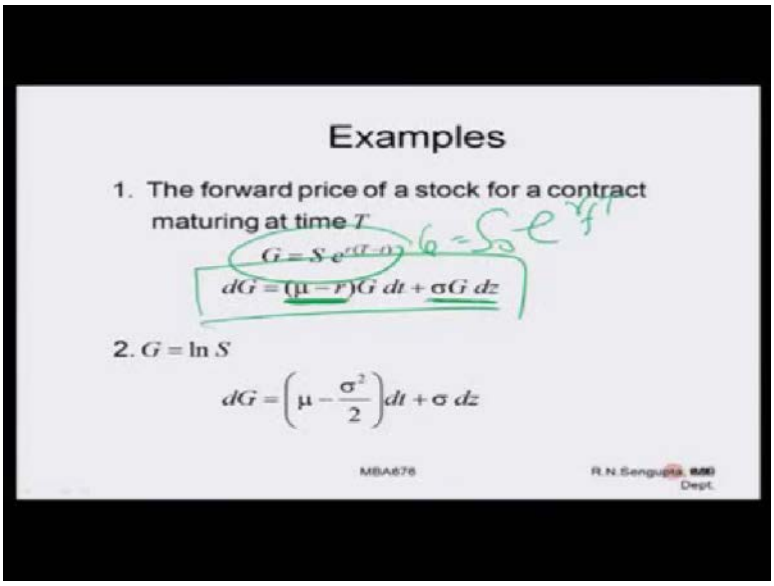

Now, if you consider the forward price of a stock for a contract, maturing at time t, is given by this; we already know. So, because it is American one and if you consider it is e to the power r f, which is a risk free interest rate into t into s 0. If you consider this G is a function of this, then if you solve it, the fraction becomes like this, which is rate of change of G is now being given by two terms, which is the excess rate of the particular stock over the risk free interest rate. If you remember we have considered that in some of the optimization problem and the volatility. So, now, it consists of two terms; excess rate over the risk free interest rate and the volatility rate and what are the other factors. One would be the time difference, which is delta t or t and one would be the simple venna process, which you have assumed underlying assumptions in the model.

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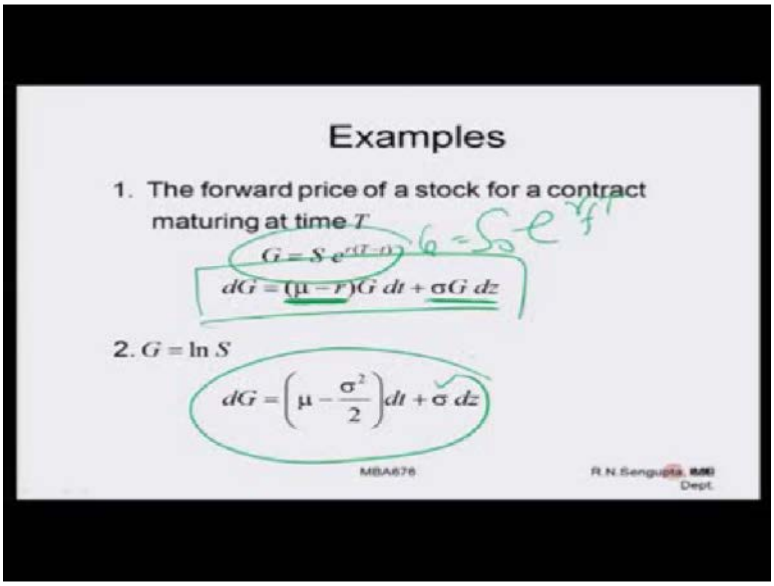

Now, if G is a function, if you solve it the overall function comes out to be this. Again, this is the average rate, but the average rate would now decrease by the term of sigma square by 2, with sigma is basically, the volatility and in the same way, the second term which is the white noise, comes continues to be remained as sigma.

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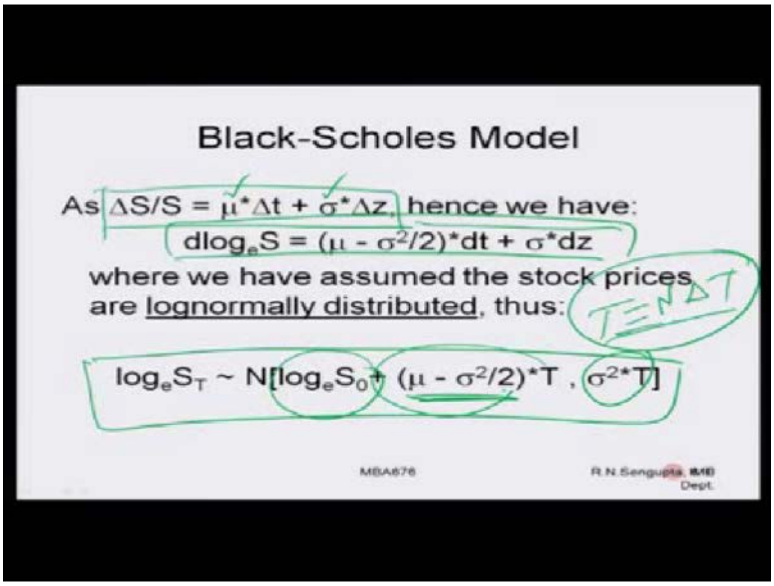

Now, consider that if we had the equation as given, the simple one, which is true; mu is the drift rate; sigma is the volatility or the white noise and on the left hand side, we have we have the rate of change, which is delta s into x, divided by s. Hence, if you are able to solve it, it comes out to be this, where we have assumed that the stock prices are log normally distributed. If it is normally distributed the prices are given by this, where the log of the spot price as of today, would basically, be a normal distribution with two parameters, which is the mean and the standard deviation. So, the standard deviation, the variance continues to be sigma square into t; t is the time difference from where, you are and where you are trying to measure, and sigma square is the volatility per unit time or volatility is just constant, if delta t is very small. Delta t and this capital t are again, different; let me again mention. This is what we are considering and the mean value would basically, depend on the mean value as of now, plus a constant term and where is this term coming from?

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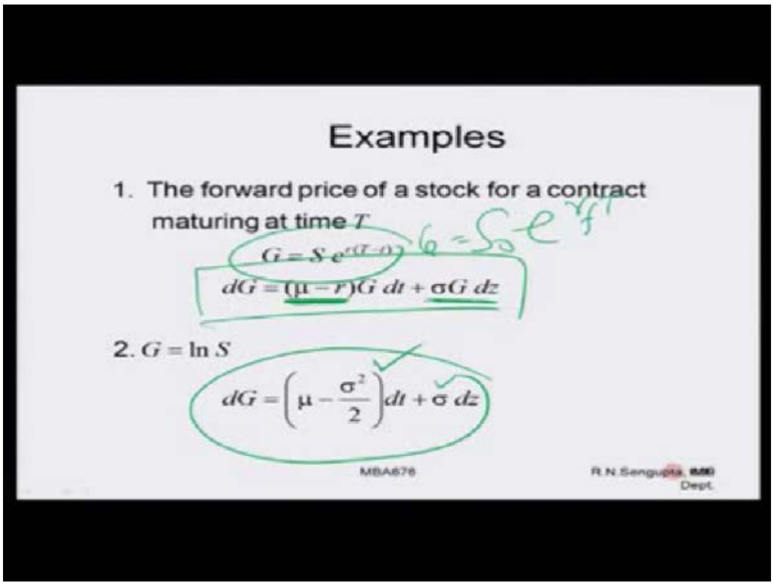

This was what we have considered in the last case; just the last case here; multiply it by again capital t. Again, capital t is given by this formula.

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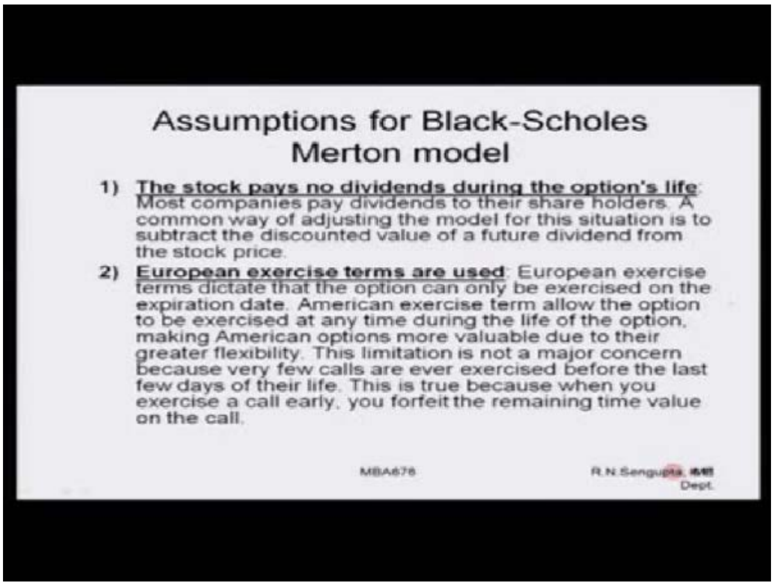

So, assumptions of the Black-Scholes; we have been able to at least, get a feel that how the stock prices using Ito’s Lemma, the venna process, the generalized venna process and the assumptions will consider about the volatility and the drift rate. So, based on the fact we have been able to simply, derive and understand the Black-Scholes model, but let us first, also try to analyze what are the important assumptions based on this model, was developed. The assumptions are very simple.

The stock pays no dividends during the option’s life. If you remember we did consider this time and again, that all this is no dividend; that means, the value of I 0 or any independent payment, which is being made and they are being calculated as of now, would not be considered. Most companies pay dividends to their shareholders. A common way of adjusting the model for this situation is to subtract the discounted value of the future dividend from the stock price as of now, which is s 0 minus I 0, but we are not going to do that in the simple calculation of the Black-Scholes model.

Second one is European exercise terms are used; it means, if you remember the time to expiration for the European one was exactly fixed at time t is equal to capital t. Those are not American options. It says that European exercise terms dictates that the option can only be exercised on the expiration date. American exercise terms allows the options to be exercised on any time during that period of time before it expires. So, this simulation is not a major concern, but it can be solved accordingly.

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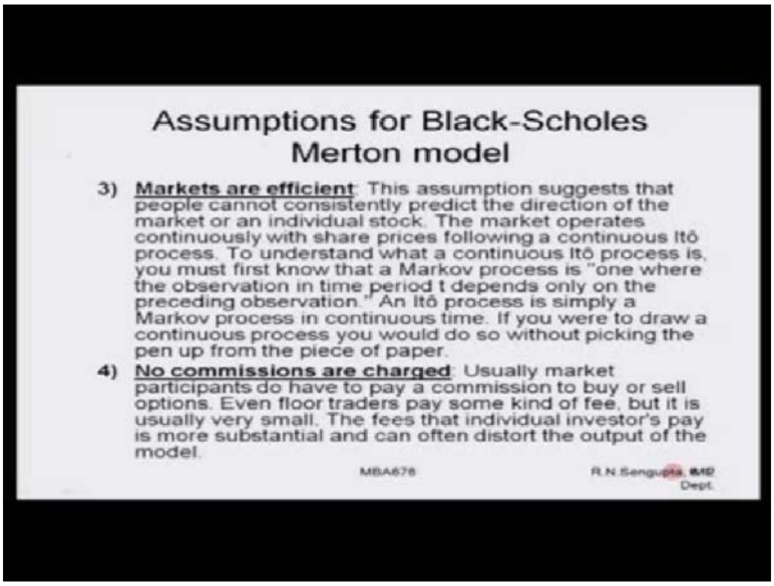

We will also consider markets are efficient; that means, the demand and supply of the prices are such that it would basically, be able to give the perfect information of the spot prices are being dictated. There is no commission charge and also we will consider there are no taxes; that means, if you are selling and buying a particular option or a stock in the market, you are using the broker self. You will consider that you are not either paying revenue amount to the broker for buying and selling this different types of stock and option.

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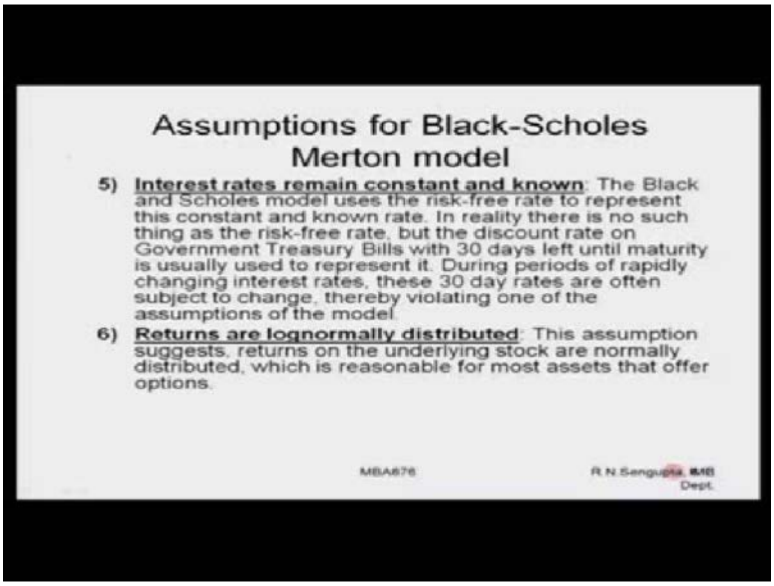

We will finally, consider the interest rates remain constant; that means, the mu value or the sigma value which are constant, will always continue to be constant with respect to time. This, I had mentioned time and again when you are deriving, and the returns are log normal distributed, such that based on the fact, the derivation which you did for the using the Ito's Lemma for the Black-Scholes model, continues to hold true.

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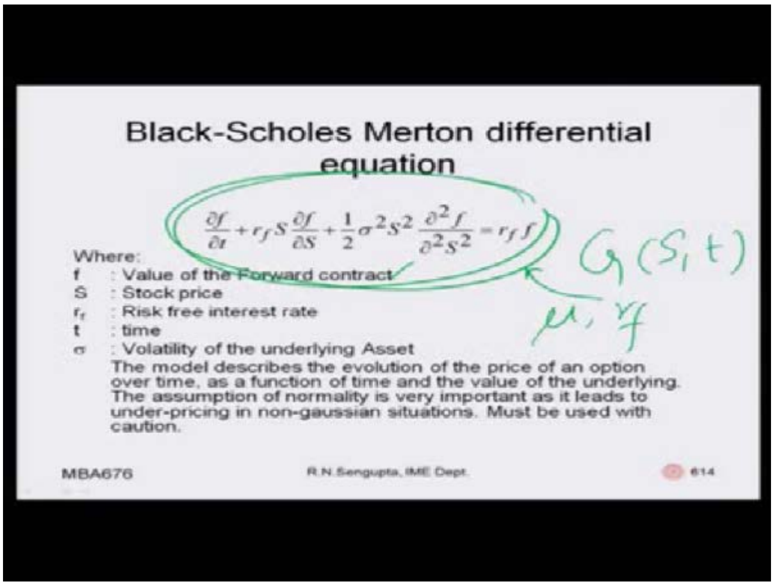

So, finally, we wrap up this class. We will take another two minutes. The final equations for the Black-Scholes model would be given by this, where f is basically, the corresponding value of the forward contract, which we are discussing in the capital f or the small f, which is dependent of risk free interest rate and the spot price and the differential equation, which we have already got. So, G was basically the function which was dependent of s and t and based on that, we found out and mu was the rate of change of the function, and r f was the risk free interest rate. So, we are basically trying to find out with respect to how big or small of the value of the rate of change of the function is with respect to r f. Based on that we will try to invest into that derivative. So, s is the stock price; r f is the risk free interest rate; t is the time and the volatility of the underlying asset is given, which is the actual stock we are trying to buy and so, remember one thing; that r f is constant as mu and sigma are constant.

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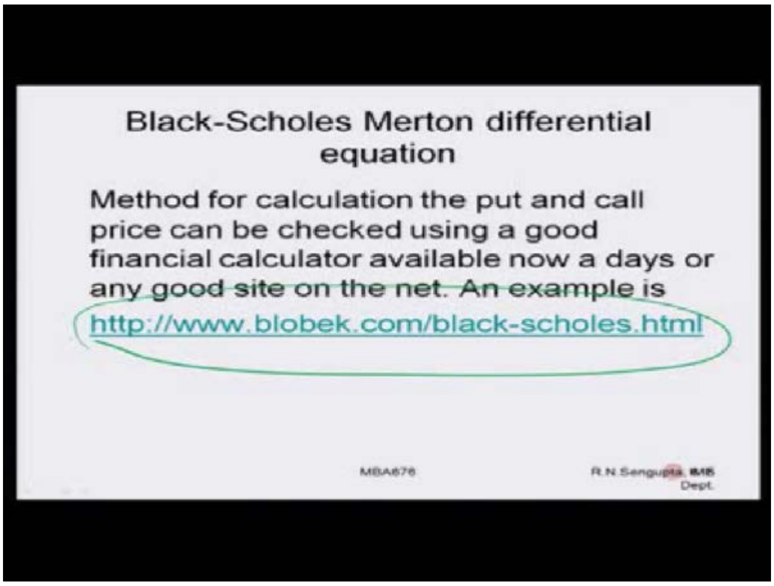

So, method for calculation for the put and call option can be found out using any financial calculators which you have, and a good site for an example in the net, can be found out here. So, we will continue our discussion with other models later on and then, come back and trying to explain that how the Black-Scholes model can be utilized in trying to find out the stock the prices of that options based on the spots, which are available.

Thank you very much.