Strategy: An Introduction to Game Theory
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Lecture – 44

Hello, welcome to another module in this online course Strategy, An Introduction to Game Theory. We are looking at auctions as Bayesians games and we have looked at the first price auction and the second price auction. Let us now look at a slightly different auction format that is an all pay auction.

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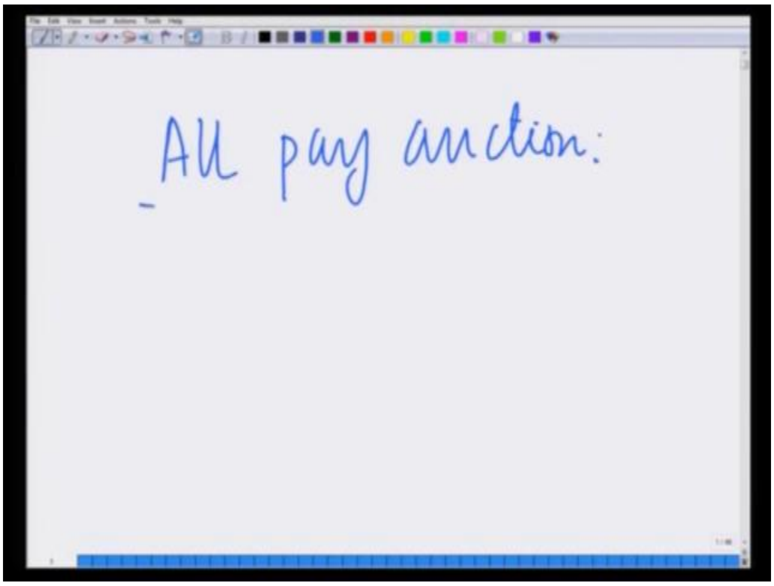

So, let us look at an all pay auction, again similar to the earlier auction scenarios.

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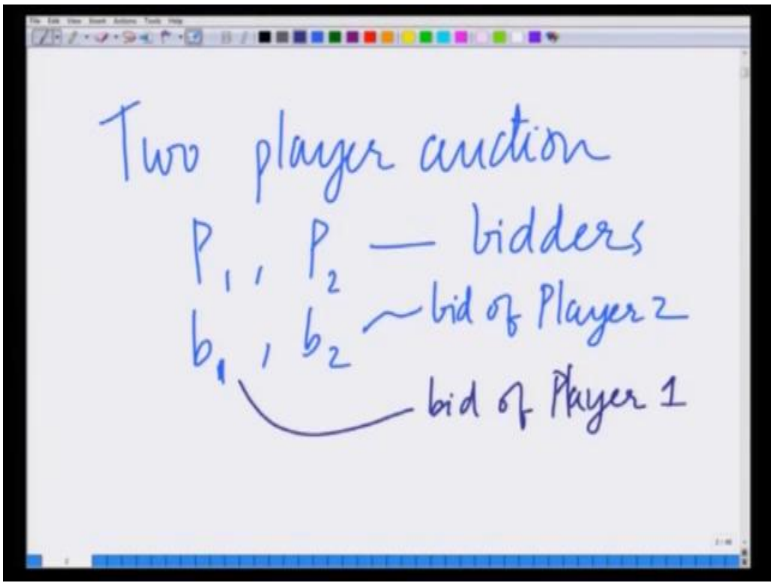

Let us consider a two player auction, player 1 and player 2 these are all two bidders and let their bits be given as b 1 comma b 2 that is b 1 is the bid of player 1, b 2 is the bid of player 2. So, b 2 is bid of player 2 and b 1 is bid of player 1, the player with the highest bid wins.

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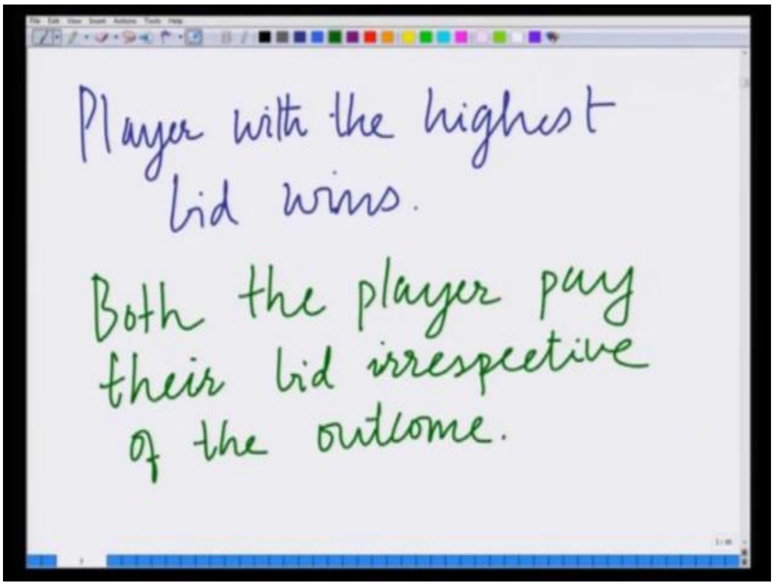

But, both the players submit or both the players pay their bids irrespective of winning or losing. However, both that is both the players pay their bid irrespective of the outcome, that is the winner is paying his bid and the loser is also paying his bid. Remember, this is very different compare to our first price or second price. In a first or second price auction we said, that only the winner in the first price auction the winner pays his bid, in the second price auction the winner pays the second price bid.

However, in this auction in the all pay auction we are saying that each player that is player 1 bids an amount b 1 and player 2 bids an amount equal to b 2 and the player with the highest bid wins the auction and both players pay the respective bids that is b 1 and b 2 irrespective of the outcomes. So, you can think of these bids as sunk cost. So, irrespective of the outcome, irrespective of winning or losing each player is paying his bid.

So, each player is paying his bid and we are also going to assume again similar to the first and second price auction. Let us assume that V 1 and V 2 are the valuations of players p 1 and p 2 respectively, which are independent and distributed uniformly in the interval 0 to 1.

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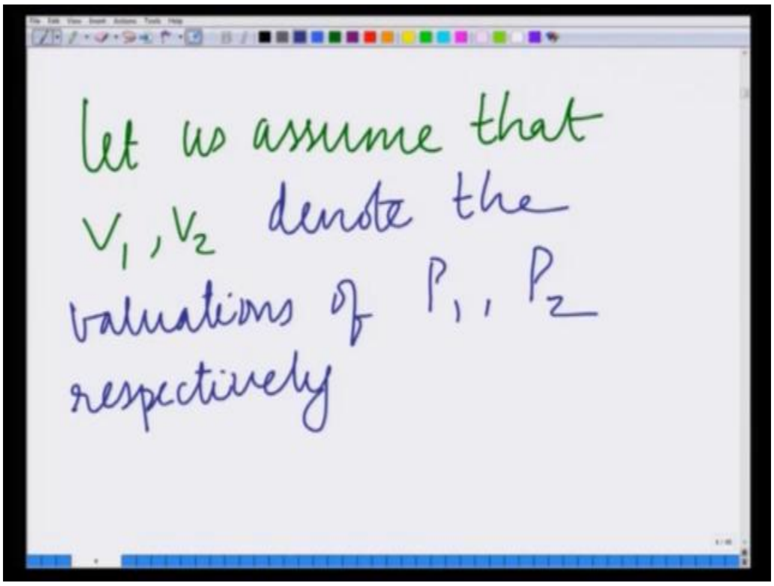

Let us assume that V 1 comma V 2 denote the valuations of P 1 comma P 2 respectively.

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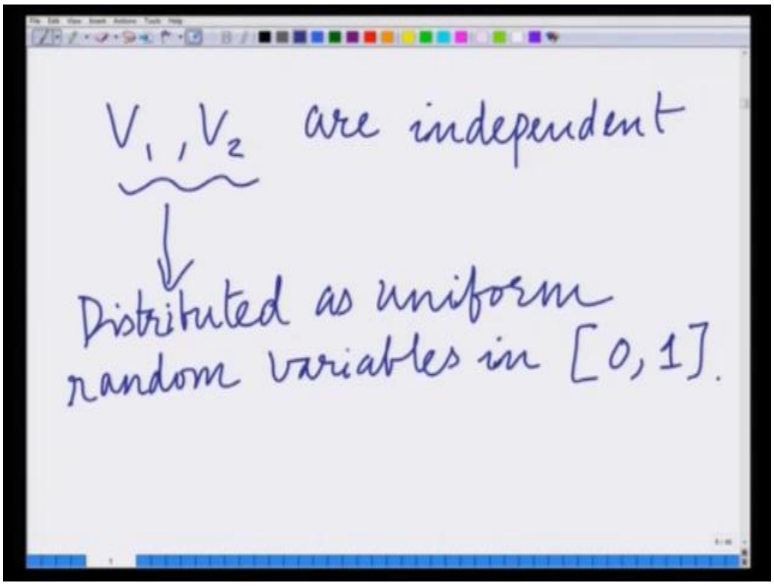

And also V 1 comma V 2 are independent and these are distributed as uniform random variables in the interval 0 to 1.

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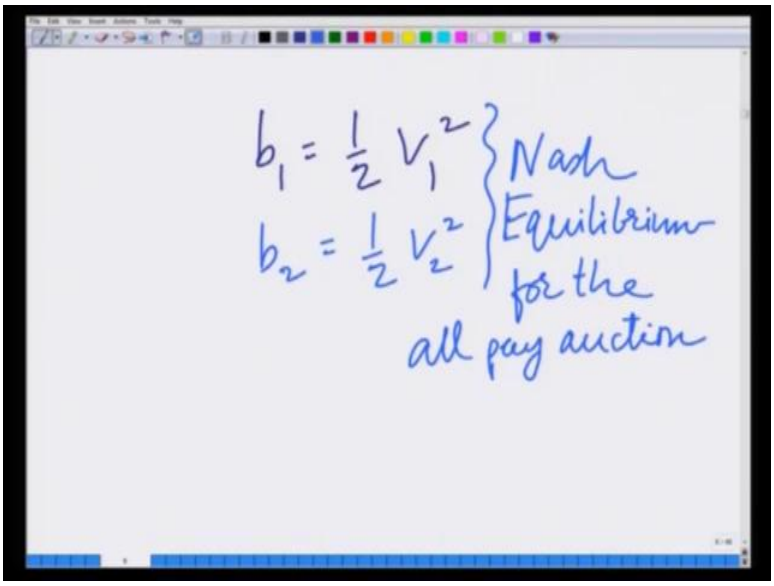

And for this, all pay auction format we are going to show that b 1 equals half V 1 square and b 2 equals half V 2 square, we are going to show that this is the Nash equilibrium for the all pay auction format with the valuations distributed as independent uniform random variables in the interval 0 to 1. Again let us follow a similar procedure, let us start by assuming that player 2 is biding b 2 equals half V 2 square and let us find the best response bid b of player 1.

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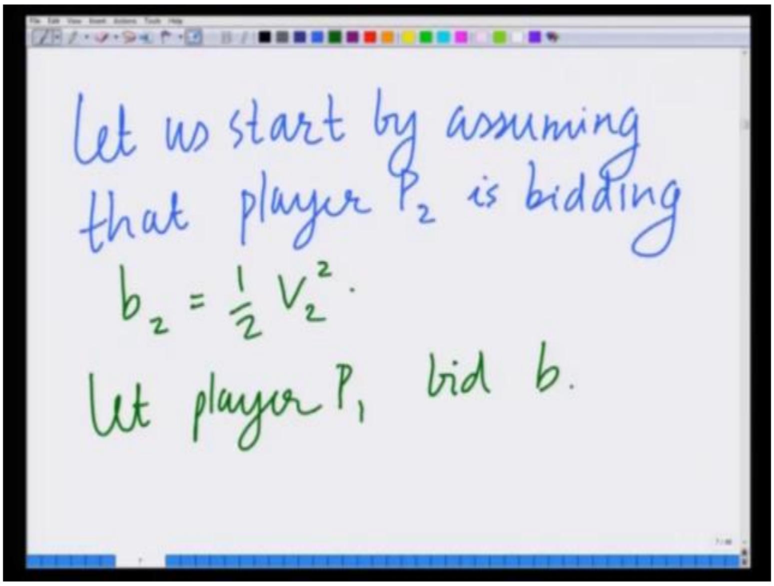

Let us start with the assumption player P 2 is bidding b 2 equals half V 2 square, we will find the best response bid b of player 1, let player P 1 bid b. Now, let us denote the expected payoff of player P 1 by pi of b.

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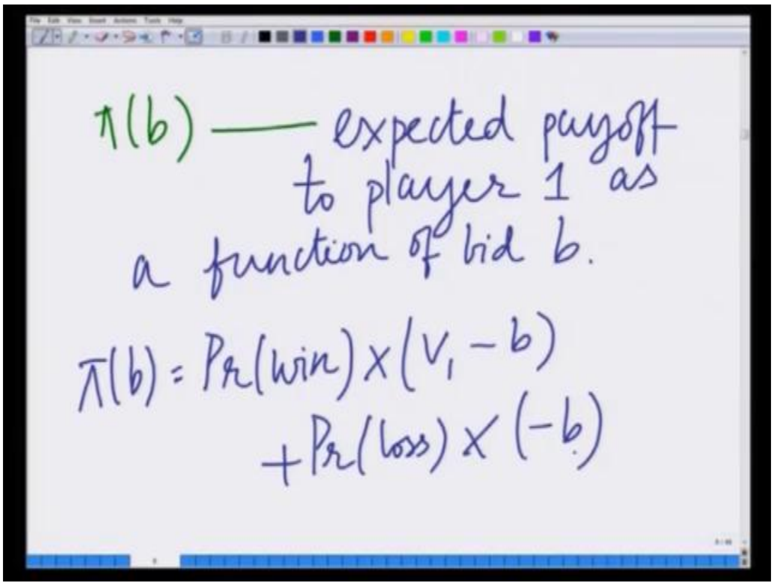

So, by pi of b let us denote the expected payoff to player 1 as a function of his bid b and therefore, pi of b is equal to the probability of win that in case he wins multiplied by his revenue on winning. Remember, his revenue on winning is his valuation minus his bid b plus probability that he loses, in case he loses that is the probability of lose times remember he pays his bid irrespective of the outcome.

So, in case he loses he does not get the object, but he simply pays his bid which is b therefore, his payoff is minus b. Because, if he loses he does not get anything and ends up paying his bid b therefore, his net payoff is 0 minus b which is minus b. So, the expected payoff pi of b as a function of b is probability of win times, when he wins he gets the object. So, his net payoff is valuation V 1 minus P plus probability of lose times simply minus b.

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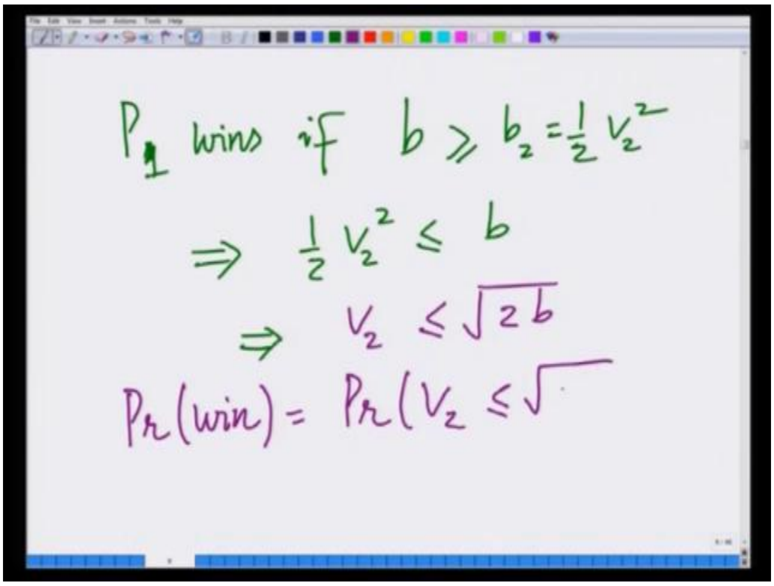

Now, let us find what this quantity probability of win is, remember he wins P 1 wins if his bid b is greater than or equal to b 2 equals half V 2 square implies half V 2 square is less than or equal to b implies that he wins when V 2 is less than or equal to square root of 2 b. So, therefore, the probability of winning equals probability V 2 less than or equal to square root of 2 b, but V 2 is a random variable which is uniformly distributed in 0 to 1.

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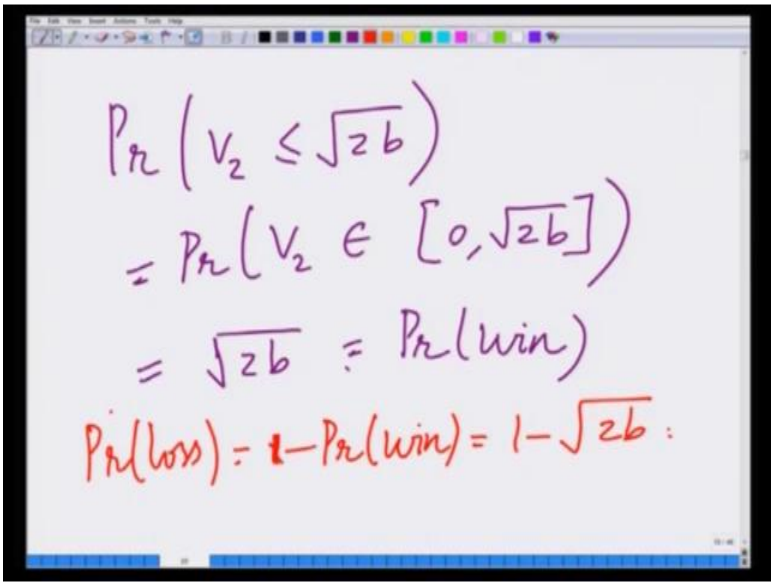

Therefore probability V 2 is less than or equal to square root of 2 b equals probability V 2 belongs to the interval 0 comma square root of 2 b. And remember, the probability V 2 belongs to the interval 0 comma square root of 2 b is simply the rent of the interval that is square root of 2 b. Therefore, the probability of winning is equal to square root of 2 b, this is equal to the probability that b wins bid b wins the auction for player 1.

And therefore, the probability of lose is 1 minus the probability of win that is 1 minus square root of 2 b. Therefore, probability of loss equals 1 minus probability of win which is 1 minus square root of 2 b.

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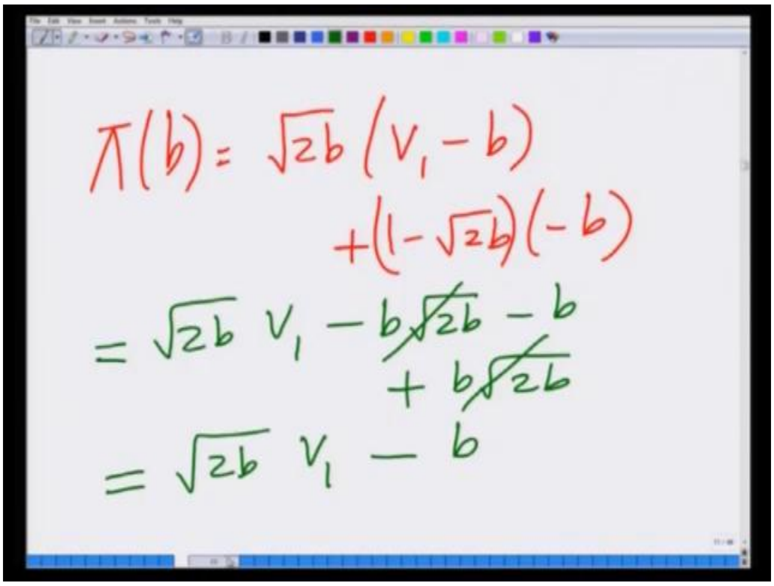

And therefore, now substituting this back in this expression for pi b, we have expected payoff pi b is probability of winning that is square root of 2 b times V 1 minus b plus probability of loss 1 minus square root of 2 b into minus b, which we can now expand as square root of 2 b V 1 minus b square root of 2 b minus b plus b square root of 2 b. These factors b square root of 2 b cancelled and therefore, what we have is square root of 2 b V 1 minus b.

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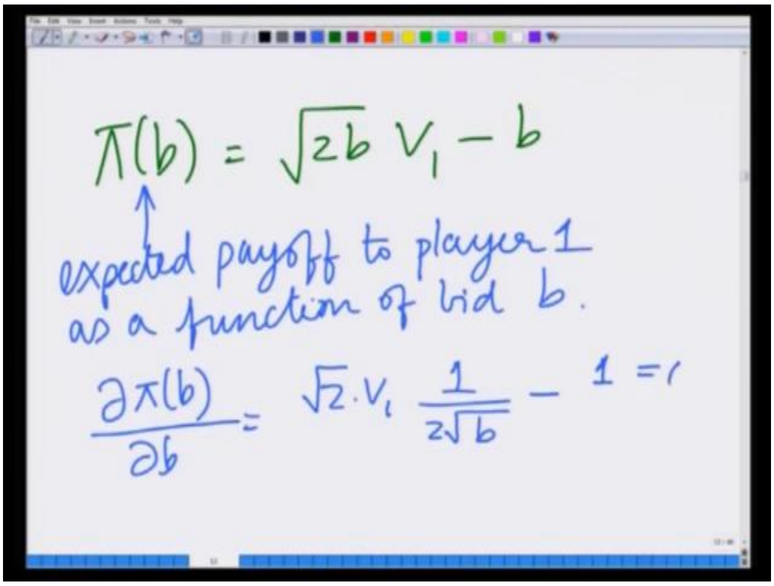

So, the probability pi of b is equal to square root of 2 b V 1 minus b remember this is expected payoff to player 1 as a function of bid b. So, the expected payoff pi of b to player 1 as a function of it is bid b is square root of 2 b times V 1 minus b. Now, therefore, to maximize this to find the best response bid b we have to maximize this, which means we have to differentiate with this with respect to b and set it equal to 0.

So, therefore, differentiating this with respect to b we have dou pi b by dou b which is equal to square root of 2 times V 1 times the derivative of square root of b is 1 over 2 root b minus the derivative of b with respect to b equal 0.

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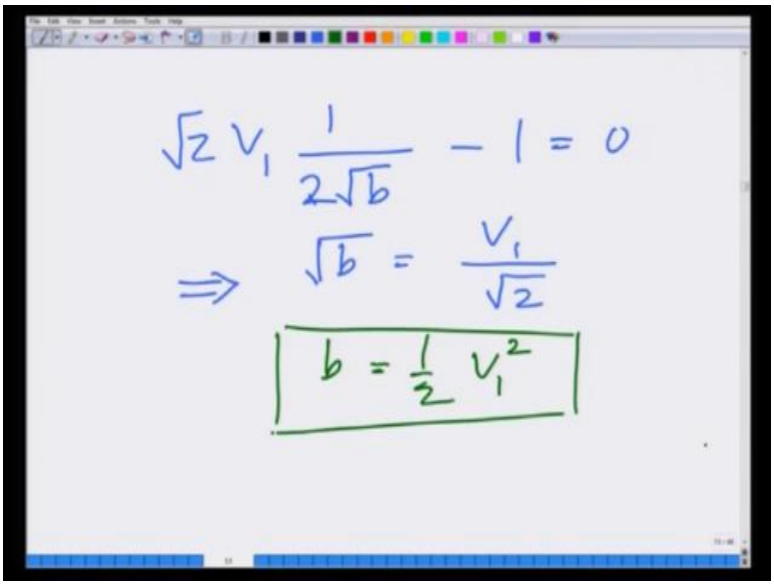

So, we have square root of 2 V 1 1 over twice square root of b minus 1 is equal to 0 which implies that square root of b is equal to V 1 divided by square root of 2 which implies that b is equal to half V 1 square. So, we started with the assumption that b 2 equals that player 2 is biding b 2 equals half V 2 square and we have demonstrated that the best response bid b of player 1 is b 1 equals half V 1 square.

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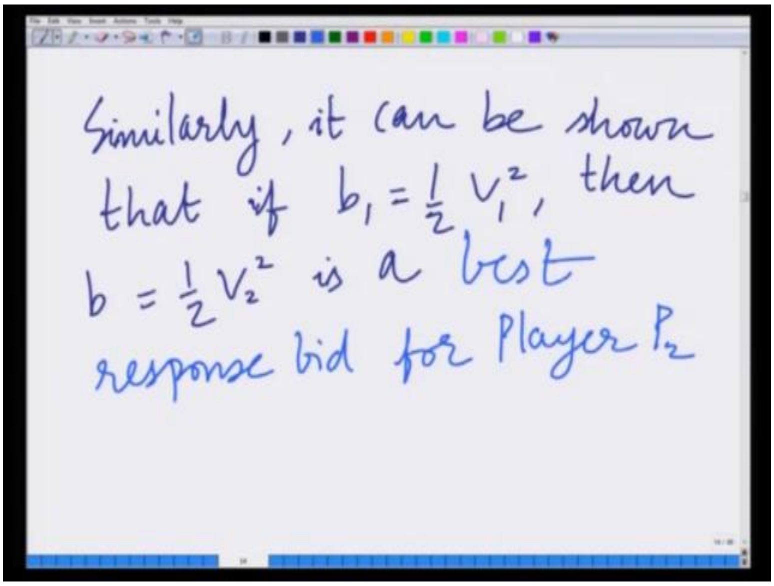

Similarly, it can be shown that if b 1 equals half V 1 square then b 2 equals half then b equals half V 2 square is a best response bid for player P 2. So, therefore, b 1 equals half V 1 square and b 2 equals half V 2 squared at the best responses to each other in this two player all pay auction game.

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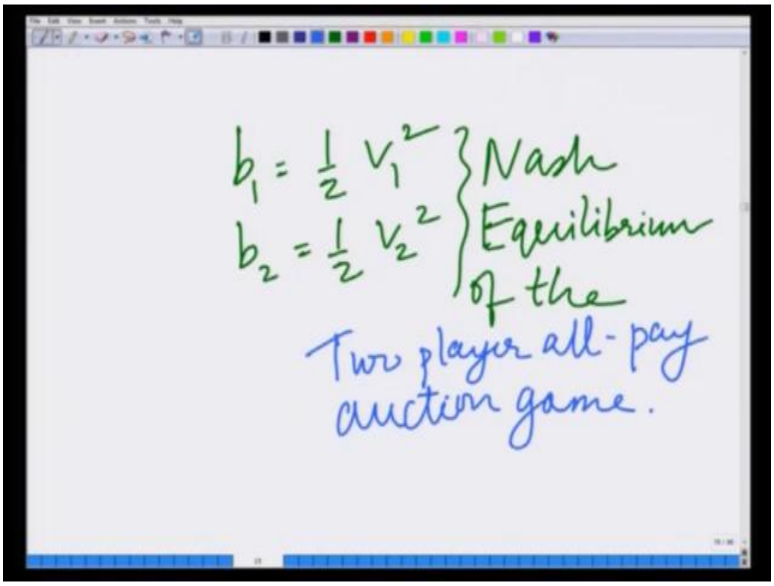

And therefore, we are saying that we have shown that b 1 equals half V 1 square b 2 equals half V 2 square, this is the Nash equilibrium of the two player all pay auction game that is b 1 equals half V 1 square and b 2 equals half V 2 square and we have also use the assumption that V 1 and V 2 are valuations which are independent random variables, distributed uniformly in the interval 0 to 1.

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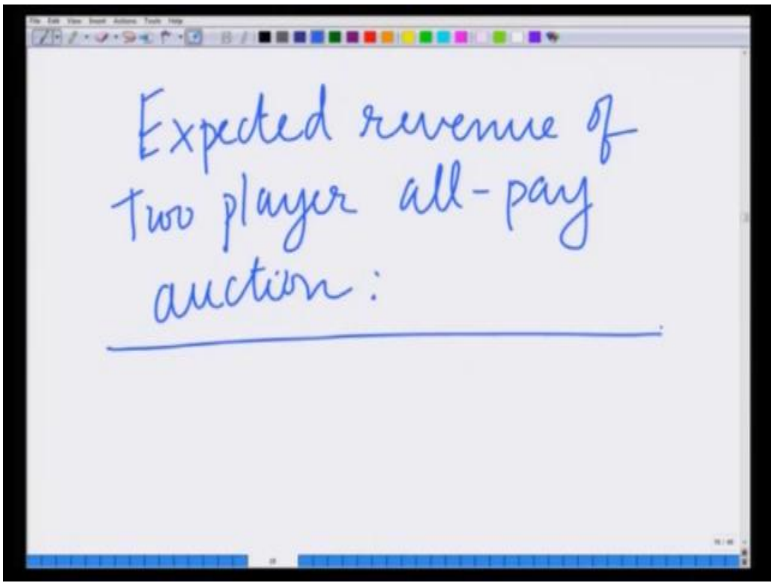

Let us now find the expected revenue of the two player all pay auction, let us find the expected revenue.

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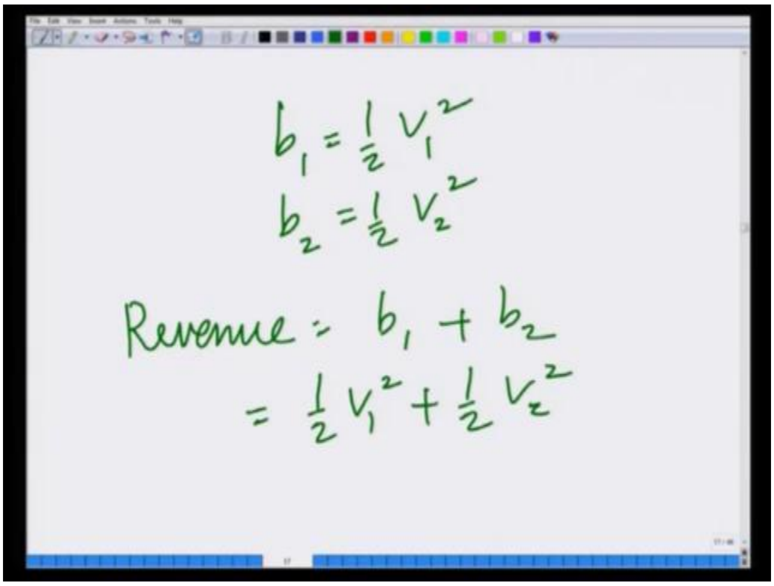

Now, we have seen that at Nash equilibrium the bids are b 1 equals half V 1 square b 2 equals half V 2 square. And remember these bids are paid irrespective of the outcome therefore, the revenue to the auctioneer equals b 1 plus b 2 which is equal to half V 1 square plus half V 2 square. Therefore, the revenue to the auction here is simply b 1 plus b 2, because irrespective of the outcomes the bids b 1 and b 2 are pay to the auction here. Therefore, the revenue is b 1 plus b 2 which is equal to half V 1 square plus half V 2 square.

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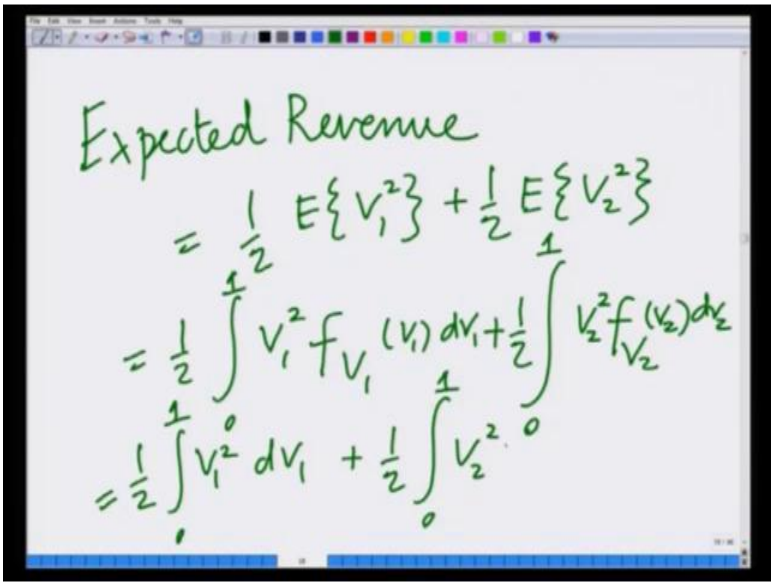

Now, we have to compute the expected revenue which is equal to half expected value of V 1 square plus half expected value of V 2 square which is equal to half integral between 0 and 1 V 1 square times the probability density f of V 1 times V 1 d V 1 plus half between 0 and 1. The expected value of V 2 square which is V 2 square f of V 2 multiplied with the probability density of V 2 times v d v 2, but V 1 and V 2 are uniform random variables distributed uniformly in the interval 0 to 1 therefore, f of V 1 and f of V 2 is simply unity therefore, this is equal to half integral 0 to 1 V 1 square d V 1 plus half integral 0 to 1 V 2 square d V 2.

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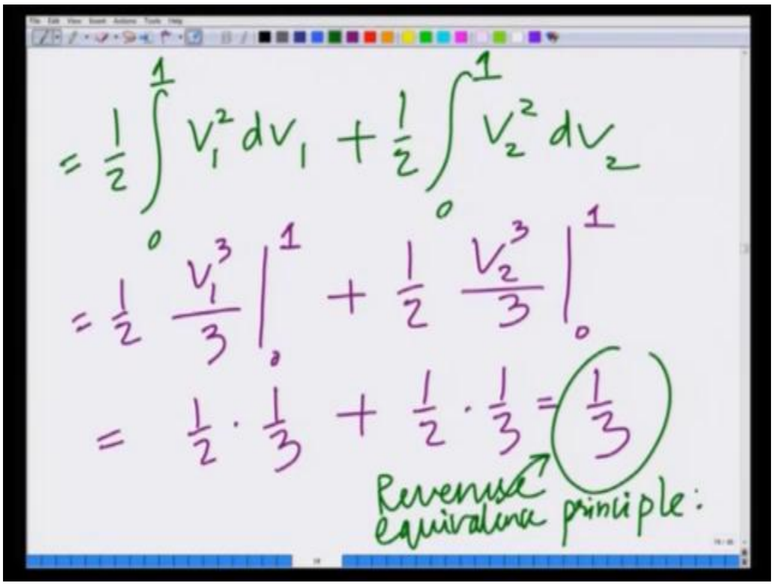

Which is equal to half integral 0 to 1 V 1 square d V 1 plus half integral 0 to 1 V 2 square d V 2, which is equal to half V 1 cube divided by 3 integrated between the limit 0 to 1 plus half V 2 cube divided by 3 integrated between the limits 0 to 1 which is equal to half into 1 by 3 plus half into 1 by 3 equals 1 by 3. And therefore, once again we are seeing the revenue of the all pay two player Bayesian auction is also in fact, 1 by 3.

And this should not come as a surprise, because we can predict this from the revenue equivalence principle. Remember, that we are talked about the revenue equivalence principle which states at irrespective of the auction format, the revenue to the auctioneer is equal. Therefore, what we are seen is the across the auction format all the Nash equilibrium bids are changing, what is not changing is the revenue to the auctioneer that is we had seen in the Bayesian first price auction, the second price auction and now in the Bayesian all pay auction that the revenue to the auctioneer is fixed which is 1 by 3.

Therefore, this is an illustration of the revenue equivalence principle. So, basically we are now what we have done is, we are looked at several auction formats, we are looked at the first price auction, the second price auction and all pay auction. And we have model these as Bayesian auctions with uncertainty in the valuations and we derived the Nash equilibrium of these various auctions and we are also derive the expected revenue and we have illustrated the revenue equivalence principles for these Bayesian auctions, let us conclude this module here.

Thank you very much.