Bandit Algorithm (Online Machine Learning)
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Lecture – 21
Online Gradient Descent

Fine. So, now, let us see the update rule turn out to be simple gradient updates, gradient descent based update.

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So, if we have this what is the regret bound we are going to get, that is of our interest, right.

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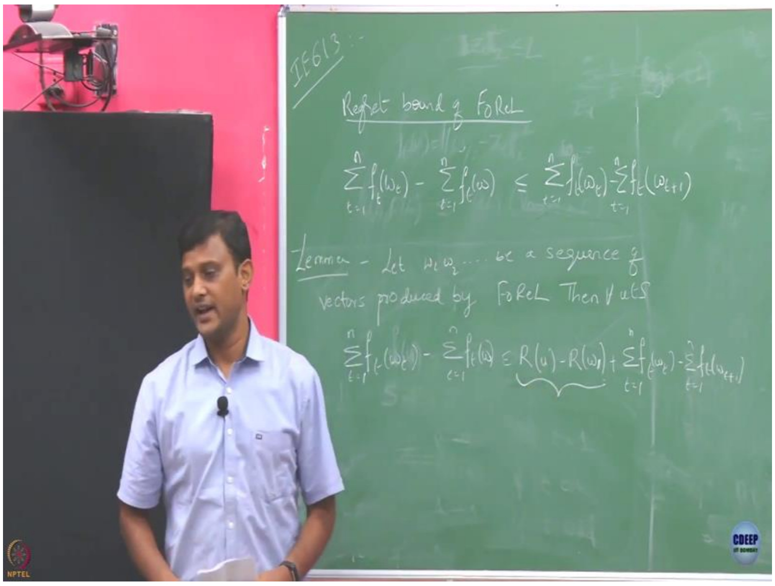

So, what is the regret bound? So, first thing when we gave a regret bound for the follow the leader, how did we do? We showed that sum_{t=1}^{n} (f_t (w_t) – $f_t(\omega)$) <= sum_{t=1}^{n} (f_t (w_t) – $f_t(w_{t+1})$).

Now, with this adding regularization function can we have an equivalent version of this?. So, and then what will be that. So, that our first lemma says that, so once we had a regularization the only difference the extra term we get is this term on the spot. So, this bound here get changed by this bound.

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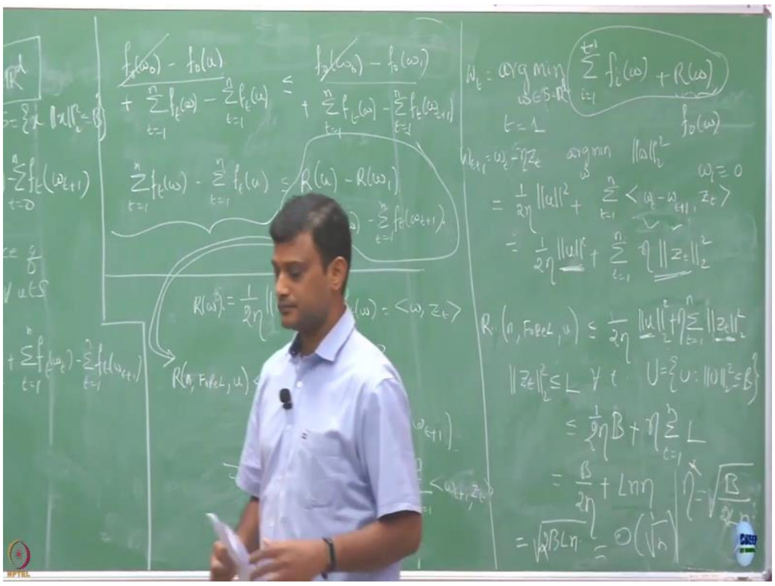

So, do you anybody see what how can I get this or why this makes sense? When I did my follow the regularized leader, right what is my algorithm?. I was doing $\arg\min_{\omega} \{\sum_{i=1}^{t} f_i(\omega) + R(\omega)\}$. This is $R(\omega)$ which is I decide, right, this is now generated by the adversary. This is totally under my control.

I am going to treat this function as some $f_0(\omega)$ that is a function which is generated in the 0th round, ok. So, if I do this, now I am basically saying that instead of starting my algorithm from round 1, I am going to start my algorithm from round 0, ok.

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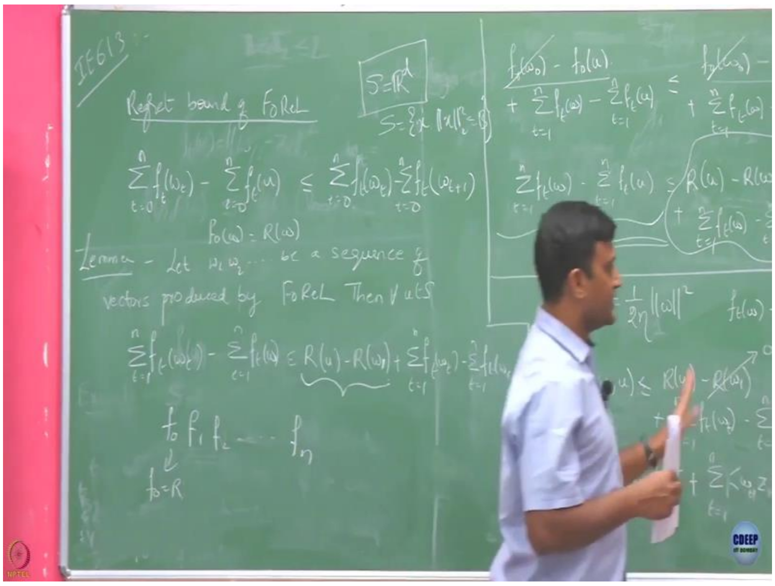

And then this same step should hold for that also, right. If this hold; and what is this? And in this $f_0(\omega)$ is nothing, but $R(\omega)$. Earlier I starting this algorithm, but now I am treating that is to be $R(\omega)$ to be another function which is generated in the 0th round. So, this whole thing should hold if I start from 0 everything, right.

Now, once you have this now do you see that this actually implies this, you see it. So, now, instead of now what I am going to treat is my sequence of function I have seen I am going to treat it as f_0, f_1,.. all the way up to f_n, where f_0 is my R function. So, just simply expand this, right.

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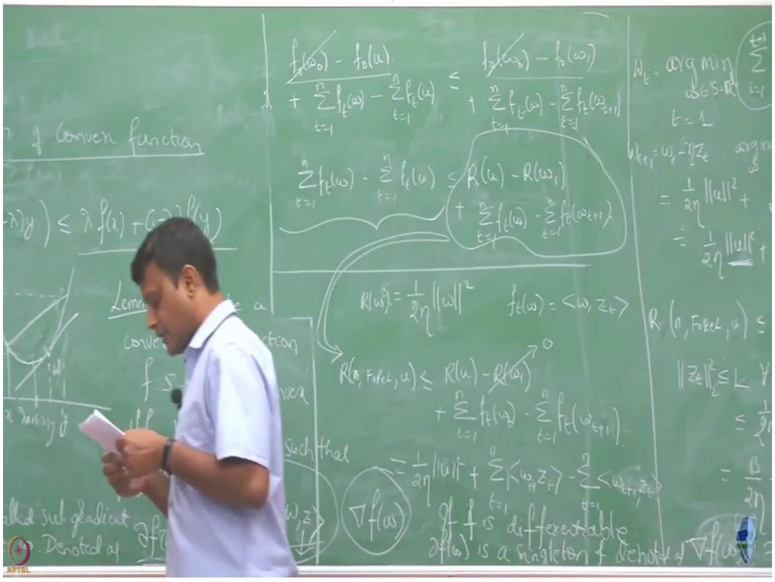

So, you take it to be separate out t equals to 0th term here, then this is going to be f_0 (ω) – f_0 (u). What was this term here? This was some u, right whatever I am interested in. My u and other terms I just keep like that t equals to 1 to n, f_t (ω) minus if this going to n, f_t (u). This whole thing is upper bounded by again I separate out the first term here that is going to be what? $f_0(w_0)$ - $f_0(w_1)$, right plus the remaining terms.

So, just try to manipulate these two terms. So, this term gets locked off with this. Now, you just bring it here, so then that is all we needed or maybe take this on the other side. So, if you do that you get whatever we have here. So, you should just do this. But now what is f_0 ? f_0 is basically R function, right. So, this you replace it by R, R(u) – R(ω_1) plus this terms.

So, if I have a regularizing function I will just get this. Yes, we will assume that this is also it is f_0, right like this. So, then only the whole of my function is convex. If each function is convex their sum is convex, so I will also make it convex by assuming this R is also convex, ok.

So, now, with this once we have like this earlier we have shown that my, ok. Now, with this what is the regret bound I am going to get?. So, let us try to work out this regret bound for a given u. So, this is my regret interested and now I need to compute this value, right, ok. Let us compute this upper bound for my case that; what is my case? I want to now take this to be R equals to norm of this and my $f_t(\omega)$ to be $\langle \omega, z_t \rangle$, ok. Now, let us compute what happens to this portion.

So, if I am going to take this; so, my regrets of my FoReL for a given u this is going to be upper bounded by this quantity, right; R of u minus R of u 1 plus t equals to 1 to t of omega minus. So, now, let us substitute this value. What are this value? This is 1 upon 2 eta and w. And what is this? Sorry, this should be u here, ok.

What is w n is going to be? So, when I have to do w 1, so this is for t, right. If I am going to compute this at t equals to 1, in that case it is still t minus 1, right, my regularized leader version. So, there is no term here when t equals to 1. What I will be interested is only on this. And this term is this quantity. What is the minimization of this quantity?

So, one more thing I have forgot to tell is let us assume that my S is R d here, the entire R d space, ok, the Euclidean space. So, because of this can be S which I am thinking to be R d. So, for t equals to 1 what is this quantity is going to be? So, this is simply saying arg min of norm of over w, right. What are these quantity? What is that minimizes this quantity? 0. So, then this w_1 is 0. So, this term is going to be 0, I do not care about it, ok.

Now, let us work out the remaining terms. So, what is this term? This is going to be w z t and this term is going to be $\langle w, z_{t+1} \rangle$. Now, what I did? Sorry. So, this is going to be w t computed at w, this is w_{t+1} computed at z_t. Is this correct? So, f_t of this is $f_t(w_t)$, right and this is $f_t(w_{t+1})$. So, I have just substituted this value.

Now, if you are going to simplify this further. So, I am just writing this as a compact form for these two things. Can I do that? So, because z_t is common it is simply $w_t - w_{t+1}$. But what I know about this? So, w_{t+1}, I got it through w_t right, through gradient descent. What was the relation between w_t and w_{t+1}? So, how did we get w_{t+1} to be w_t - eta.

Student: (Refer Time: 12:17).

And, that is z_t, right. So, can I substitute here? So, if I substitute here, this quantity is simply zi t sorry, eta z_t. So, after substituting this I can write it as norm of z_t whole square. Lineup product between them. Is this correct? So, finally, what I got is the regret bound of on a particular n is upper bounded by 2 eta norm of u square plus this quantity.

Now, let us again use the same condition we used earlier, right. So, let us assume that this parameter z_t in every rounds are such that let us say this norm of z_t are less than L for all t. So, the gradients here z_t denotes the gradient in round t, right let us say they are all up like this. So, now, what is the bound I am going to get? 2 eta. So, this guy is also going to be L, right because this is also u is coming from the same space as that.

Now, like; so, what is this u? So, this zitas are the gradients, ok. What now I am assuming is these are the gradients and I am going to assume that they are upper bounded by some quantity L, ok.

And what is this u? u is one of my reference point, ok. So, let us assume that this reference points they are coming from a set u, right, I am also going to define this u to be such that u where all of this u's they are also bounded, they are also coming from some bounded interval, ok. So, only thing I am doing is instead of let us say instead of considering all the points like this I will consider some ball, ok.

So, one possibility to do this is let us all; what is this points x such that norm of x is. So, what does this set denotes if I write it like this? So, assume dimension is two circle, right of radius B in this case, ok. So, I am just going to assume like even in this dimension I am going to consider some ball of radius B here.

Now, because of that this guy is going to be some B plus I will have this eta term, is equal to eta and this guys I have been assumed to be upper bounded by L, so this is going to be L. So, finally, what I end up with this is B by 2 eta plus L n eta, n is coming because I am adding L for n terms.

Now, we are seeing this kind of bound earlier also, right. What are we are going to do now? What we will do? So, eta is a parameter of power S, right. This is a regularizing, this is a parameter which we used in the regularizing function. So, how; now it is up to us how you want to choose it. Can I choose it in some specific fashion here?.

Now, treat this upper bounded is the function of eta, right. Is this a convex function in eta?. This part is linear in eta. This part is 1 by eta is what?. So, it is a convex function, just you just do the second derivation two differentiation, right.

1 by eta square sorry. It is going to be what? If you differentiate just 1 by eta to twice, you will see that it is having a positive slope. You have positive second derivative if eta is positive. So, if you now just differentiate it, and try to find a point eta and plug it back and tell me what is the bound you are going to get, so basically optimize this with respect to eta. What is the bound you are going to get? So, if you are going to choose eta to be B L times square root 2 n, it is in the denominator. But, what is that? This L should be in the numerator or denominator?

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Just differentiate this and tell me what is that optimal value of eta here?

Student: (Refer Time: 18:05).

Under root.

Student: B by (Refer Time: 18:09).

B by?

Student: (Refer Time: 18:14).

Now, if you plug back in this what is the bound you are going to get? So, can you tell me what is the final bound we are going to get? Suppose, if I choose my eta specifically like this it is going to be what? Square root of, ok; just do this (Refer Time: 18:37) square root B, square root of the z square root B will be here then becomes B L n will be there and I will have 1 by square root 2, 2 by 2.

Student: B L n.

Will you get this? Square root 2 B L n. And what is this? Now, because B and L n are constants which are chosen in this fashion and then this is like order square root n. So, you see that now even for the linear function, linear loss function if I use my regularization in this fashion I will end up my follow the leader, that is follow the regularized leader it is going to give me a regret bound which is order square root n, ok. That means, this is going to this will give me a sub linear regret, ok, fine.

So, fine. We so far looked into two types of convex function, one is linear function and another is the quadratic function we are looked at. But, what about the other convex functions? Is there we can do something about this?. It so happens that studying other convex function is almost same as doing studying this linear functions here because of the property of a convex function which allows us to represent this convex function with a lower bound which has a linear (Refer Time: 20:25). So, let us discuss that.

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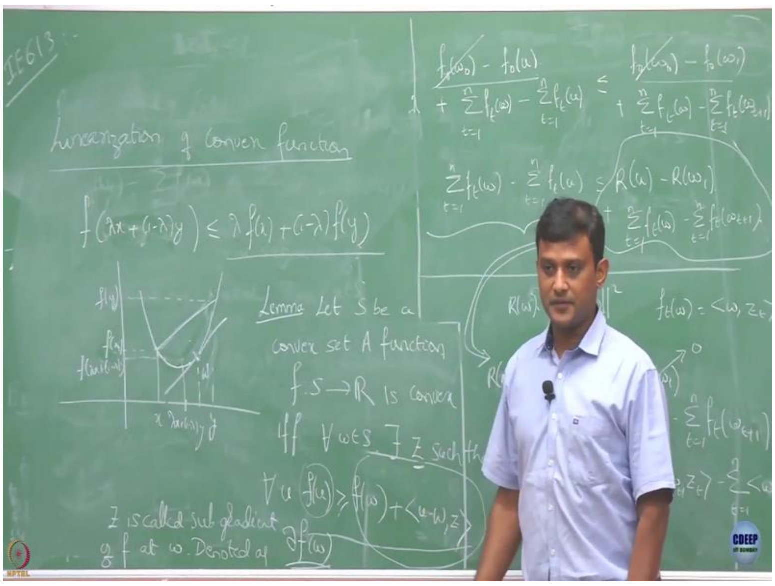

So, linearization of convex function. So, how many of you know definition of a convex function? What is a convex function means? $f(\lambda x+(1-\lambda)y) <= \lambda f(x) + (1-\lambda)f(y)$.

Now, what is lambda? 0 to 1, right. So, we have already seen it in other class, but anyway let us try to again do it here. So, suppose let us say my function is like this and I have two points here, one is at x and another at y. So, this is my f of x, right, and this is my f of y. So, where is lambda x plus 1 minus lambda y is going to lie? So, it is going to be somewhere in between, right because lambda is between 0 1, and we are also assuming that this function f is defined on a convex set, ok.

So, if you take a linear combination of any two points of x domain, it is also going to lie in the domain. So, this is going to be lambda x plus 1 minus lambda y and this is let us say this is its function. And what is this quantity here? Lambda times f of x, 1 minus lambda f of y. What is this? So, as you vary lambda I am going to (Refer Time: 22:58); I mean scrape through this line, right like from here to from lambda equals to 0, I am going to be at here, and lambda equals to 1, I am going to be at here.

So, what is this saying? It is saying that my function value at this point; that is what is this? This is $f(\lambda x+(1-\lambda)y)$ at this point is always going to be lesser than the line joining these two. So, this is the standard thing we know about convex function.

But another interesting property about this convex function is, you take any point, let us take this point. I will can come I can have a tangent at this point on this convex function which acts as a lower bound for this entire function, that is maybe I should be drawing the slightly better one, ok.

So, let us say take a point here another point. I can draw a tangent. So, tangent here is the point where it touches my function only at this point, ok. And this, so whatever this point and I can have a tangent here which is like a lower bound for my entire function. Like if I have this if you look at any point the value on this line is going to be always smaller than the corresponding value on this function, ok.

So, in this case then let us say this point is some w. Now, what is the property of a convex function is, so let us say let S be a convex set of function if and only if for all there exists such that. So, it is saying that if S is a convex set then my function f is convex if and only if it so happens that you take any point w at that point, if I can come up with a another point z such that this relation holds.

This is true for all u, that is you tell me a point w, I will be able to come up with a lower bound on my function f. So, this is my function f this is true for all u, right; that means, this is a lower bound on this function.

And what is this lower bound? This lower bound is now defined in terms of the value at that point w and also another point we are saying z which exists at that point z. And if at all that exists then that function must be convex, ok. And now, this z here whatever we said exists this z is called sub-gradient of f at w and it is denoted as del of f. So, anybody has question about?

Student: (Refer Time: 28:17).

That z need not be unique. All we are saying is that there exists in z and that z could in fact, depend on the w the point at which you are looking at, and. So, then if that such a z is going to be called as sub-gradient of this function f at omega. So, they are not going to prove this is like a standard result in convex theory of convex function. And, but what we are going to do is we are going to exploit this result to linearize my convex function, ok.

The way I have drawn my convex function here, do you think this is differentiable at every point? Right, because it is smoothly changing at every point. So, in such case, this if this such this sub-gradients can be unique and if at all it is unique that is you tell me a w there exists only one particular z for which this relation hold then, but that particular z we are going to call it as a gradient of my function at point w here, ok.

If at all this property holds with a unique z, then that z defines my gradient of my function f at that point. And in that case if z is denote they are going to denote it as del f of omega. So, when we say this, when this is uniquely defined the gradient.

When we say this is not uniquely it is satisfy for a unique z, but there are there could be possibilities we are, in that case this del of f omega can be a set, right; because. So, what we are saying? This z is called sub-gradient, right and we are just denoting it.

If there are more than one z that satisfies that then all of that will be called sub-gradients at omega and that will be denoted by this notation. And whenever it is unique then we are just going to write it as with this capital del of f omega. Is that clear?

So, what I want to say is if f is differentiable then this del f of omega is a singleton and we denoted as f of omega, ok, ok. Now, let us see this y. So, can you come think of a case where the sub-gradient can be a set that is it can have more than one elements in this?.

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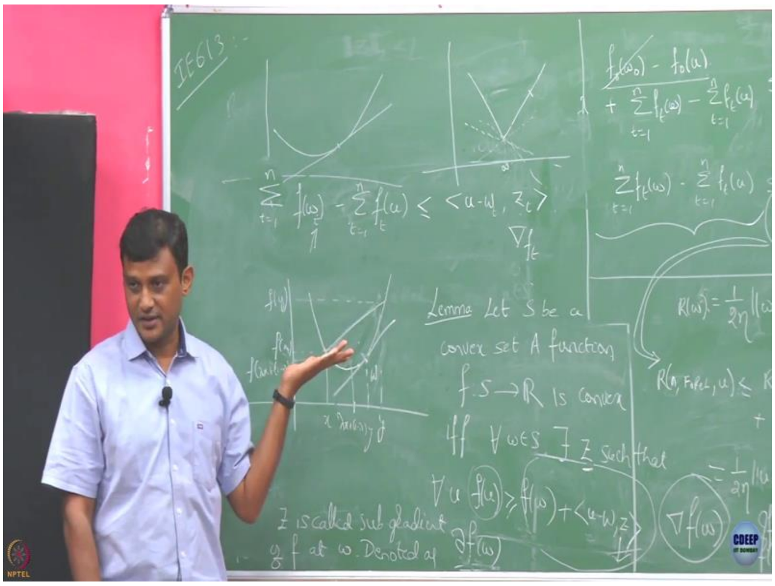

So, anything like whatever I have some let us like a convex function like this. This is differentiable at every point? Yes, that, this is smoothly changing. So, at any point I can have only one uniquely one unique tangent that is passing through that line, ok. But, now take this; is this a complex function? Right. It holds our property, right. If you take anything and my function value is going to lie between these two that is the property of convex lines.

But now, suppose is this function differentiable at this point. This function is going to not differentiable at this point, right because when I approach it from the right it has a the negative slope and when I approach it from the positive side it is having a positive slope. So, at this point it is not going to have a it cannot be differentiable. But, is there a z here? So, let us take this is the one particular w which I am interested in.

This relation can be satisfied by a unique z or there could be multiple z that could satisfy this relation. So, it so happens that in this case there could be multiple z's that could be satisfying this, one possibility is this, one possibility is this. Maybe you can think of many lines which are all kind of touching this point only at one point and they are like a tangent here.

They are also lower bound to this function, and but the there is no unique line there are so many lines, right. And each of these lines can corresponds to one z, right.

So, in this case at this point w my delta f omega can have multiple points. So, here the sub-gradient is a set it is not a single term. But if you look into at this point here my function is differentiable. So, in that case my sub-gradient will have a singleton which I am going to call it as simply the gradient of my function at that point, ok. So, now, what basically shown? How can we exploit this? Right.

If you are going to reorganize this function here what we have is $f(\omega) - f(u)$, ok. So, earlier, ok; so, let me write it in terms of this we earlier we had this function, right this is my regret. Now, take a particular t and look at the difference $f(\omega_t) - f_t(\omega)$. If my f_t is a convex function, I am going to appeal to this function here.

Then, what I can do? Then, this is going to be what? Is this is true? If my function is convex can I upper bound this by like this using this relation? So, what I am doing is in this case I am looking at my sub-gradients at w_t , my points w_t and this z_t is the gradient of my function f at the point w t, ok.

So, if done, I know that this relation has to hold for some z_t, ok. And if my function is convex and let us say it is differentiable everywhere then this z_t, I can as well replaced by f of t, right. So, what we have done basically is we have given, here f is any arbitrary convex function, right. What we have done is we have replaced, we have upper bounded its regret by. So, what is this? This is nothing but, if I reorganize this, this is nothing but $\langle w_t, z_t \rangle - \langle u, z_t \rangle$.

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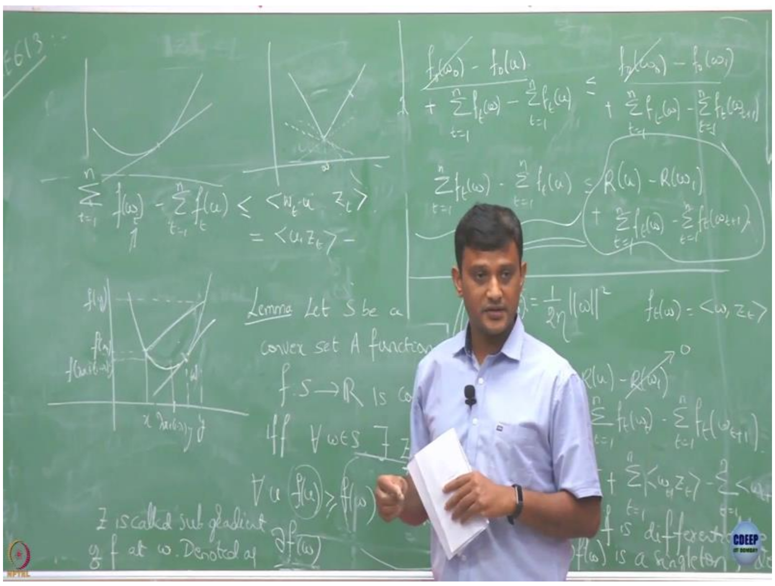

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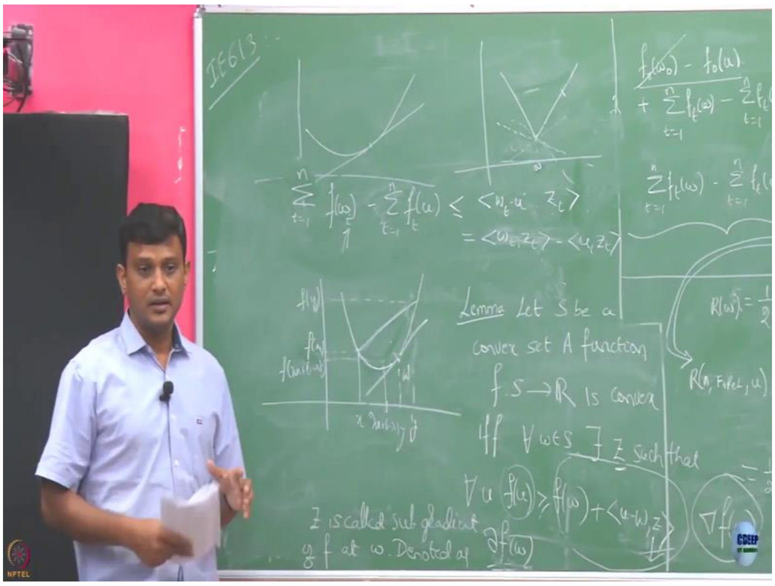

So, if you write this then this is $<w_t - u, z_t>$. So, what we have basically now see is; so, this could be a arbitrary convex function, but if we can linearize it I can upper bounded in terms of a linear function, where my z_t is the sub-gradient of my function at that point w_t, ok. So, once I have this, what are all the things I have done here for my linear functions I can appeal to this and get a bound here, right.

So, that will give me a bound here, but this is already upper bound on my regret. So, that bound also holds on this regret, ok. So, with this I just want to write this pseudo code, then we will leave.

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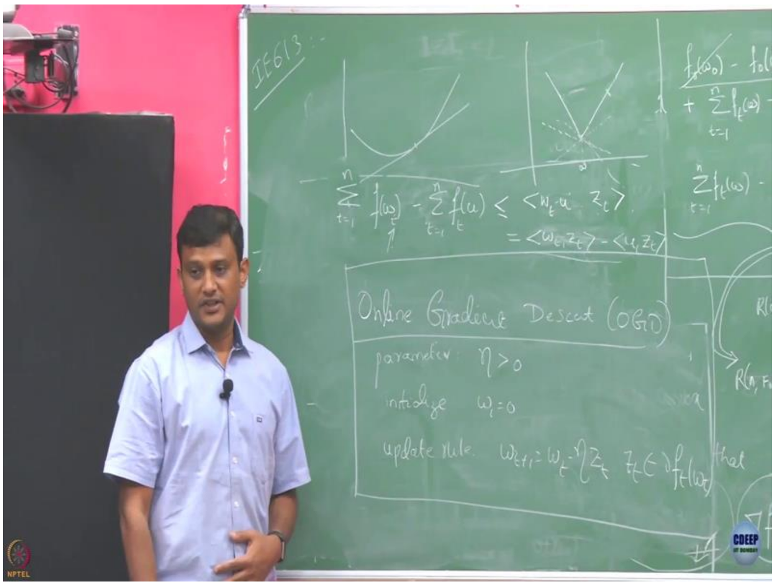

Then what we finally, ended up with is what we are going to call it as online gradient descent algorithm. And how does it work? It takes a parameter eta and then it initialized $w_1 = 0$, because w_1 I do not have any control and then the update rule is $w_{t+1}=w_t$ – eta*z_t.

And what is z_t here? z_t is the gradient of my function at w_t. So, this is what we are going to, we have simplified our follow the regularized leader for the specific case of L_2 regularization to be this online gradient descent algorithm. So, we have already discussed why this is gradient descent, right, ok.

So, let us stop here.