Strategy: An Introduction to Game Theory
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Lecture – 12

Hello everyone, welcome to another module in this course Strategy, An Introduction to Game Theory. So, we have looked that variety of game so far in the different modules up to this point, I hope you are learning and enjoying the theory of games. So, let us start looking at a slightly more a different or slightly different variety of games with slightly different favor, then what we have been looking at so far. So, we are going to start looking at what are known as games with mixed strategies or randomized strategies.

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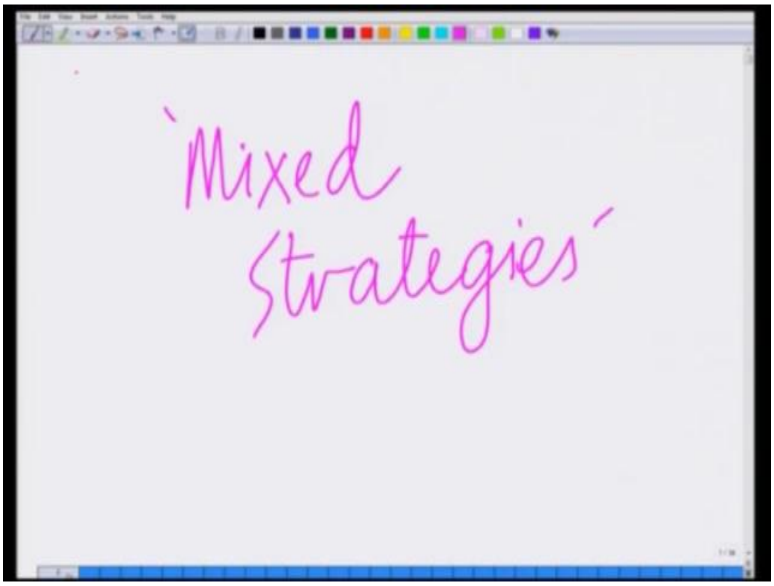

So, we are going to start looking now at mixed strategies or also what are known as randomized strategies. Now, to understand what are mixed strategies, let us take the example of a simple game, this game is also known as matching pennies.

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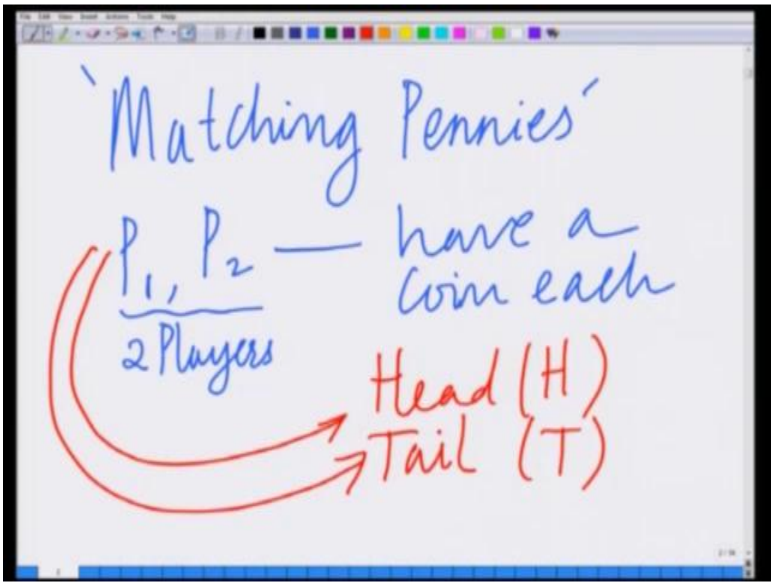

This is a game which is titled as matching pennies and the idea is as follows that is there are two players, let say there are two siblings P 1 and P 2 who have a pen or a coin each. P 1 comma P 2 these are the two players, they have a coin each, they can show one face of a coin that is they are free to choose any face of a coin. They can show one face of a coin, they can either choose a head; obviously, which we are going to denote by H or they can come up with the tail face that is at the head face or the tail face, that is each player P 1 and P2 can show either a head or a tail.

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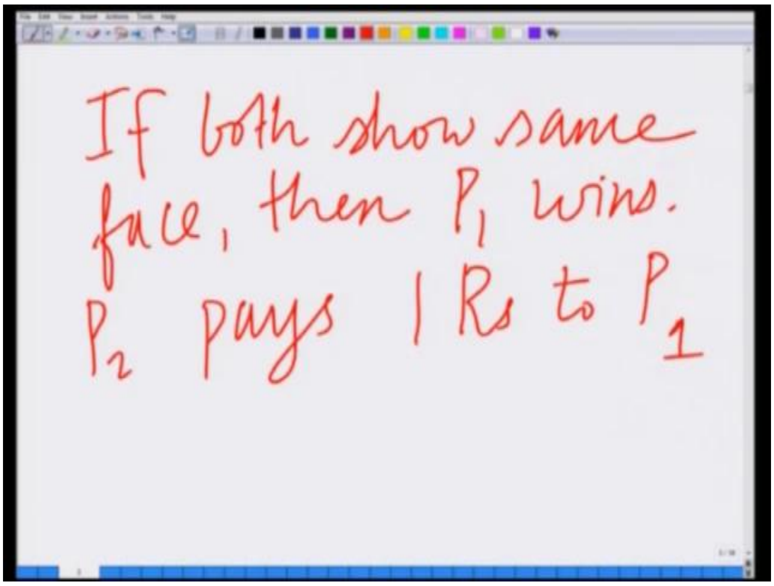

Now, the catch is that, if both show the same face that is, if both show same face then P 1 wins that is player 1 wins and P 2 pays, let say 1 rupee to P 1. So, if both show the same face, then P1 wins and P 2 has to pay 1 rupee, let say 2 can amount to P 1.

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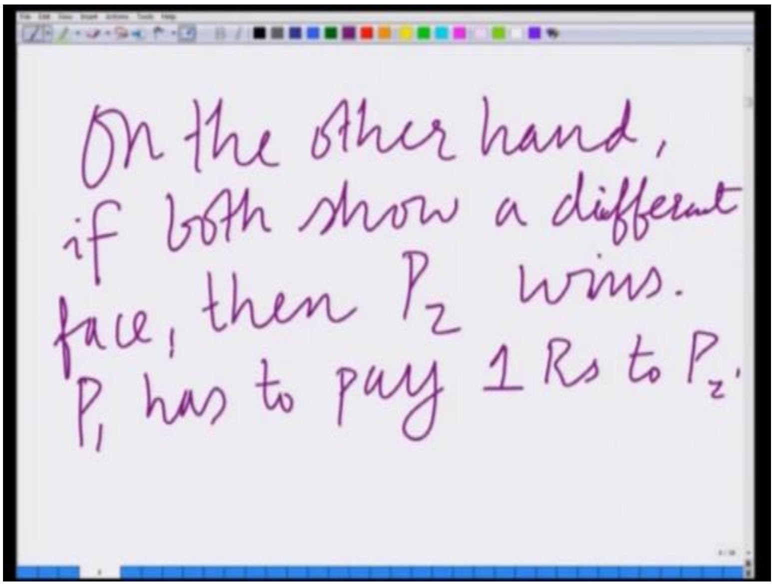

On the other hand, if both show a different face then P 2 wins and P 1 has to pay 1 rupee to P 2. So, let me just repeat the key idea P 1 and P 2 have a coin each, they are free to show either face that is either head or a tail, either the face they show is the same that P 1 wins and P 2 has to pay 1 rupee to P 1. On the other hand, if both of them come up with different face, that is P 1 shows heads and P 2 shows tails or P 1 show tail and P 2 shows heads, then P 2 wins and P 1 has to pay 1 rupee to P 2. So, that is the game and again let us similar to what we have done previously.

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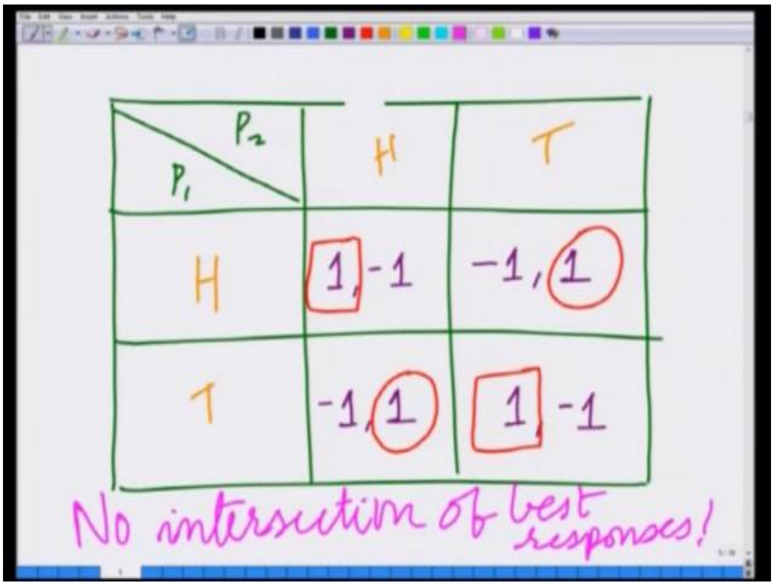

Let us formulate this in terms of the game table, which is a very convenient way to represent the game which all of you should be very familiar with so on. P 1 and P 2 they have two options to either show heads or tails, heads or tails. If both show heads, then P 1 wins, in which case he gets 1 rupee and P 2 gets minus 1; if both show tails then again P 1 wins, in which case he gets 1 and P 2 gets minus 1.

On the other hand, if P 1 shows heads and P 2 shows tails, P 2 wins P 1 has to pay 1 rupee to P 2. So, P 1 gets minus 1, P 2 gets 1, if P 1 shows tails and P 2 shows heads again P 2 wins in which case P 1 gets minus 1 and P 2 gets 1. So, this is the game, this has row player P 1, column player P 2 each has to choose one action that is heads or tails represented by heads or tail.

Now, of course, let us try to find the Nash equilibrium again by the best response, if P 2 chooses head, best response of P 1 is to of course choose head. If P 2 chooses tail, best response of P 2, P 1 is to choose tails, because P 2 wins when he matches P 1. On the other hand if P 1 chooses heads, best response of P 2 is to choose tails and if P 1 chooses tails, best response of P 2 is to choose heads, because heads gives him 1, tails gives him minus 1.

And therefore, now you can see in this game there is no intersection of best responses. So, we are now seeing a game table in which there is no intersection of best responses. So, we cannot find the Nash equilibrium of this game using the techniques that we have learnt so far and this is a specific example of a game, where each is considering a single strategy, such a strategy is also known as a pure strategy.

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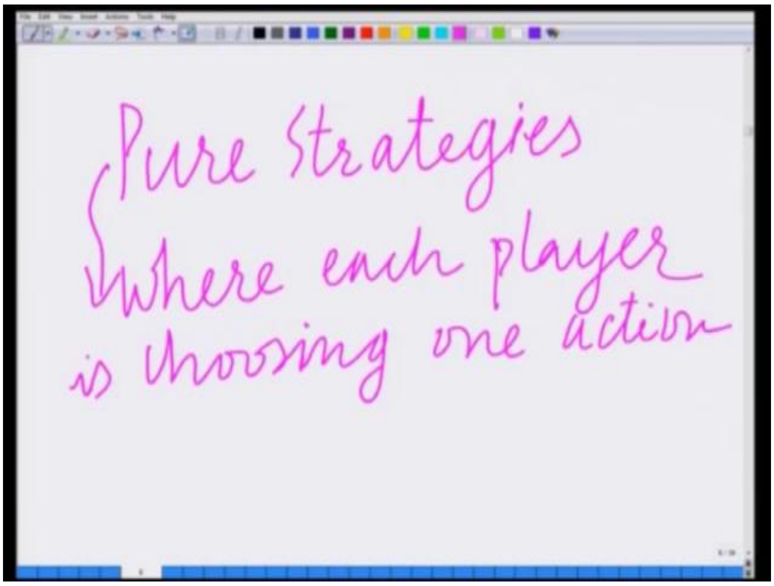

So, far we have consider pure strategies, where each player is choosing one action, that is in the prisoner's dilemma each prisoner is choosing either confess or deny. Similarly, in the case of the coordination game, each one is choosing either the deer or the rabbit. What we are going to now start considering are a different set of games, in which the players no need to choose one strategy at every instant. But, they can randomly choose between two different strategies with a certain probability for each strategies.

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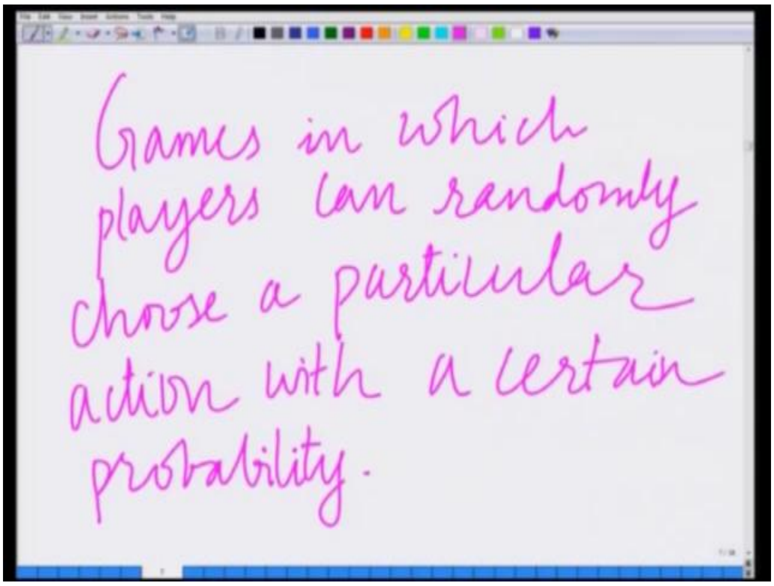

That is we are going to consider games in which players can randomly choose a particular action with a certain probability, that is that we are considering a game in which let us take an example of this matching pennies game itself that is, a player can randomly choose between heads and tails. With a certain probability he can choose heads, with a certain probability he can choose tails and such a game or such a strategy is known as a mixed strategy or a randomized strategy.

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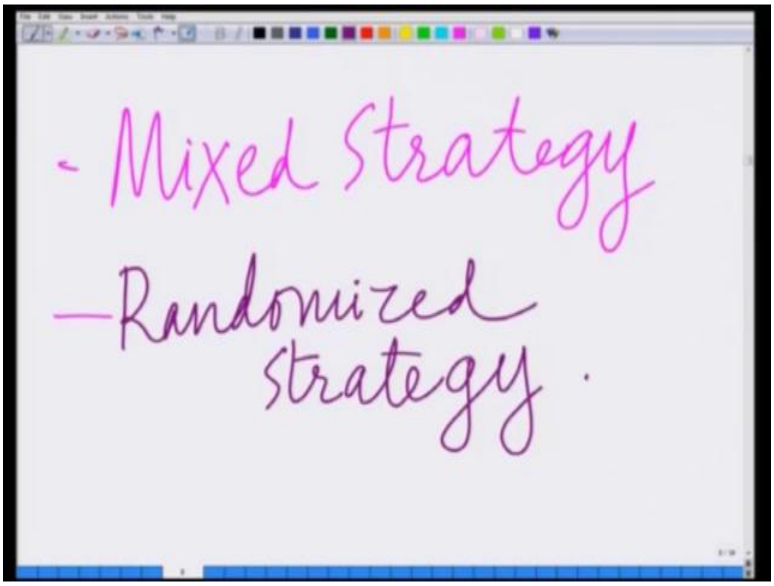

This is known as a mixed strategy, because you are mixing between two different actions that is you are mixing between heads and tails or a randomized strategy, because you are randomly choosing one action versus the other. This is similar to, so as to say randomly choosing one action versus the other. So, this is known as a mixed strategy, this also known as a randomized strategy, where you are randomly choosing with a certain probability, you are randomly choosing or with certain probabilities, you are randomly choosing one action versus the other.

So, let us now try to analyze this game in terms of randomized strategies, let us go back to this game ((Refer Time: 09:31)), let us now try to analyze this game in terms of randomized strategy. Let say player 1 is choosing heads with probability p and therefore, naturally he is choosing tails with probability 1 minus p. To people who are not familiar with the basics of probability, let me just give a brief recap of the probabilities, the probability is a positive quantity which is less than 1.

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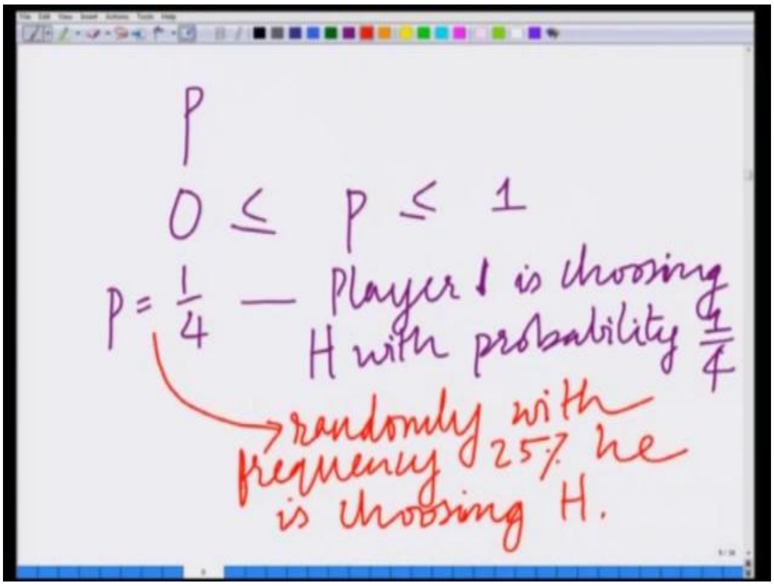

Therefore, the probability p has to lie between 0 and 1 let us say for example, p equals 1 by 4th which means player 1 is choosing heads with a probability 1 by 4th, which means players 1 is choosing H with probability 1 by 4. That means, randomly with frequency that is 1 by 4th of the times randomly 1 4th of the total number of the times the game is played, he is choosing heads approximate 1 and average 1 4th of the time he is randomly that is frequency 1 4th that is 1 4th of the total times the game is played, he is choosing heads.

So, randomly or you can say with frequency 25 percent, so randomly 25 percent of the time he is choosing heads, which means 75 percent of the time he should be choosing tails, which means the probability of tails is 0.75 or the frequency is of tail is 0.75.

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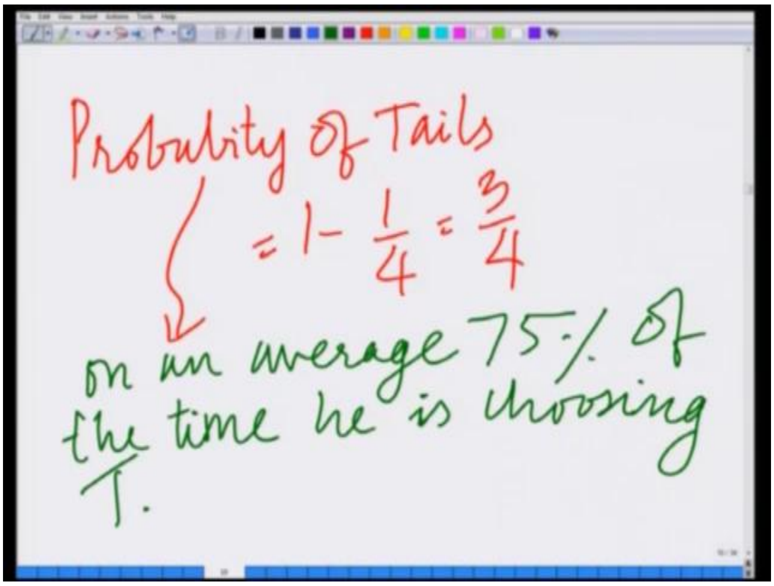

So, which means the probability 1 minus 1 by 4th is equal to 3 by 4 that is to say on an average 75 percent of the time he is choosing tails. Therefore, p equals 1 by 4th means about 1 4th of the total time instants that is 25 percent of the time is choosing heads and 75 percent of the time randomly he is choosing tails. Similarly, if p equals let say 0.6 it means 60 percent of the time randomly he is choosing heads and 40 percent of the time that is the p equals 0.6 1 minus p is 0.4 which means 40 percent of the time randomly you choosing test.

So, as p various between 0 and 1 the relative frequencies of this heads verses of the choice of this heads verses tails is also varying appropriately. So, what we are saying is we are going to start considering a different kind of games in which the players do not need to choose a fix action. But, randomly choose between a set of actions and we have define a probability that in this matching pennies game, let say that the p is the ((Refer Time: 13:17)) p denotes the probability with which player 1 is choosing heads then naturally 1 minus p is the probability with which player 1 is choosing tails.

And this quantity p is probability which means it is a positive quantity greater than 0 and less than 1. So, 0 lies between that therefore, p lies between 0 and 1 and therefore, what we already said is, let me draw this table again just to illustrate this.

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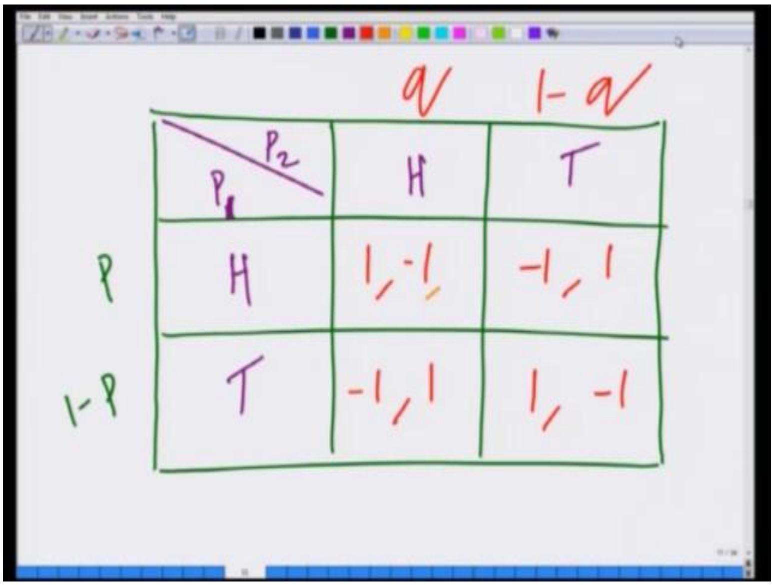

Clearly, we are considering matching pennies game with two players p 1, P 2 each has an option to choose head or tail, head or tail and the payoffs are 1 comma 1, 1 comma minus 1, minus 1 comma 1, minus 1 comma 1. Now, let say player 1 and now we also set that player 1 is choosing heads with probability p, tails is with probability p. Now, let us try to compute the payoffs of player 2, if player 2 chooses heads then with probability 1 p he will meet player 1, he will find player 1 playing heads.

In which case is payoff minus 1 with probability 1 minus p that is an average frequency 1 minus p he will find player 1 player in tails in which cases is payoff is 1. So, fraction p of the time he will get a payoff of minus 1 fraction 1 minus p of the time is going to get a payoff of 1.

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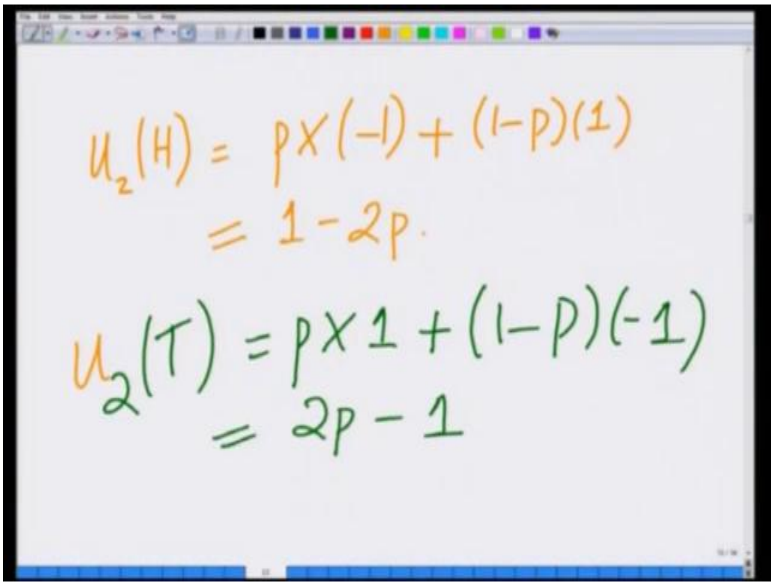

And therefore, the average payoff of player 2 is that is if player 2 chooses heads is average payoff is fraction p of the time he gets minus 1 plus fraction 1 minus p of the time he gets 1. So, therefore, is a average payoff is p times minus 1 plus 1 minus p times 1 which is basically equal to 1 minus 2 p ((Refer Time: 15:44)). So, let me repeat this argument again, because this is a at least initially for those you are not very familiar with probability this can be a slightly trig argument and it also confusing one.

So, I let me repeat it, so that you understand it thoroughly, we are assuming the player 1 chooses heads with probability p and tails with probability 1 minus p and if player 2 is always choosing heads, then we have with probability p he meets player 1 are encounters player 1 playing heads. So, with probability p is payoff minus 1 is probability 1 minus p he encounters player 1 playing tails in which his payoff is 1. So, fraction p of the time his payoff is minus 1, fraction 1 minus p of the time this payoff is 1. So, is average payoff is p into minus 1 plus 1 minus p into 1 which is equal to 1 minus 2 p.

Similarly, now on the other hand if player 2 is always choosing tails then fraction p of the time his payoff is 1 fraction p of the time when P 1 choosing heads is payoff is 1. Fraction 1 minus p of the time is payoff is minus 1 therefore, the average payoff this time is p into 1 plus 1 minus p into minus 1 which is equal to 2 p minus 1. So, what we are saying is, if player 2 is always choosing tails then fraction p of the time is payoff is 1, fraction 1 minus payoff the time is payoff is minus 1 therefore, is net payoff is p times 1 plus 1 minus p times minus 1 which is equal to 2 p minus 1.

So, these are the two payoff that player 2 receives if we chooses either heads his payoff is are minus 2 p, if you always chooses a tails his payoff is 2 p minus 1. Now, this is the situation then player 1 is using probabilities p and 1 minus p. Now, when is if you think about it when is player 2 going to randomly choose between heads and tails, because we are assuming that both the players are using randomized or mixed strategy, when is player 2 randomly going to choose between heads and tails.

For instance, his payoff from heads is 1 minus 2 p his payoff from tails is 2 p minus 1, if the payoff from heads that is 1 minus 2 p is greater than the payoff from tails, then player 2 is always going to choose heads. Similarly, if the payoff from tails to player 2 is always greater than the payoff from heads that is 2 p minus 1 is greater than 1 minus 2 p then he is always going to choose tails. And therefore, he is going to randomly choose between heads and tails only when the payoff from both of these are equal and this is the important principle to understand.

One would randomly mixed between two actions a 1 and a 2 only if both the actions yield and identical payoff, another way of saying this is that the player is in different both these actions.

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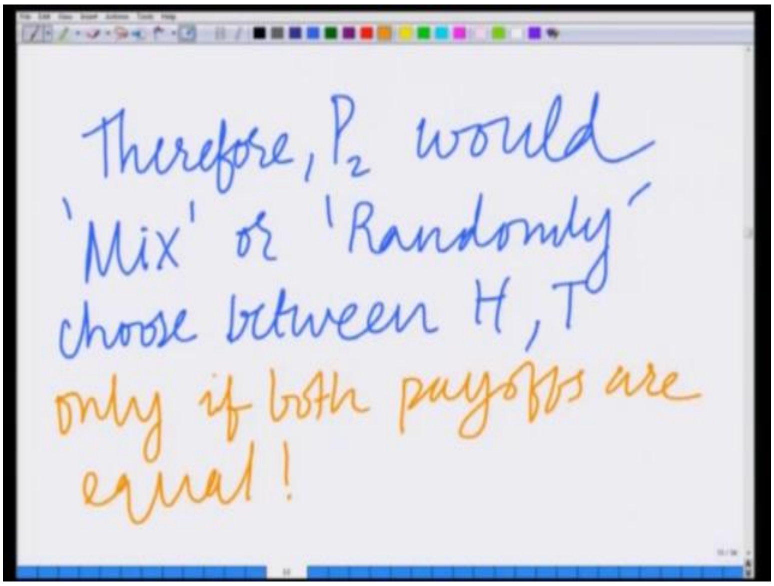

Therefore, P 2 would mix or randomly choose between H comma T only if both payoffs are equal. Therefore, you randomly choose between head and tail only if both payoffs are equal. The payoff from head is greater than tail then you would always choose head, if the payoff from the tail is greater than head then you would always choose tail and he would randomly choose between of one verses the other only when the payoff of both of them are equal or is in different to both of them.

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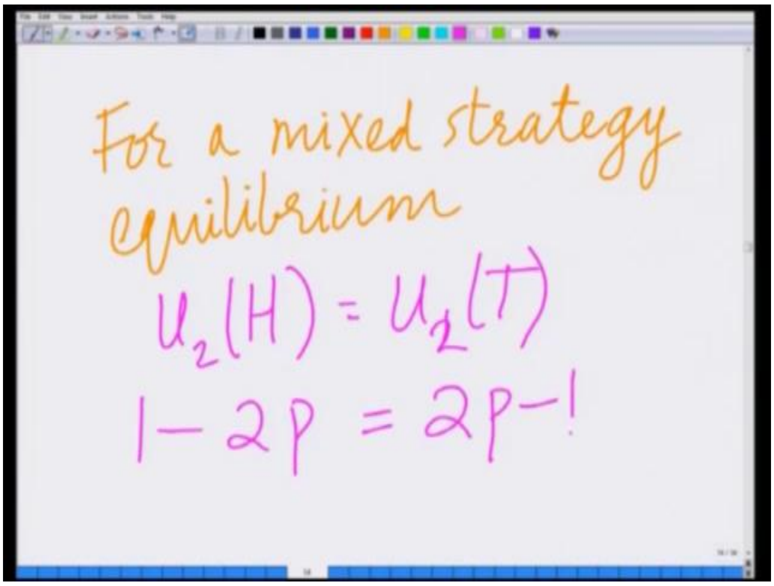

And therefore, for a mix strategy Nash equilibrium for a mix strategy equilibrium that is for player 2 to choose randomly between heads and tails it must be the case that u 2 of heads equals u 1 or of u 2 of tails which means I have to have 1 minus 2 p equals 2 p minus 1 which means I have 1 minus 2 p equals 2 p minus 1.

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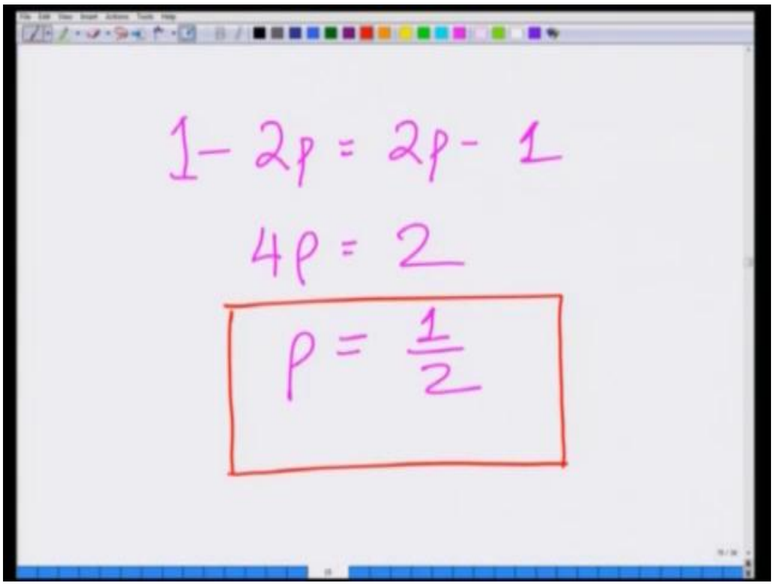

And solving this I get 1 minus 2 p equals 2 p minus 1 which means 4 p equals 2 which means p equals half. And therefore, what the says if p is equal to half that is when p is equal to half when player 1 is choosing heads with probability half and tails with probability half that is randomly mixing are his randomly choosing heads 50 percent of the time and tails 50 percent of the time it make sense are P 2 is going to imply a randomized strategy. Because, in only that scenario his payoff from heads which is equal to 1 minus 2 p is equal to his payoff from tails which is 2 p minus 1.

And therefore, it makes sense for him to choose it imply a randomized strategy. So, intern we can also say that the strategy employed by P 1 for a mix strategy Nash equilibrium should be such that are the probability chosen by P 1 should be such that P 2 or player 2 becomes in different two choosing between heads and tails and only then is going randomly choose between heads and tails. And therefore, we get calculating equating the payoff 2 player 2 from heads and tails, we determine that p is equal to half.

And this is an interesting aspect, determine the mixer of player 1 that is to determine the probability p of player 1 we have to look at the payoff of player 2 and we have to find that point at which player 2 is in different between these two actions and that is what we said and he is in different between these two actions, when the payoffs are equal that is 1 minus 2 p equals 2 p minus 1. Now, let us similarly find the mixer for player 2, let us assume player 2 is choosing heads with probability q and tails with probability 1 minus q.

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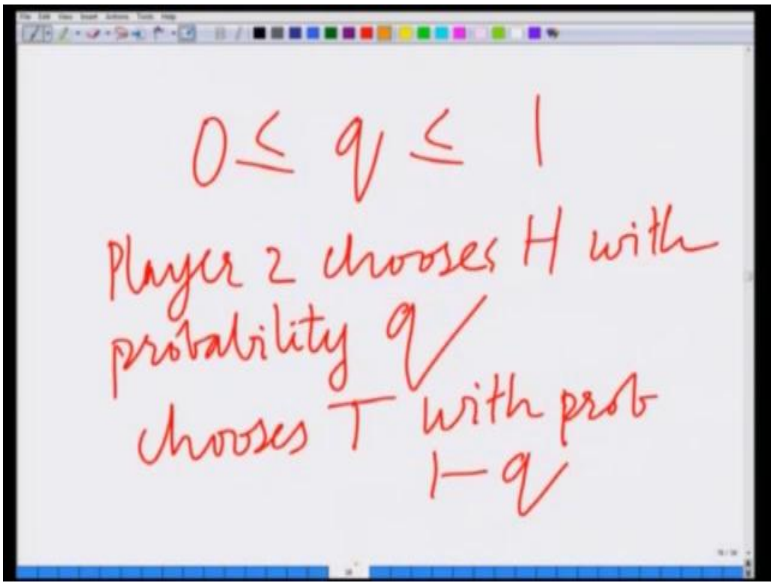

Where again q is a probability therefore, we have to have 0 less than equal to q less than equal to half. So, player 2 chooses H with probability q I am chooses T with probability

1 minus q. Again to determine q remember we have to look at the payoffs of player 1, determine the parameter the mixer of player 2 we have to look at the payoffs of player 1. Now, if player 1 chooses H his payoff to choosing H is with probability q p encounters player 2 playing H in which case he gets 1 with probability 1 minus q he encounters player 2 playing tails in which case he gets minus 1. Therefore, his payoff that is fraction q he gets a payoff of 1, fraction 1 minus q he gets a payoff of minus 1. Therefore, his average payoff is q times 1 plus 1 minus q times minus 1.

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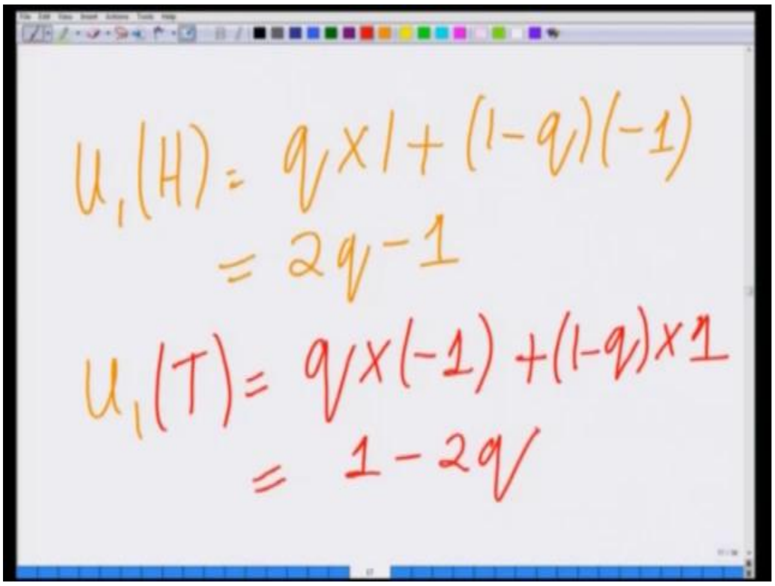

So, average payoff to always choosing heads, the payoff of player 1 always choosing heads is q times 1 plus 1 minus q times minus 1 which is equal to 2 q minus 1. Let me repeat that argument again, if player 2 is using a mixer q 1 minus q that is the probability q is choosing heads, probability 1 minus q is choosing tails then the average payoff of player 1 for always choosing heads is q times 1 plus 1 minus q times minus 1 which is basically 2 q minus 1.

Similarly, if player 1 chooses tails with frequency q are with probability q he is going to encounter player 2 choosing heads. In which case is payoff is minus 1 and with probability 1 minus q is going to encounter player 2 playing tails in which case is payoff is 1 therefore, his average payoff is q times minus 1 plus 1 minus q times 1. So, average payoff of u 1 if he is always using tails is q times minus 1 plus 1 minus q times 1 with probability q he finds player 2 playing heads in which case he loses 1 rupees, so it is probability, so is average payoff is q into minus 1 plus with probability 1 minus q he finds player 2 also playing tails in which case wins 1 rupee.

So, that contributes 1 minus q times 1, so is net average payoff is q times minus 1 plus 1 minus q times 1 which is equal to 1 minus 2 q. So, as a function of the probability q or the mixer q employed by player 2, the payoff the player 1 for always choosing heads is 2 q minus 1, the payoff the player 1 for always choosing tails is 1 minus 2 q. Now, of course, again repeating the same argument, if 2 q minus 1 is greater than 1 minus 2 q then player 1 will always choose heads. On the other hand, if 1 minus 2 q is always greater than 2 q minus 1 then player 1 will always choose the heads.

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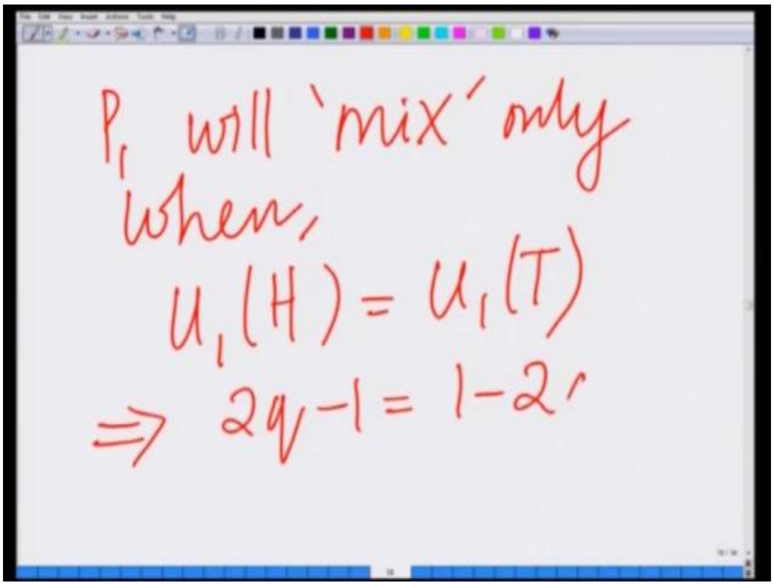

He will randomly choosing between heads and tails only when his in different to one verses the other that is when he will randomly player 1 will mix are employ a mix strategy only when his payoff u 1 from heads is equal to u 1 from tails implies that 2 q minus 1 equals 1 minus 2 q.

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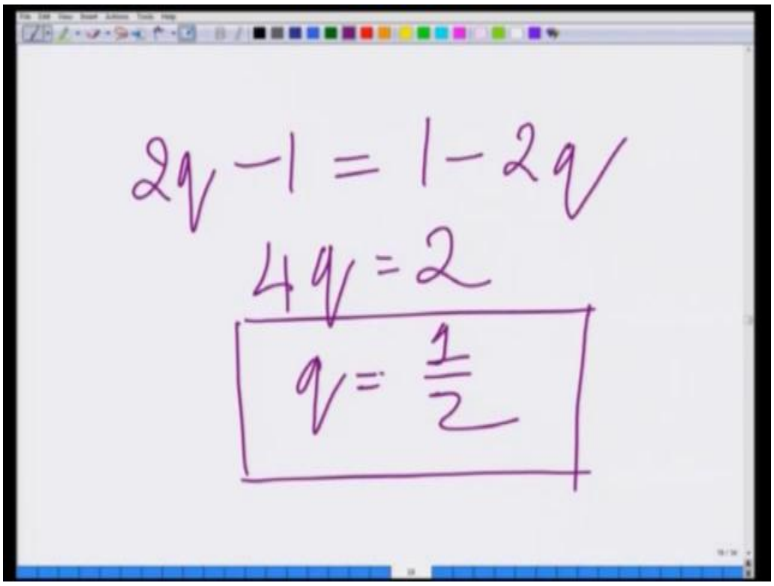

And solving this we have 4 q equals 2 implies q equals, so again what we are finding is that for player 1 to be in different between heads and tails to use randomized strategy, we are mixed strategy we have to have q equals that is the mixer implied by player 2 should be q equals half which means 1 minus q equals half therefore, he is mixing heads and tails with probability half and half.

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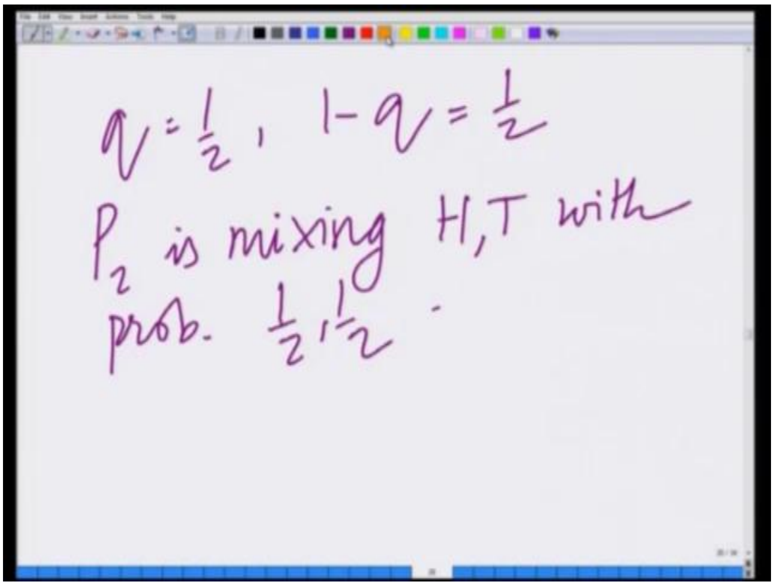

So, we have q equals half 1 minus q equals therefore, also half therefore, P 2 is mixing H comma T with probability half comma half.

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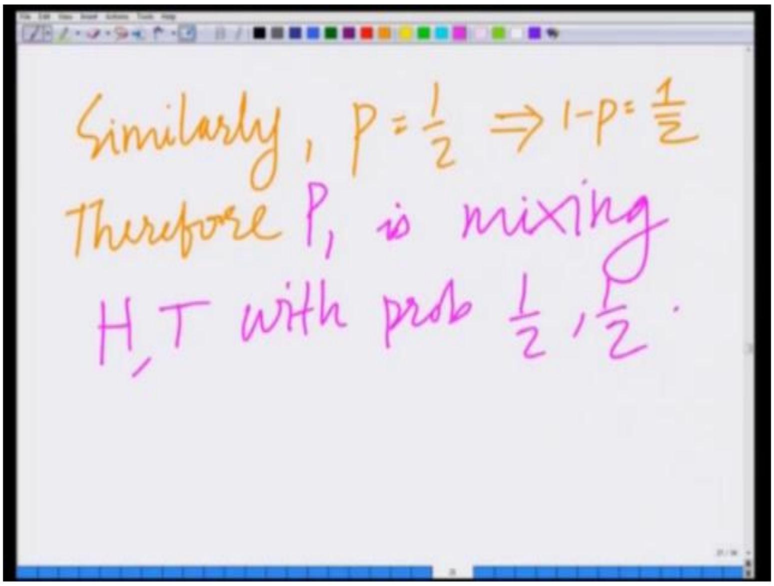

Similarly, we have p equals half implies 1 minus p equals half therefore, P 1 is mixing H comma T with probabilities half comma half.

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Therefore, the mix strategy Nash equilibrium for this game can now be formulated as the mix strategy Nash equilibrium are the randomized strategy the mix strategy Nash equilibrium for this game is now given as well, we have to give one mixer for each player that is half comma half for player 1 half comma half for player 2. So, this is the mix strategy you are player 2 or player 1 and this is the mixed strategy of player 2.

What do we mean by mix strategy? A mix strategy basically denotes the probabilities with which each player is mixing the various actions. So, half comma half says that player 1 is randomly choose between heads and tails with probability half, each and half comma half player 2 says that player 2 is also randomly choosing between heads and tails with probability half edge and these two mix strategies or part of this mix strategy Nash equilibrium.

So, at Nash equilibrium player 1 is randomly choosing between heads and tails with probability half and his choosing and player 2 is randomly choosing between heads and tails with probability half and this is the mixed strategy Nash equilibrium for this game. So, today we have learn to new concept which is a mix strategy Nash equilibrium in comparison and in contrast to previous games where we had looked at pure strategy Nash equilibrium there each player is choosing a pure action in this in a mix strategy Nash equilibrium each player is randomly choosing a one action against the other.

And we get considered a matching pennies, remember we said that this matching pennies game does not have any Nash equilibrium in pure strategies. On the other hand it has Nash equilibrium in mix strategies, where each player is mixing heads and tails with probability half. And also it is worth mentioning at this point let it turns out that the mixer is exactly half that is the probability is both the action is are equal and the mixers at the same for the both players.

But, these need not be true for a general tips that is the probability is for all the action for the different actions can be different and also the mixers for the different players can be different. And we are going to look at some examples of games that have this property that these probabilities are different in future examples. So, this is again in different kind of a game I hope you have understood, please go through the lecture again and try to understand it thoroughly with their some aspects which are not clear.

Thank you very much and we will conclude this module here, thank you.