Bandit Algorithm (Online Machine Learning)
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Lecture – 49
Construction of Confidence Ellipsoids – II

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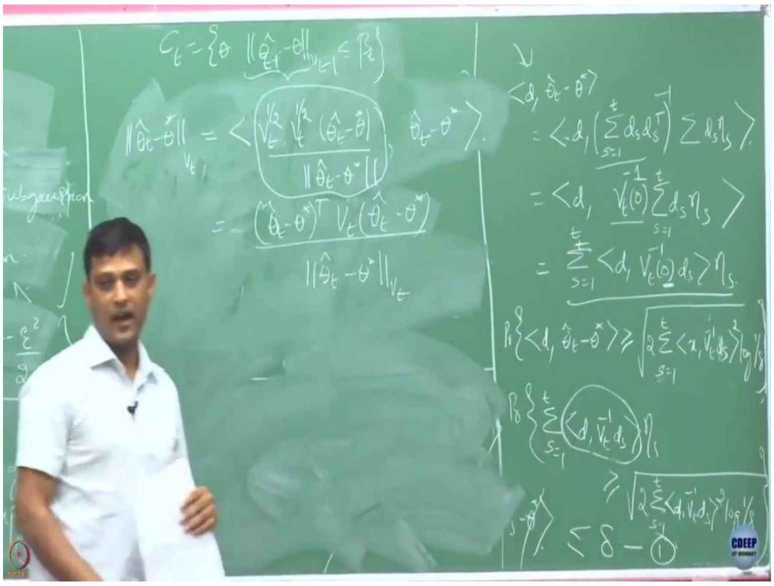

So, now you see that this quantity what I have done is basically I have express it like some quantity here. So, what is this? This quantity if you look at this finally, this just needs a vector right. So, this is a matrix product, this is a matrix and now this is a vector. This product it gives you a vector and this denominator is some constant.

So, this quantity is a vector and now you are looking at its inner product with this quantity. Then it is like similar to what I was basically doing it here right, but instead of d I have replaced it by this entire quantity; instead of this vector d I have another quantity here.

So, if I want to now apply some bound probability bound on this like in this fashion the way I did, I need to just then consider this and then use whatever I have done here. But, there is an issue here, when I did take this inner product here right; this d is a particular arm and I took the inner product on this error term here. But now here this quantity whatever which I want to treat it as a d, it is not fixed quantity. It is a random quantity. Why is that?

Student: θ^*.

Because θ hat has come into picture. So, now, if I want to apply any tail bound on this like I want to consider what is the probability that this guy is going to be greater than less than or equals to δ something of that sort; I cannot directly use what I have done here.

Because here to do this I have to deal the projection of a random arm on my error term. So, how to now account for the randomness in this quantity? Ok. So, for that we have to do some more some more applied some more tricks and that trick comes something called covering argument.

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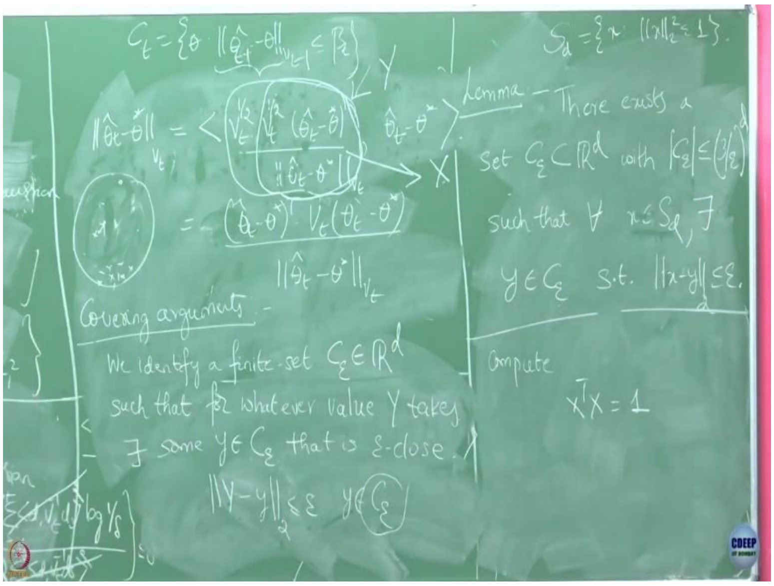

See this quantity here is random and this will lead to an column vector which could be any, it can take any possible value in my R_d space right.

Because, this $\widehat{\theta_t}$ is a random quantity because of this, this quantity can be anywhere in my R_d space. I do not know what is this value is it depends on the value of $\widehat{\theta_t}$ and also other quantities θ* and what was the matrix V_t at that time. So, to do this it can take all possible values many possible values in the space R_d.

We are going to first discretize this possible values and now try to focus on the discrete possible values. So, this is this can be any value in R_d space which could be like uncountably large. So, what we are going to do is then we have such a situation right, we usually go for and apply the union bound right we consider.

So, right now this quantity has become a random quantity for us because of this estimated term here. Whenever we have a randomness quantity, if you remember when we did it in the multi arm bandit; so, you remember in the multi arm bandit the number of the times R pulled is its the random quantity right. How did we and when we wanted to apply Hoeffding's inequality how did we do it? We consider all possible values of the number of pulls and then took a union bound on that.

So, but there applying a union bound was because then the values taken by the number of pulls was only finite that is 1, 2 all the way up to t, but here this quantity can take uncountably many values. So, first what we are going to do is we are going to discretize it and then we are going to see that whatever possible value it can take. We will we are going to say that there exists a point in my discretize space which is going to be arbitrarily close to this.

So, we will make that notion precise and now once we have discretized we have only finitely many points in that set and then we apply a union bound on that. So, what we mean by. So, we identify finite set C_ϵ belonging to R_d such that. So, let me call this entire quantity Y for whatever value Y takes there exist some y belongs to C_ϵ that is ϵ-close.

So, what I mean by this? For whatever value of Y, there exists some y such that this holds where y belongs to my C_ϵ. So, understand this what I mean by covering this I am basically saying that see my setup points that is the possible values Y can take it could be uncountably many. Now, I want to make an approximation of this by a finite points such that whatever value this guy takes there exists a point in my set such that is going to be ϵ-close to that to value y.

That is whatever value Y has taken there exists a y from a set C_ϵ such that this difference is going to be less than ϵ. This is what basically I have done the discretization right. Now, the question is whether such a $C\epsilon$ exist? And if at all it exists and of course, that C_ϵ I have denoted as with the subscript ϵ to mean that depends on the size of ϵ. So, naturally if you are expecting a small ϵ what you expect the size of C_ϵ to be?

Student: larger.

It is going to be large right.

Student: Yes.

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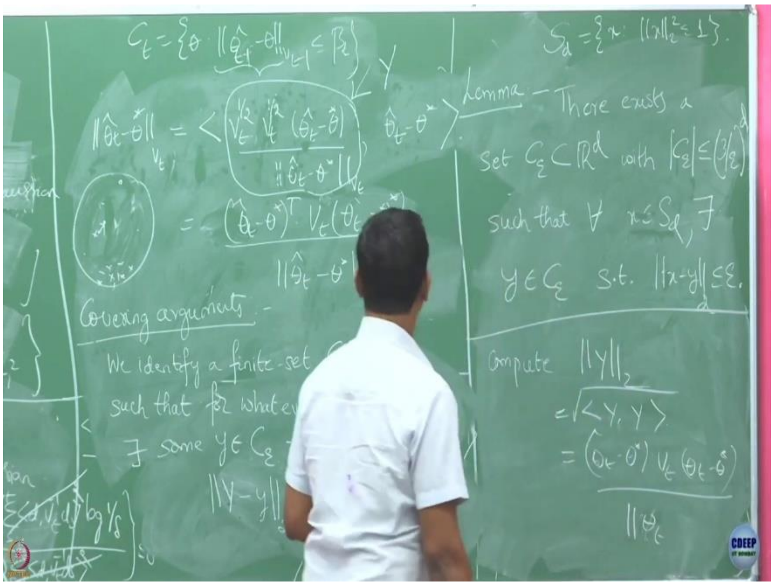

Here is a lemma which says existence of such a covering set. So, I am going to denote a set first $S_d = \{x: ||x||_2^2 \le 1\}$. S_d is such of all x such that their norm is less than or equals to 1, I mean basically it is a unit ball in my dimension d.

So, now there exist a set C_ϵ with cardinality of C_ϵ upper bounded by $\left(\frac{3}{\epsilon}\right)^d$ such that for all $x \in S_d$, there exist y belong to C_ϵ such that; you follow what is this lemma is telling? It is telling that if you have a unit ball then that has that can have uncountably many points in that right.

Now, we are saying that I can come up with the discretized version of that set which will have at most $\left(\frac{3}{\epsilon}\right)^d$ point such that if you give me any point x in my in your in this unit ball. Then I will have a corresponding y in my set C such that their difference is no more going to be ϵ.

So, that is basically you have. So, this is your ball let us say now you have discretized it with some finite number of points. Now, we are saying that if you give me any point let us say this is the point you have given, I will be able to come up with some point which is going to be within ϵ distance from that point. So, I should be able to do it for any point given in my ball, I should be able to come up with some point from my this discrete set such that this quantity happens ok.

So, this is true and we are just going to take this lemma, I am not going to prove this ok. So, this is a and this is a pretty handy lemma to know you can use it elsewhere also. Like whenever you want to do a discretization and by the loss you are going to incur by doing a discretization will be at most ϵ; if your number of elements in your discretization is going to be this much. But, one has to be also careful that how where this discrete point are going to lie.

You cannot assume that in this all my discretized point only lie in some small region. Then if I choose a point here, I will not be able to find a good approximation to that point ok. So, of naturally this discretized points has to kind of cover this entire region. So, that is why we are looking for a cover for this set, but that consist of only discrete finitely many points in it.

Fine. So, now, the question is this lemma holds only for the case where my point I can cover points which are coming from a unit ball right. But, now the set Y whatever this Y does this come from a unit ball?

Student: No.

That is whether this Y belongs to the set S_d, it will just let us just compute just compute let us see what this gives you. So, this should be what is this? This will give you ν. So, what is this? This is nothing, but $\langle Y, Y \rangle$ itself right.

Student: Multiplied by.

Let us see I only want to take this part x. So, let us leave this one V_t to the power half ok.

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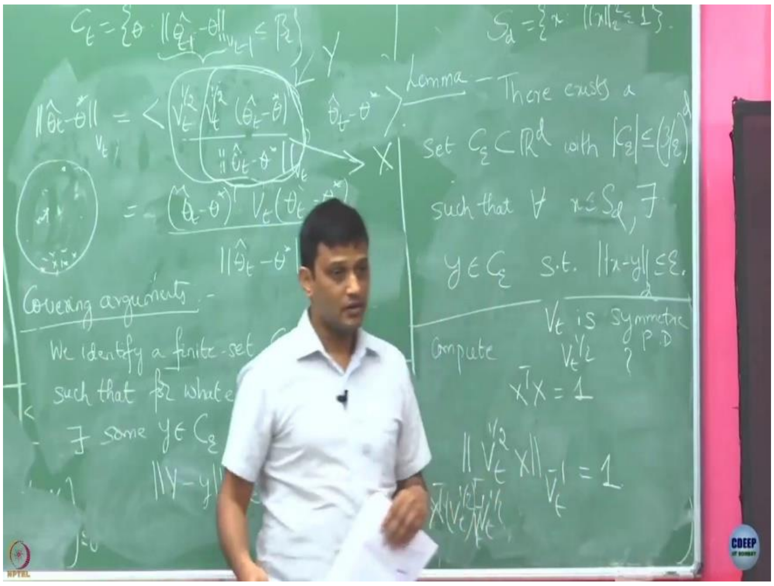

Now just take this X. So, now, let us see whether my X what is this X? Is this X going to lie in my unit ball? So, just check that this guy X transpose X is going to be 1.

And now this X entire thing ok, then I want you to check what is this quantity half of this X V t inverse ($\left\|V_t^{\frac{1}{2}}X\right\|_{V_t^{-1}} = 1$); can I check what is this quantity is going to look like? So, this is that is nothing, but the norm of X right, this is again going to be 1.

Student: 1.

Fine. So, let us have this let us see ok. So, I have this properties. So, now let us come back to this. So, if I am going to denote this quantity as X is that clear that this quantity is $X^T X = 1$ and then V_t the norm of this quantity is 1.

Just see what is the meaning of V_t half means every element is just being taken square root.

So, suppose let us say first consider the case when every element is positive and then I am going to define V_t to the power square root t. Then this is the same thing right, this is again going to be symmetric.

Student: Is it positive definite?

And now the question is what about $V_t^{\frac{1}{2}}$ is it the same. So, let us continue now I want to use this after doing this now our goal is what? Our still goal is to still find out whether if I can bound such a probability ok.

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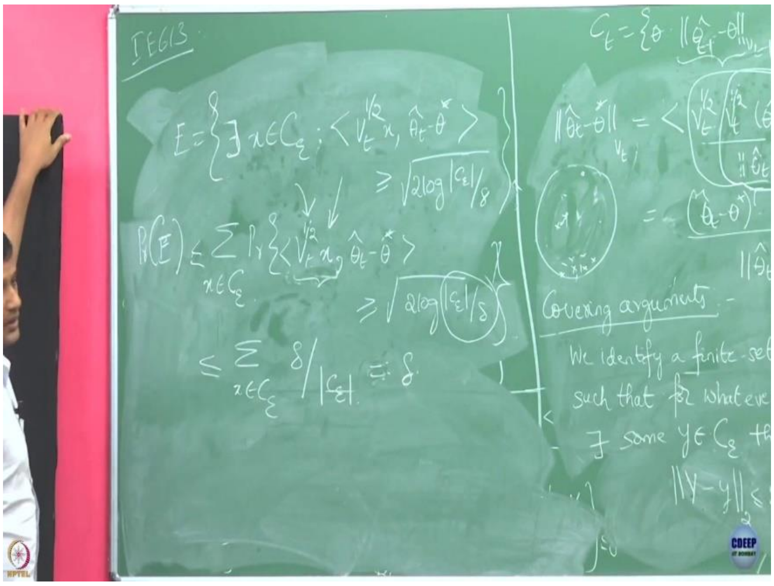

If I have to use some δ there, now I am going to show that if I have to define.

Student: (Refer Time: 18:22).

Set we will see that there exist C belongs to C_ϵ such that now I want to do this set.

Student: (Refer Time: 18:40).

Now, see what I am basically doing is now I am looking for an event where I am looking for a presence of an x in the set C, C is now a finite set for me which violate this condition which we in which this inner product is going to be larger than this ok. So, this if I want to look at the probability of this event, what is this?.

Can I write it as $\Pr(E) = \sum_{x \in C_\epsilon} \Pr\{< V_t^{\frac{1}{2}} x, \widehat{\theta}_t - \theta^* > \geq \sqrt{2\log\left(\frac{|C_\epsilon|}{\delta}\right)}\}$ is this correct? Ok.

So, I am interested in this event where I am looking for there existence of an x for which this condition holds, then I can always.

Student: (Refer Time: 20:17).

Upper bounded by this quantity right. Now, everything x is deterministic, I am looking for one possible element in this discretized version. Now, already now I also know that this V_t to the power half into x, where x is of this form; I already know this is has a norm of 1; that means, this guy is this, this points here they are already coming from a unit ball.

The same V_t, I am taking there also right V_t to the power to the power of half.

Student: So, then norm is with respect to V_t.

The norm is with respect to V_t^{-1} just a minute that is right. So, earlier when we applied this result right what would we say? We took it any. So, when we started with we just started with saying that d inner product with some quantity right. So, d inner product with some quantity and on so, that was like projection of my arm d on the error, I was trying to bound that.

So, now this is now whatever this quantity is let us say this is now a vector I am trying to project it on this ok. And, we had in the first step we have demonstrated this, this can be written as a linear combination of my noise and then I had applied a result to this quantity right. So, what would I show for this? So, now, this all these quantities says deterministic, there is nothing randomness about this quantity x ok.

So, now what is this probability is I have shown you that if we have this quantity, we had earlier I had $\frac{1}{\delta}$ here, but now I have cardinality of $\frac{|C_\epsilon|}{\delta}$. So, if you just go and work this is going to be like x belongs to C_ϵ. This is now nothing, but $\frac{\delta}{|C_\epsilon|}$. This probability is going to look like this.

So, I erased it, but whatever the bound we have earlier if I am just going to look into that, we will have this quantity. And now this is nothing, but equals to δ. So, what now we have basically done is suppose if I restrict take a set C_ϵ which is like a discrete version of all the points that could have arise from points like Y here. So, on that even if I have to remove the randomness here, how I am going to remove the randomness here?.

By considering all possible x's that are coming from my discrete set C which has only finitely many elements. Even if I am do that, that quantity that this probability of error is going to be still δ only. And, now I have to now translate fine this probability is upper bounded by δ. But, there was an error when I went from points here to a discretized points in this set right; now we have to worry about that part as well.

So, now, you will see that fine. So, at this point I have not actually used that this guy has to be with this norm of this vector has to be 1. It can be any vector here $V_t^{\frac{1}{2}}$ x can be any vector. Now, for that we know this upper bound holds ok.

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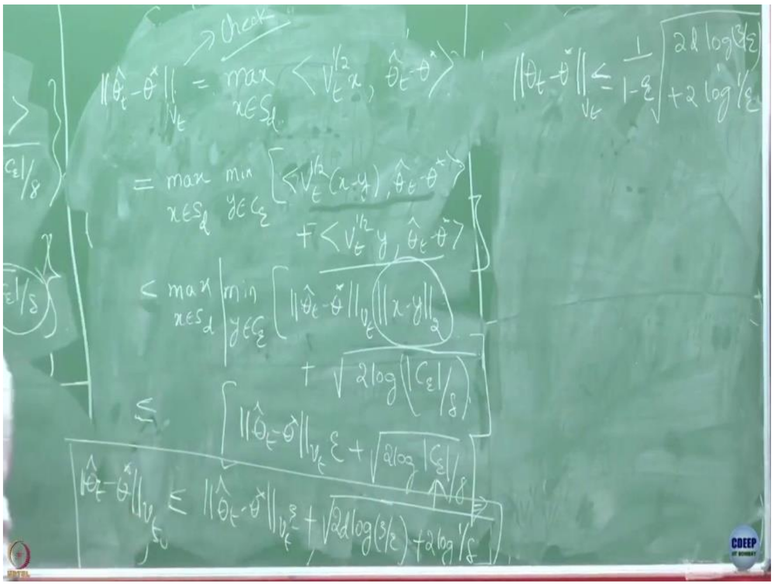

So, now, let us see where I am going to use the error part $\| \hat{\theta} - \theta^* \|$. So, this is another result I am going to write which you need to verify. So, this equality holds, good. I am just writing this right away, but you can verify this I am just saying that norm can be written as. So, this is going to be V_t here. This matrix norm can be written as a maximization problem over unit ball ok. So, how is this please check.

So, that term earlier k because we are dealing with the summation there. Now, that sum is not there on this part, if you now just carefully look into; we were there looking it summation for s 1 to t.

So, it is just the same thing, but instead of the summation I am only looking at one term here. So, that is why it is still going to be lambda here and now I am saying that take this inequality for granted. So, you notice that we had these straight result and this result we are just stating.

Student: (Refer Time: 27:42).

This ok. So, we just take it for granted them. Now, I am going to do quick manipulations ok; why is this so long. We are almost done.

Student: Sir.

So, now with these half I am just going to do this quantity plus ok, then I am going to do it as $\max_{x \in S_d} \min_{y \in C_\epsilon}[< V_t^{\frac{1}{2}}(x-y),\ \widehat{\theta_t} - \theta^* > + < V_t^{\frac{1}{2}} y,\ \widehat{\theta_t} - \theta^* >]$. So, what I have done here? I have simply this, this minimization is an artificial here because I have just subtracted y from x and added y quantity here. So, this y has actually has no effect on this, but I am just introduced it and trying to minimize it over this quantity.

So, these are the manipulations I need ok. Now, if you apply some Cauchy Schwarz inequality on this quantity, what I am going to get $\max_{x \in S_d} \min_{y \in C_\epsilon}[\left|\left|\widehat{\theta_t} - \theta^*\right|\right|_{V_t} \left|\left|x-y\right|\right|_2]$. Can you just quickly check this? This quantity if I am going to apply Cauchy Schwarz inequality, I am just going to get this inequality here.

And, then whatever the quantity that remains here we already know that this quantity being greater than this quantity is very upper bounded by a small probability δ. So, we are going to with high probability we are going to replace.

This quantity by this upper bound which is ok. Now, let us see this what I will do is now this quantity x here we know that they were falling in a unit ball right. So, and now this y's we have chosen they are coming from a unit a discretized version of this; the corresponding value of y and see I am picking a minimum here. So, I know that if I am going to the minimum value I am going to get on this is going to be at most ϵ.

So now, if I do this quantity is nothing, but $[\|\hat{\theta}_t - \theta^*\|_{V_t}\epsilon + \sqrt{\{2\log\frac{|C_\epsilon|}{\delta}\}}]$ and now just get rid of this maximum because, now there is no more x here.

And, now let us try to plug in the quantity C_ϵ. What is the size of this C_ϵ? We know that the C_ϵ is at most upper bounded by $(\frac{3}{\epsilon})^d$. So, this quantity is. So, what this we have finally, gotten is $\|\hat{\theta}_t - \theta^*\|_{V_t}\epsilon + \sqrt{\{2d\log\frac{1}{\delta}\}}$.

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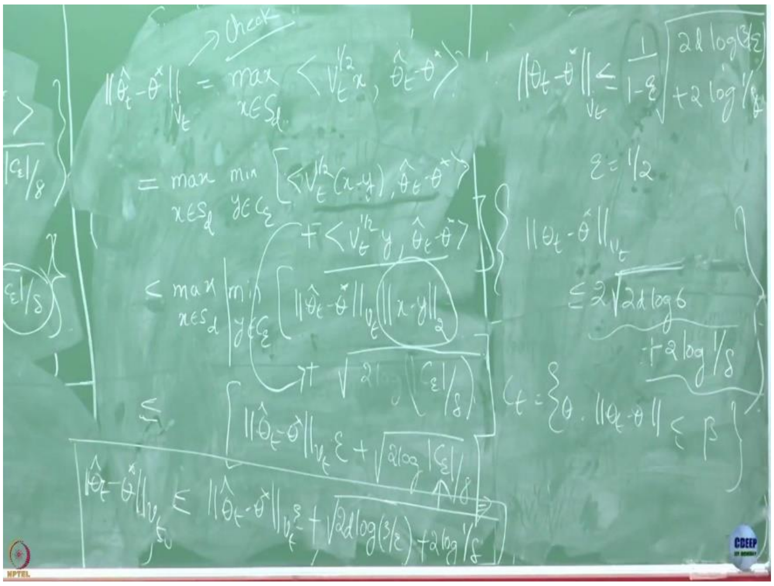

So, if I am going to further simplify this quantity what I will finally, end up is $\|\hat{\theta}_t - \theta^*\|_{V_t} \leq \frac{1}{1-\epsilon}\sqrt{\{2d\log\frac{3}{\epsilon} + 2\log\frac{1}{\delta}\}}$.

And, this is true for any ϵ I am going to choose right. So, you see that I have an upper bound on the difference between these two quantity. And, this is going to hold with probability at least 1 minus δ because I have used this fact, that this quantity being larger than this lower bounded by this quantity is δ. So, this quantity being upper bounded is going to be at most 1 minus δ.

So, that is what I have done here. So, this quantity holds with probability at least 1 minus δ and now if this ϵ is what I have chosen, ϵ just told how I discretized my unit ball right. So, if I am going to choose ϵ small what will this quantity is going to be like? It is going to be large right, but whereas, this quantity if I choose ϵ small this ϵ comes in the denominator. So, this can shoot up.

So, we have to choose ϵ appropriately because one term there is a tension between these two terms here right in terms of ϵ. One guy is increasing another guy is decreasing. So, how to choose ϵ appropriately you can tune it appropriately, but let us say for time being we are going to just set ϵ equals to 1/2 then $||\widehat{\theta_t} - \theta^*||_{V_t} \leq 2\sqrt{\left\{2d \log 6 + 2 \log\frac{1}{\delta}\right\}}$.

This is 1/δ. We will now see that this is upper bounded by we just do this simplifications and just replace this we are going to get it as something 1/2, this is going to be 2. So, now, you see that if my β_t is or δ capital T like has to has been chosen like this, then I know that my θ* is going to be satisfying this inequality ok.

That means my θ* is the norm of this difference with respect to δ_t is bounded by this with probability at least 1 - δ. Now, what we have basically demonstrated is if I am going to choose a δ in such factor I can and look for a set C_t. So, how was my C_t? Let us say if I just do this and this quantity is same thing here which I am going to call it as some δ.

I know that so, this is just θ, the θ* is going to lie in the set with probability at least 1 - δ right. So, what we have basically demonstrated is through the sequence of steps under some assumptions may be very restrictive as of now, we will be able to come up with some δ_t such that if I am going to look at this confidence ellipsoids. It is going to contain my true parameter θ* with high probability and this is true for any t.

Now, the assumptions are made; they were mostly non paratactical right. Like I mean they are not possible in the sense that, I am not going to select an arm in an deterministic fashion without ignoring what has been observed. But, if I am going to select an arm in each round observing what has been offered, the things are more complicated.

There is more coupling that is going to happen, because all the action you are going to choose; it has been based on all what has been observed. And, further there is a coupling through this θ* which is common for every reward or that has to be taken into care. But, once you have to take into all those coupling you have to still go and use some still sophisticated machinery.

This machinery to use this they have mostly used only simple linear algebra. And then we have exploited sub Gaussain trail behaviour right. But, now once we have bit more coupling of the rewards we have to use I mean the analysis that has been shown in the book that uses martingale properties and mixture methods.

And, then you can able to show that indeed even if you relax all the assumptions you made, it is still possible to come up with such a δ_t here which will guarantee that you can come up with an ellipsoid which contains θ^* in every round with high probability. And, once that happens we already know how to analyse right. In the first class when we started this stochastic linear bandit, I already give a recipe how to get the regret bound; if once we can construct this confidence ellipsoids.

So, there I had assumed how let us say such a confidence ellipsoid exist and then we showed a regret bound. Now, you just demonstrated under restricted assumptions that yes such a confidence ellipsoid indeed exist. But, we can go further and say even under if you remove all the assumptions we have made in this not all, but few we are still able to cover this confidence ellipsoid. And then once we have that we also have the regret bounds ok.

So, I will leave you to read into that, that is bit more mathematical. I mean we will get easily loss with that. So, just read how to apply the martingale properties, you need to have at least basic need to know what is the basic martingale is that should be enough; I think rest is all algebra. Just you try to go into that. So, the study of stochastic linear bandits we will stop here.

But, now what we do is from the next class is we will continue to study linear bandits, but now we will go with the reverse gear and we will go back to adversarial setting. So, you remember we first started studying adversarial bandits, then we have shift to stochastic bandits. Now we continued study of stochastic bandits for the linear setting, but now we will go back to study this linear bandits for the adversarial case in the next class ok.