Bandit Algorithm (Online Machine Learning)

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Lecture - 11
Proof Weighted Majority

Today I just want to cover the Proof of the Weighted Majority algorithm, we discussed last time. So we will just do that part. And from the next week onwards, we will start with more a setup called adversarial bandits, but let us try to complete the proof of weighted majority algorithm today.

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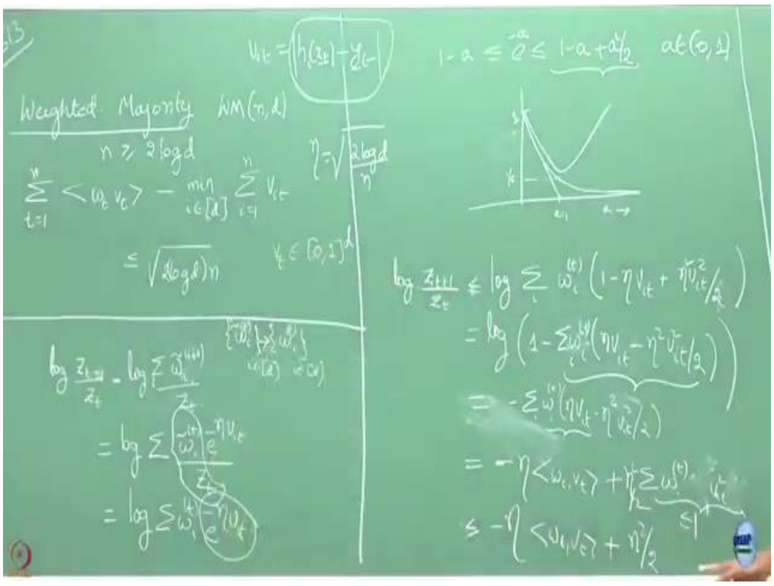

So, if you recall the weighted majority algorithm we did in the last class, it was in a setting we called as prediction with expert advise right where in each round environment assigns a loss value to each of the expert.

And the learner picks one of the expert according to some distribution which he keeps updating in every round based on the past observation. And we defined in this case is expected regret to be; so this is says loss in round t and this is the total loss incur incurred (Refer Time: 01:50) and we compared this with one of the d experts. And we said that this is going to be upper bounded (Refer Time: 02:14); $\sqrt{2(log d)n}$ so today let us briefly discuss why this is true.

So, one of you asked the question; is it necessary that in this case we have to assume that the adversary generates the label according to some hypothesis h; that h not necessarily be in my hypothesis class, but is it required? So, now we said that whatever the setup we are going to study we said that and that could be a mapped to the setup of this prediction with expert advice right.

And now we have said that the loss that you are going to incur that is the loss assigned to expert. We had defined it to be in our online classification setting to be if I apply i th hypothesis on this, what is this value? And this yt; so here we said that now when we were in this setting, this Vt is a vector of losses; we said that in the this is arbitrary generated, I do not know how it is being generated by the environment.

In the binary classification, this was the loss, but I could take this to be my loss, but I can take this to be more general version when I looked into expert setting; I took to be any Vt right. So, if you said that this Vt is coming from Vt $\in [0,1]^d$.

Now, since this t are going to be getting translated to my xt and yt were in this fashion in the binary hypothesis classification, but now I am allowing myself is Vt to be anything. So, fine if you define going to be loss to be like this; but if any other loss is also fine my entire analysis go through.

In that sense, how this yt's are generated; I really did not worry, is it going to come from some specific hypothesis or he is following some arbitrary role which no hypothesis can define; I do not care about that right because this Vt could be arbitrary number earlier right. So, because of this; even in the binary hypothesis class it makes more sense if you assume that adversary is also generating these labels according to some hypothesis, but which need not be my hypothesis class, but we could also relax in that condition.

We can say that maybe adversary generating this labels; according to some rule which I cannot even characterize with some specific hypothesis ok. Now let us try to prove this. First thing this proof this bound we are going to prove under the condition that my number of rounds is larger than 2 log(d)

Remember, I told you that n and d are input to my weighted majority algorithm right. So, let me; so weighted majority I am going to write with inputs n and d; what was d? Number of experts and what was small n? Number of rounds; we are saying that its a number of rounds happens to be twice the logarithm of d, then this bound holds ok. If it is not; I, I do not know and also notice that we have defined a parameter eta how in the weighted majority algorithm.

$\eta = \sqrt{\left\{\frac{2logd}{n}\right\}}$. Now under this condition n is going to be greater or equals to 2log(d); what will be this value of η?

It is going to be less than 1 right ok? So, we are going to prove this condition holds; under the assumption, under this setup where η is going to be less than equals to 1. So, the way we are going to do proof is we are going to consider this quantity and we are going to find a lower bound on this and find an upper bound on this; then manipulated to get what we want; finally, we will end up with this relation here ok.

So, this Zt +1; what is, how did we define Zt + 1? So, Zt is nothing, but the weights of Wi tilde right? You remember that Wi tilde where if you are updating every; every round and then we took their summation to get this.

So, usually we call this Zt here as the potential in round t plus 1; that is the sum of all the weights we have in that round; those weights. So, we had two set of weights one is Wi tildes and another is Wi. From this to this we went when we went from this to this; this became a probability distribution right, this was not a probability distribution. But Zt plus 1 in roundth t plus 1 is nothing, but the sum of all these things. So, there was an index t right here; how did we write the index t? Superscript or subscript?

Student: Superscript.

So now, let us substitute the value of only Zt plus 1 ok; what is this Zt plus 1? That we know as Wt plus 1 tilde; t by Zt and now further. So, let (Refer Time: 09:18); this is i here and I am summing it over i.

Now, let us substitute the value of Wi; sorry this should be t plus 1 here right because I am looking at t plus 1 index here. How did I; now let us write Wit tilde plus 1, in terms of Wit. What was the relation between Wit at t plus 1 and at t? We have defined it as Wit; e to the power minus nu($e^{-\eta V_{it}}$) Vit right; is this correct, this relation?

Then you have to; this is already defined in the weighted, this is how exactly the weighted majority (Refer Time: 00:00) algorithm was working. And so and it is tilde here right and now write it Zt; now, if I look into this ratio by definition what is this quantity for us?

Wit, right; without tilde; so let us I mean we are going to do a series of manipulation on this set up now before we come to this. I mean by the way like when you haunt to prove this bound by looking at the algorithm its apriori, not clear what is the right intuition to go through what steps ok.

But somehow once you write down these steps; it is clear that you will get this bound, but looking at the algorithm what is the; the right steps to go through this, it is not clear. So, the proof here is though quite simple in apriori; it is not clear how to get this ok.

But, so these are some of the classical proofs which we are going to use many many times; so just try to follow all the steps we are doing here ok. Now, let us try to apply some inequalities to get a bound on this. The first bound I am going to use is; if I have e^{-a}; this quantity is bounded by $1 - a + a^2/2$ for $a \in (0, 1)$ and also this quantity will be bounded by 1 minus a; is this true ok? Let us see.

So, this is my; if you have my a; this is going to be my e to power minus a. So, now if you are going to look your a equals to 1; what is this quantity is going to be?

Student: (Refer Time: 12:21).

1/ e right; now is this clear this this quantity is going to be bonded by 1-a? So, how this the graph of 1-a look like? So, 1-a is going to start from this is going to be 1 here and it will reach 0 here right and it will be falling linearly. So, it is going to be like this and this is going to be a lower bound on this quantity in the interval (0, 1); now what about this ok? So, can we see will this quantity will be decreasing or increasing till point a=1?

So, fine can you just check whether it is a convex or concave in a? How do you check concave, convex?

After differentiating what is the quantity here?

Student: It is positive here.

It is positive. So, if it is positive, what it is?

It is convex right. So, convex means it has a minima at one point; what is, where is the minima happening?

Student: (Refer Time: 13:33) 1.

At 1 after that it is increasing, right? So, it is like something like something like this quantity. So, you can see that we have a tight and they will be in this range 0, 1; they are pretty tight in this. So, we are going to use this bounds to proceed bounding this ok.

Now, let us take this; I am now going to treat this whole quantity as e^{-a} ok; a = ηv_{it}. I will note is that η by my assumption; it is less than 1 and Vit also be less than 1 right because they are coming from (0, 1) interval. So, this quantity ηv_{it} is already quantity less than 1 and also it is positive quantity. So, I can apply; I am going to apply this upper bound on this.

So, if I apply this upper bound; what I get ok? Now, further now Wi^t; this is these are probability values right. If you now; if I take it inside with this one and sum it over all i's; this is going to be 1. Now, I have this quantity, but with a minus sign here.

So, this should be minus here right; now minus; I have taken inside and put everything in the bracket. So, now let us define this quantity to be new a for us and now let us try to see. So, my argument is now even this quantity here is entire quantity a is again between 0 to 1 ok. Let us argue why is that.

So, η is less than 1, Vit is less than 1; so this whole quantity squared by 2 is going to be less than 1 and this quantity is η Vit is again less than 1. And now this is I am going to take these are probabilities right; if I am going to take expectation; that means, basically weighing with the probability that is going to be less than 1. Fine, less than 1 being cleared; is it greater than 0? We also want a to the between is it greater than 0 is this quantity non negative, why?

Because this is this quantity, this quantity is going to be smaller than this quantity right because this is less than 1; if you square it, it is going to be further smaller and you are further dividing it by 2. So, now can I apply this other direction bound? A lower bound on this; if I apply because of this negative sign, I will still get an upper bound on this quantity.

So, I am going to treat this as like 1 - a; if I do this what I am going to get? Log of e to the power, this entire quantity here; what is that quantity? And because of this log and this exponential we cancel; we will just end up with this quantity here.

Fine, now I am going to just simplify this. So, this first quantity here is I can write it as $-\eta < W_t, V_t > + \frac{\sum W_i^t \eta^2 V_{it}^2}{2}$; now I will do a one more simplification. So, I will just pull out η square from this; η square by 2 and whatever the remaining, $W_i^t V_{it}$; that quantity is still going to be something less than 1 right. So, I will just ignore that; then I will still get an upper bound.

Because if I just pull it outside here; this quantity here, whatever remains this is going to be less than or equals to 1. So, that is why I have only written this and remaining quantity upper bound and 1; so I get an upper bound here fine. This is we have done it for this ratio; this is true for any t, now what we will do is; we are going to add it over all t starting from 1 to t.

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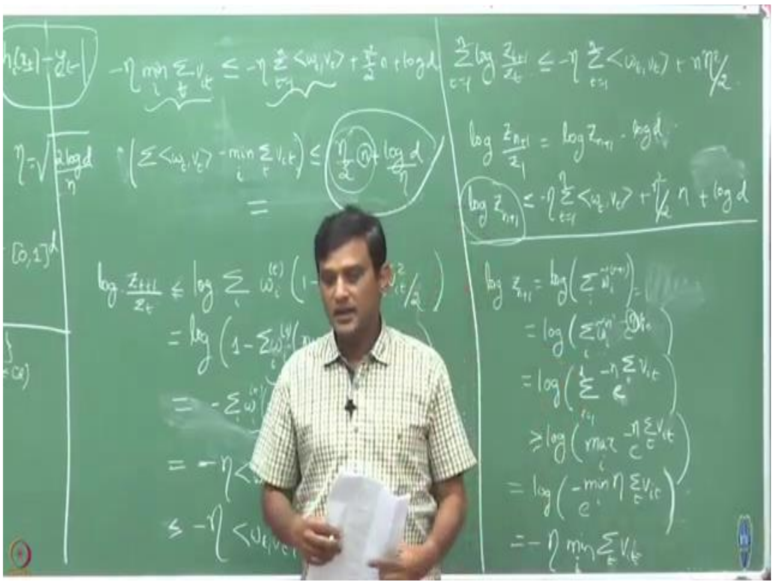

So, if I am going to add; what I will get? I am going to get upper bound is this fine? Now look at this term, I am adding log terms right. So, if you simplify this what you are going to get? So, if you, this is the summation of the logs right. So, I can write it as product of log of products. So, if I do log of products; see this ratios how they are.

They will cancel out and what remains finally?

If you simplify it, what you will end up with Z_{n+1}/Z_1; the left hand side which is nothing, but $\log Z_{n+1} - \log Z_1$. Now, look into Z_1; what is the definition of Z_1? Z_1 is summation of Wi tilde 1’s right. How did we define Wi tilde 1’s?

We said initialize this quantities right? What was this value? 1 and Z_1 is sum of all these quantities. So, because of that what is Z_1?

Student: d.

It is going to be d.

So, this is going to be d ok. So, from this relation what I have finally get is; $\log Z_{n+1} \leq \sum_{\{t=1\}}^{\{n\}} < W_t, V_t > +\frac{\eta^2}{2}.n + \log d$. I will just going to simplify this.. Now, what is by definition log Z_{n+1} ?.

This is summation; Wi tilde at n + 1 which is nothing, but log Wi. So, this is summation over i right and if I write it log summation i; what is this by our definition? This $\log(\sum_i W_i^{n+1} \, e^{\{-\eta \, V_{i,n+1}\}})$. I have just substituted the definition of Wi tilde with this; is this correct?

Now, I am going to keep on defining this; I know that this $W_i^{n+1}$is defined in terms of the previous quantities right. And if I am going to keep on going back this repeating and going backwards repeating; what I will eventually get is, $\log(\sum_{i=1} e^{\{-\eta \, \Sigma_{\{t\}} V_{it}\}})$.

I am just this I have done for; suppose if you just express W_i^{n+1}, you can write it in; in terms of W_i^n; then one term of $e^{\{-\eta \, \Sigma_{\{t\}} V_{it}\}}$ will come. Now, go back and replace the W_i^n withW_i^{n-1}; that will give you another term of $V_{i,n-1}$. So, you can keep on going till backwards; so that is why we are getting sum of all this Vit's here.

This is just by definition ok. So, because of that; now we will end up with this summation. So, now let us try to play with this summation; now I want as you said all want a lower bound on this right. So, now see that this is a summation over from i equals to 1 to d; instead of taking summation over all of them, if I only take one of the index and retain it and throw everybody else; will I get a lower bound or even if I take a maximum one; is this correct?

I am taking; I was taking instead taking the sum I am taking just the max element in that among sums in the sum; so this is going to be true. And this is again if instead I can write it $\log(\sum_{i=1} e^{\{-\min_i \eta \, \Sigma_{\{t\}} V_{it}\}})$.

So, I have just taken; take max of this all quantities is same as e to the power max of this quantity, but there is a minus sign right. So, if I am taking this minus outside that becomes min of this quantity; is this clear this step of the manipulation finally, what I will do is log is there right.

So, this is going to be simply going to be minus of going to just simplify this $-\eta \min_i \sum_{\{t\}} V_{it}$. So, now on the same quantity log of Z_{n+1}; I have in this upper bound and I have this lower bound, now I will simplify it to get the desired bound ok. Just let me erase this figure part here; what we have $-\eta \min_i \sum_{\{t\}} V_{it}$ .And now this is upper bounded by what? This quantity ; I just used this equation and this equation. Now, let us readjust this to get the desired quantity we are interested in. We are interested in deference of summation of the similar product with the minimum quantity right. So, I have; I want to find the difference of these two quantities. So, I will take it on the left hand side, I will eventually end up with.

So, now what I will do? Fine, I got this quantity; now I want to bound, I want to show that this quantity here which I have will be eventually I can right this; get a bound like this on this ok. So, first I will do is I will divide throughout by η; we know that η is a positive quantity right. So, I can divide both side by this quantity and the relation we still hold, I have just divided (Refer Time: 30:59) alright ok. Now, we have taken this η to be some specific value right; what is this η? η is taken to be?

Can you substitute that value and compute what is this quantity is? Just substitute η in this quantity; did you get the same quantity as this?

$\sqrt{\{2\,(\log d)\; n\}}$. So, we you might be wondering ok; why did in the algorithm at all we prefer to choose this ok. So, now let us say you got this quantity; this is an upper bound that holds true. So, n and d are given to you; let us say η is your design choice that you want to set. How we are going to choose η here? You want to bound this; this is your expected regret right.

You are; you have gotten this upper bound, naturally this upper bound you want to make as small as possible because you want to make the regret small right. If you have to make your regret small and η is your parameter that you have to choose; how you are going to choose this η? So, you would like to choose an η which minimizes this quantity right. So, now let us take this function this to be a function in η; now can you find an eta that minimizes this quantity; how you will do it?

Find differentiation and see; what is the value of η you will get that minimizes this quantity? So, you will see that if you try to minimize this with respect to eta; this is exactly the η that it is going to that will minimize this. So, that is why the η has been set to be like this in your algorithm.

So, in a way; η is kind of controlling how much you want to give importance to the losses you have observed right. Like, when I am going to update this weights right; the way we are updating this weights from n plus 1 th round; we are going to take eta times Vin; what are the loss I have observed, I am not taking that value, but I am weighing it by η factor.

So, this is the importance since how much I am going to give the importance to the samples I have observed; while I am going to update this. And one has to carefully choose that weight; if you are not going to careful choose that weight, I mean you may not get a good performance right.

So how to choose this; you see that this bound can change as eta changes there right. Suppose, you chose η to be very large quantity in that this quantity maybe large, but this quantity maybe large small. For the simple case, let us take η you took to be; so this is getting multiplied by n right.

So, if you are going to choose eta to be very close to 1. What is happening? You are regret, you are saying is upper bounded by n; almost right order n which is of no use to me. And if you are going to take eta to very close to 0; you are trying to make this quantity smaller, but what is happening is this quantity is blowing up ok.

So, in a way this η is kind of balancing what we call as exploration and exploitation right. In the first class, we discussed a little bit which we are going to talk more a bit later this parameter saying these are the losses I have been observing from this.

But it may be like initially I observed few loss small loss on something, but I need not necessarily latch onto that. I need not to start assigning high weights to that, I will be cautious about that, I will only take its value with this much weight; eta weight ok. So, this is this parameter eta that find; gives as a fine balance between how I am going to do explore in exploit and that has to be carefully chosen.

And you see that if my n is going to be large right what I am basically doing I am setting a small η small setting a small eta means I am giving less significance to the weights I am observing; that means, am I forcing exploration here or I am preferring exploitation when η is small? You are basically forcing exploration right because you are not giving too much importance to the samples you have been already observing.

So, when η is; let us say small, how will be the distribution look like? So, when η is small right; this quantity is like almost like constant like because if this η small, this quantity is going to be small that is e to the power something small; that is almost close to 1 right.

You can realize later if you want, but eta is small this Wi's; all the W i tildes become kind of equal; that means, you are giving equal importance to all the experts; that means, you are basically forcing more exploration. But if your n is large; that might be right because if we have lot of many many rounds to play, you may be to do a little bit more exploration.

Before you kind a figure out what is that, but if n is large small; you do not have that luxury to do lot of exploration initially. So, you want to start right away thinking about taking the observation you have made bit more seriously; that is by giving them good weight ok.

So, that is why one has; this η has to be very carefully balanced and that has to then necessarily depend on how many numbers, number of rounds I am dealing with ok. If you have lot of numbers, you may be more free to explore. So, I do not care initially because I have a lot of rounds I will eventually find out, but you have less rounds; you have to be more careful ok. So, this algorithm is exactly trying to do this kind of balancing, exploration, exploitation; well by choosing this eta appropriately.