Bandit Algorithm (Online Machine Learning)
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Lecture - 27
Concentration Inequalities

So far I said we are restricting ourselves to stochastic bandits right. So, stochastic bandits they expect stochastic bandits to easier learn compared to the adversarial or the adversarial are easier there compared to the stochastic ones.

Student: Stochastic to be.

Why?

Student: (Refer Time: 00:37).

So, fixed distributions are there. So, if somehow if you can and once you have distributions, possibly they will be governed by some parameters right. So, if you can somehow figure out these parameters, then you know the distribution according to which your environment is drawing these samples right. So, you can start selecting the best arm.

And what is that if we are trying to say like our regret is defined such that, we want to identify an arm which has the highest mean. This mean is not going to change right like because the distributions are fixed this means are going to remain constant throughout for all the arms.

So, somehow you can figure out this means, you know already which is the best action to take. Now the question is how can you figure out this means right. So, the whole stochastic bandits now boils down if your aim is to minimize this regret, how quickly you can identify the means of each of these arms right?

Now, what is a good way to identify this means of the arms? So, somehow like instead of what is the sample what is the number, but you see that like I if I get it from some arm, if I collect some reward and take their average and if I have sufficiently many samples you should give me the mean value why is that?

Student: Strong law.

Strong law of large numbers right we know from strong law of large numbers like if I take samples from each from this particular arm many times and I am going to average it that should give me the mean value.

So, for this law this result to hold what kind of assumptions I need on the sampling? I need to be. So, for the fixed arm the identical distributions already there right because if your fixed arm, the distribution is not going to sit our only need is that the samples gone are independent and that is one of the assumptions we are going to make in this.

When I said the environment if you are going to play an arm I_t in round t and if I said the environment is going to draw an sample corresponding to that arm, that sample is going to be drawn independent of the past samples from the same arm not only independent of its past samples from that arm also independent of the other arms ok.

So, we are making going to make this assumption that the sample when you are going to pull an arm, you are going to get a sample which is drawn from the distribution associated with arm and that sample is going to be independent of the past pulls from that arm and also the pulls of the other arms. That means, to say that when you pull that arm you got a sample which at all which did not have the past samples and the other arms did not influence that reward you got from that arm that point.

Now, so, because of that, it looks like I am perfectly by making this assumption I am perfectly in the requirement of the strong law of numbers criterias right. To apply the strong law of numbers you need this iid assumptions and that I am assuming to hold.

Now all you need to know is when I do an estimate of my mean value, how quickly I can get a good estimate this? (Refer Time: 04:33) Large number says if you get many many samples and if you take an average, you will get; you will get the correct value, but I am not going to do this forever right I may be ending this only after a certain number of rounds.

Now how to get that or like at least how many samples I need before I can get to have a good estimates of this means. So, for that we are now going to make a slight detour to understand concentration bounds ok. Now it is clear for you right in the stochastic bandits if I have to identify the r with the highest mean I need to quickly identify the means and the way to identify the means is take the average of the samples I collect.

Now, I want to understand how quickly the average of the samples I am going to take, they are going to concentrate around the true value of the mean values. So, I want to now basically understand how quickly my averages concentrate around the mean value. I know that if I take infinitely many samples, it is going to be exactly that value of the mean, but I do not have that luxury.

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So, I want to see how quickly I go closer to the true value of the mean and for that we are going to study some concentrations or how quickly my average concentrate are holds the true value or let us call it concentration of measures or some also books also call it as concentration of inequalities.

So, we are going to talk about this, we are going to revisit or some discuss some of the main concentration properties. So, let us X_1 , $X_{2,..,}$ X_n are iid random variables with mean $\mu = E(X)$ and $\sigma^2 = Var(X)$.

So, our natural estimator is I am going to if I want to estimate this mean quantity, I am going to do is I am going to take average of this n samples. So, what? So, this μ hat its a random quantity right? Because its a average of n random variables. So, what is the expected value of μ hat? What is the expected value of μ hat?

Student: (Refer Time: 08:21).

Expectation of X. So, expectation of μ hat is expectation of this quantity right? But I can also write it as 1 by n summation of expectation of X_i . What is expectation of X_i ? So, here it should be it mean let us call this.

So, this is the common mean because these are X_1 , $X_{2,..,}$ X_n are iid random variables right? So, this is let us call their common mean and this is the common variance. So, this quantity expectation of X_i is going to be μ and because of that this quantity is going to be μ right? So, when this happens? What we call this estimator as? So, this was an unbiased estimate we already talked about this right in the adversarial case also.

So, fine it looks like and this is true for any n, even if you have one sample this is the case, even if you have two sample this is the case. So, this is three sample this is the case, but it says that in expectation this is going to give me the true value, but what about the variance?

If I am going to look at the variance of μ hat what is this quantity is? Why is that? So, this is the variance right. So, what is this variance this is going to be? 1 by n. So, if I please do this. Is this correct? I have just pulled out this 1 by n square outside and cross multiplied it with here it becomes n μ, but now I have pull this μ inside the summation.

Now, you can verify that because these guys are identically distributed and also independent when we expand this, this becomes summation and once you simplify this, you are going to get it as sigma square by n ok. So, now, as n increases, this quantity is going down inversely in n right.

So, what it is saying is fine even though its an unbiased estimator, the expected value of μ hat is μ is same for all n, but the error the mean squared error that is the basically the variance is going down when you have a large number of samples ok.

So, if you have a small number of samples your error that is the value of this μ hat being away from the true mean could be very large ok. So, you want. So, that is why the number of samples are important only when you have sufficiently if you want this mean squared error to be small, you need to guarantee you need to have n to be some number you cannot get a small mean squared error for any n, you need to have n large even though it is an unbiased estimator even for any n.

So, fine then this error is going to go down only when n is going to be infinity right this mean square error going to be 0 only then n equals to 0. But often we would be happy like yeah not necessary that μ hat has to be the same as the μ quantity, but as long as μ hat is very close to μ, then it is fine.

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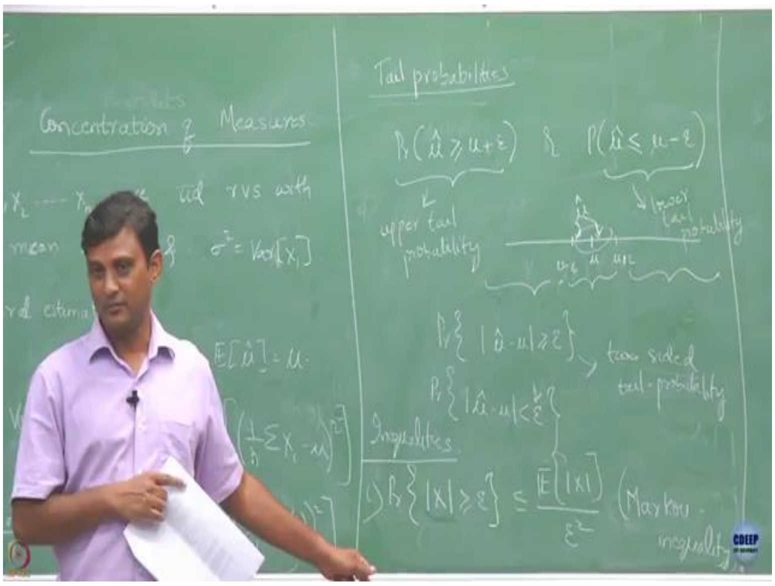

So, because and then you may be interested in this tail probabilities, where you may be interested in asking this question what are the probabilities that μ hat like this or μ hat is.

So, what we are basically asking is, you are asking that μ hat is going to be larger, then μ plus epsilon and here you are asking that μ hat is less than the true value plus epsilon ok. And now this is kind of μ minus epsilon. You can take it less if you want it to be less than epsilon wait a minute. That is right like you want. So, what is that you are asking?

You have your true value to be here and take this to be μ plus epsilon and take this to be μ plus epsilon sorry other way round and now you want that your μ hat not lying in this region, that is your μ hat is below this guy or your μ hat is in this region or in this region.

These are the bad cases for you right like you are basically asking that you are. So, this whatever your μ hat, you would be happy if it is there or here, but if it is in this or in this region you are not happy and you are basically asking that question. So, this one is called upper tail probability and this one you are going to call a lower tail probability right.

So, when you are going to ask whether your estimate belongs to this, you are basically asking a lower tail probability and when you are asking your μ hat belongs to this we are going to ask for upper tail probability or you can combine both of this and ask this question, what is the probability that your μ hat minus μ is going to be greater than or equals to this.

So, that is what is the probability that it will be either in this region or this region and this one you are going to call it as two sided tail probability. So, basically then we will be interested in how small is this quantities, what how are these probabilities is this probabilities and how they did depend on the number of samples?

Is it that if I have enough number of samples of course, this could be this probability could be small, but how small they are?. So, now, we are going to now we will be interested in bounding these probabilities ok. So, this is one, I mean the complement of this is like you will be asking μ hat minus μ is less than epsilon that is in this region.

So, what I want we may be interested in asking this? How many samples I need to collect so, that if I take the average of them the estimate I get is within epsilon neighbourhood of my true value right. In that case if this probability is very small then I can be very confident that my estimates are in this region whatever and I may decide like how to just choose this epsilon, we will discuss that later I mean depending on whatever epsilon you pass on this is the kind of results we are interested in ok.

Now, inequalities we are interested in now we want to see how we can bound them right how many of you know Markov inequality? So, what is the Markov inequality? If I have a probability if I want to a bound this. So, what is the bound?

Student: (Refer Time: 18:46).

Expectation of.

Student: (Refer Time: 18:52).

Its just X or mod X here, I want mod exact because I want this quantity to be positive valued random variables. So, I have deliberately put mod here this should be equals to expectation of mod X by epsilon square this is my Markov inequality. So, what about Chebyshev equality? How many of you know Chebyshev inequality? Others do not know Chebyshev inequality? So, 6 1 1 guys do not know Chebyshev inequality and how to get this bound Chebyshev inequality from this?

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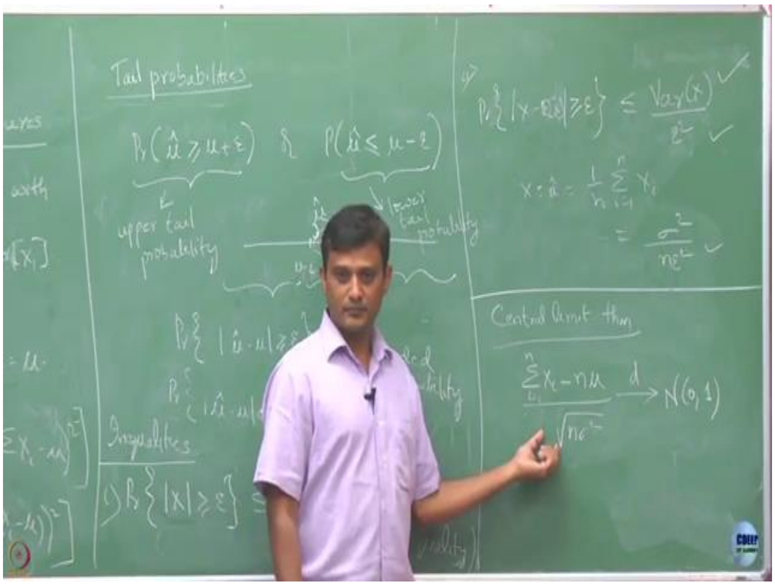

So, if I have X minus let us say μ and I want to get this what is the bound on this?

Student: (Refer Time: 20:08).

Variance of. So, it is epsilon square or epsilon here and is this correct variance divided by epsilon square? So, we already have one. So, what is X here ok? So, I will just write it correctly here. So, here X, this is expectation of X here is this. So, we know that already the expected value of μ hat is μ right and we are interested in knowing what is the probability that this is going to be. So, by taking this X random variable to be my μ hat I have already one bound here right.

So, let us compute it what happens my X equals to μ square here μ hat. So, can you just quickly compute and tell me what is this value is going to be? What is the variance of μ hat? So.

Student: (Refer Time: 21:41).

Variance of μ hat you already computed it to be sigma square by n. So, this quantity is like sigma square by n epsilon square. So, what this is we have? Suppose if you fix an epsilon and then as you increase n, this probability is becoming this bound is becoming smaller and smaller right then if you have more and more samples, that the difference between that my estimator will be away from this interval is going to be come down come down or it is going to shrink as we are going to increase n.

Now, let us fix an n and now let us look into epsilon. So, once you fix an n, this upper bound is like inversely changing with epsilon square right. So, if you want; if you want to take this epsilon to be small, what is happening to this bound? This is becoming bigger and bigger.

So, if you are asking for this epsilon to be small like you want the small interval around this μ, you can your this upper bound is also going to be very large. In the sense you are saying that this concentration bound is also large that is you are not this probabilities can be is at least you are not guaranteeing it to be small, it can be a large value when epsilon is small right.

So, we are saying that you fix an n and now you shrink this μ make it smaller, now I am asking what is the probability that this μ hat lies in a smaller interval. I can only guarantee with smaller and smaller probability right that is because this guy is now going to be larger because when I because epsilon as I make smaller, this guy will become larger and larger ok.

So, what we are saying is if you fix this interval and increase n you have a good this event happening is going to be very small that your estimates will be away from this interval is going to be very small, but if you fix n and you want this μ to be happening in a smaller interval, if you want to reduce this interval then your you saying that μ hat lying in the small interval, you are going to say that is going to happen with smaller only smaller probabilities fine.

Now, the question is this the best bounds one can have? Is it like is it like it just that this error goes down just like inversely n or it can go much faster? Because this after all this is a just an upper bound right we do not know whether this upper bound is tight or is that the one can get a smaller upper bound. So, the whole there is a lot of study that probabilistic do just to get a better bounds on such quantities ok.

And once you can get a better bounds maybe that will help you to understand what is the smallest number of samples you can have so, that this happening this event happening is small ok. Now let us then the question is what is the better we can see, we are going to discuss one more bound on this using central limit theorem. What is central limit theorem?

So, if I have again I have a sequence of iid sequence, if I am going to take their sum centre them by subtracting the mean value and then normalize them like this what is this converges to?

Student: (Refer Time: 26:50).

Normal distribution with what parameters?

Student: (Refer Time: 26:53).μ

And this convergence is in distribution right. Now our μ hat is also of this shape right that average of n. So, we also have this term here let us see that from this we can using the clt results, we can get a bound on this quantity here μ hat minus μ e greater than μ fine. So, what is the clt result say? If you let your number of samples to go to infinity, the average when you scale it in this fashion then that is going to converge this is asymptotic result.

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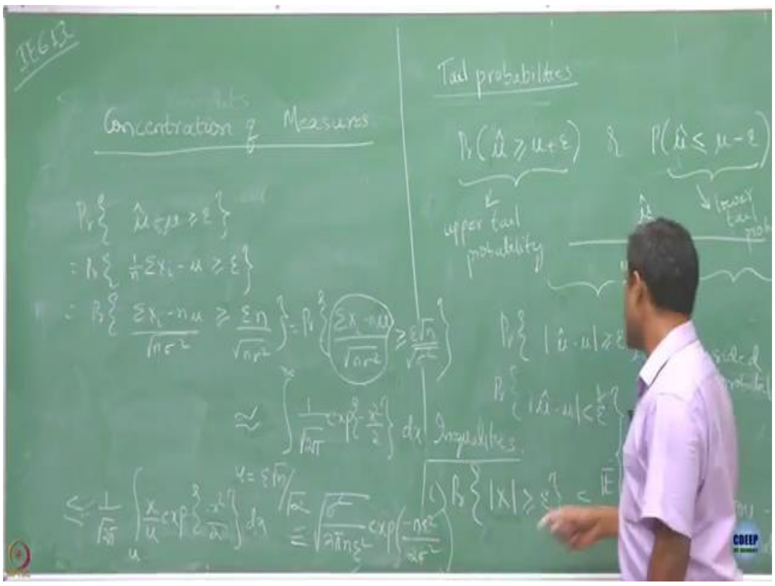

So, now let us ask the question, what is the probability that μ hat minus μ? I would let me ask only one sided question probably to the with μ plus or μ minus μ is going to be greater than or epsilon μ.

So, let me plug in this quantities what is μ hat? μ hat is 1 by n summation X_i μ right this is epsilon, I just do manipulation and write summation of X_i n μ being greater than epsilon n and now I want in this form. So, what I will do is I will divide and multiply divide both sides by n square root sigma square n square root sigma square right.

So, this is same as probability that X_i minus n μ by summation n sigma square being greater than or equals to epsilon square root of n by sigma square ok. So, I know that this guy here well n is sufficiently large is going to look like normal distribution with mean 0 and variance 1.

Now, pretend that n is sufficiently large and this is already Gaussian distribution with parameter 0 and 1. So, then what is this probability? This probability is basically asking that, what is the probability that the tail distribution what is the probability that my normal random variable is going to take value greater than epsilon square root n divided by sigma square right.

So, then I am going to write I am just going to plug in my Gaussian distribution, it is going to be between epsilon square root n by square root sigma square to infinity 1 upon 2 pi, sigma square is going to be 1 because this has asymptotically this has variance 1 and exponential minus x square by 2 dx right what are they? This is just a pdf of a Gaussian random variable with mean 0 and variance 1.

So, let me call this value simply for notational purposes to be u. So, what is this basically? We have a special name for this right and also special symbol for this what was that?

Student: (Refer Time: 31:13).

1 minus I of u and in general this expression is integration do not have a closed form expression right? This is hard to integrate. So, what we will do is we will just look at an upper bound on this, what I will do this u here right and I am integrating it over the

quantities from u to infinity; that means, I am integrating over region which is larger than u ok. So, I will do. So, u.

Student: (Refer Time: 31:55).

Yes.

Student: (Refer Time: 32:07).

We are doing epsilon root n, yes this we are saying that this is going to be Gaussian only asymptotically, but what we are saying that for any given n just assume that this is sufficiently large, but finite still and this has already converged to a Gaussian distribution just pretend that and then apply this.

So, because of this maybe we can just say this is an approximation here ok. So, now, what I did is basically here everything remains the same instead of I have just added this extra term x by u and now your integration variable x is going to be larger than u. So, because of that this we will get an upper bound, but usually this x exp minus x this is easy to integrate once you are going to integrate you are going to get this value as. So, after integration you are going to get I just keep the integration you can verify this.

So, now notice this? What is happening here? This probability which of which was our interest now this is let decaying exponentially in n, earlier we could only get it here inversely na. So, this is exponentially in n. So, this is going to decay much faster right, but the carryout here is we make some approximations here just assume that is na sufficiently large. So, we are saying that if na is good enough this convergence can happen actually this probability can decay like exponentially in n not just linearly in n.

So, based on this result we are going to prove some more concentrations along this which. So, this result basically gave us an intuition that actually this probability can decay exponentially in n not like linearly in n, even though we made some approximations here. So, in the next one we talk about Hoeffding’s and other inequalities where we get exponential decay ok.

So, let us stop here.