Introduction to Stochastic Processes
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Lecture 06
Continuity of Probability

To make this other notions a bit more formal for the proof we will just try to understand what we mean by continuity of probability? We just now understood what is continuity of function? But probability is we say probability is also a function but probability is function on what?

Student: On the right angle.

Professor: Probability is defined on what?

Students: Events.

Professor: Events? Probability is a function from events to?

Students: (())(01:08)

Professor: Let say 0 1 or real number if you want to make it more general. But now when we are talking about probability it is defined on sets not points.

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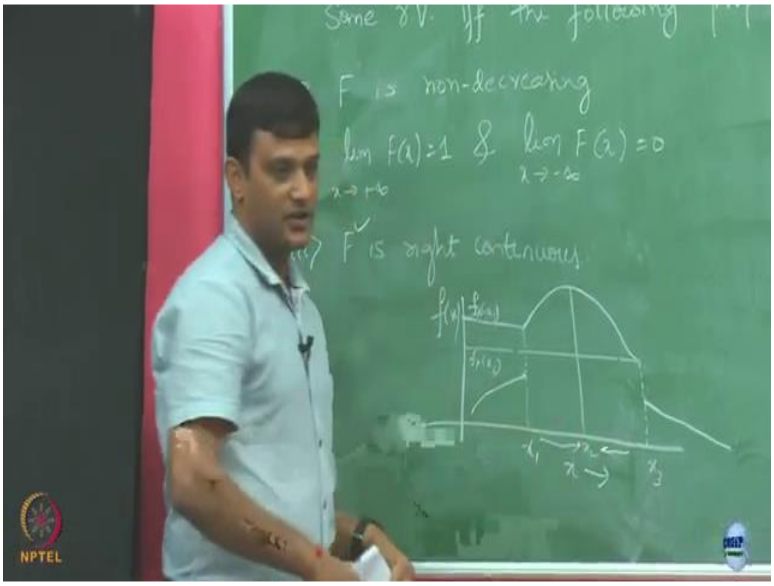

Now when you are going to, so when we define continuity in at least in this one dimension we define on points and our definition included that right continuity, left continuity and all rights to define right continuity and left continuity we had a sequence of points. So like here when we say it is continuous points that means we took right limit sequence of points converging to x2 from left and right and then verify they coincide.

But now if you want to define continuity of probability we need to have a similar notion like of this convergence of points but define on sets. Because probability itself is defined on sets. Now how are we going to have this what is the notion of convergence in sets.

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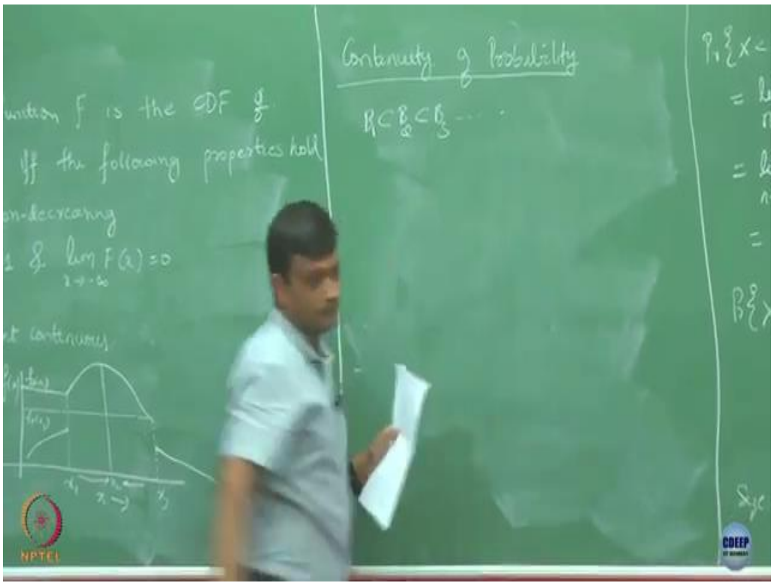

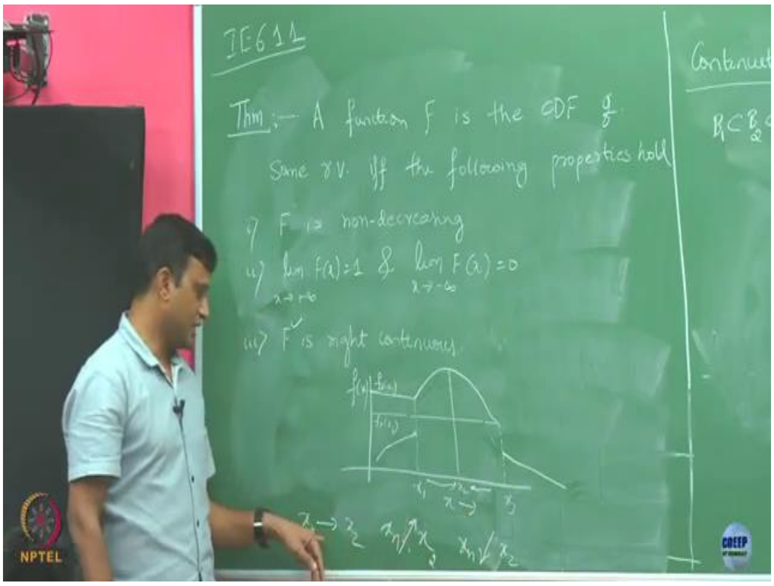

So we are going to, so let us say we have the sequence of sets so when we looked at sequence of points I looked at points which are, so let us say I want here I want a sequence xn converging to x in this case. I can get the sequence xn converging to x in multiple ways. One possibility is this xn convergence to x in this fashion what I mean by this, xn is increasing sequence and it converges to x right that means I have a sequence which is approaching this point x2 from left.

So all the sequence here are increasing and eventually they are convergence to x2. Other possibility is I could approach the same point x2 from?

Student: Right.

Professor: Right and that sequence I can right it as like this or here I am coming so when I am defining this limits I am looking in this case the right and left limits, a sequence which are monotones here right like I am going to get a monotonously increasing sequence when I look for left sequence and I am going to get monotonically decreasing sequence when I look at the right sequence that is converging to the point x.

So let us say we also look at the sequence of sets and we are going to define the limit on this monotonically increasing sets when I say monotonically increasing sets that means they become larger and larger B1 is contained in B2 and B2 is contained in like this. Now in this case what is notion of limits.

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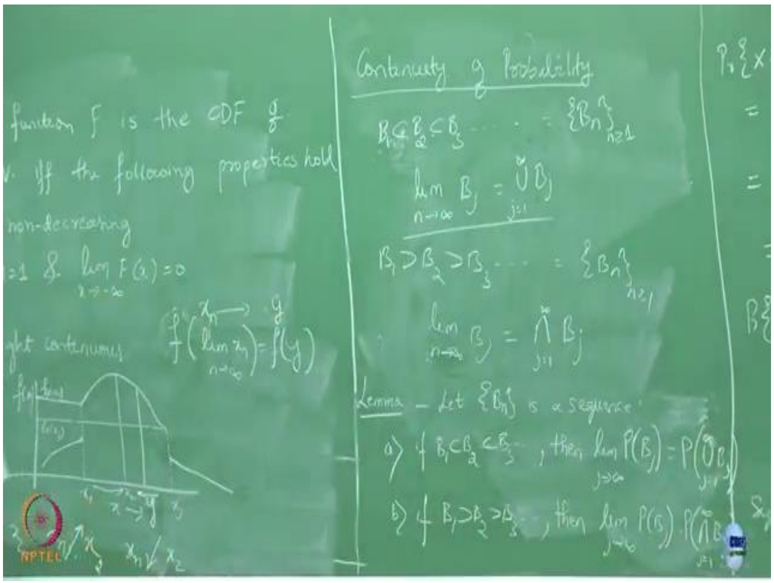

So this is nothing but I have this sequence of Bn.

Student: Sir B1 is equal to B2?

Professor: Yeah so we can consider that there it could be included. So that is what I said right when it is included I am not going to write when it is not included then I am going to write it like this, if I just write like this that means B2 could be same as B1. Now suppose if you have a sequence like this, what you expect the limit to be?

Student: (())(5:30)

Professor: Why sample space?

Student: Because that is the biggest one?

Professor: The biggest one how does the biggest one look like? it could be possibly union of all this right so I could right limit of n tends to infinity Bj as union of Bj j equals to 1 to infinity. So Bj are increasing such that anything… j tends to infinity what are the set we have here that should be at much you can verify this like so how we are going to verify?

Like if you are going to let j equal to infinity whatever the set in that limit if Omega point is there is that omega point also belongs to here and now if you take omega point in this set will that be also belonging to this sequence as j tends to infinity. So because of that we can verify that I mean this is just like definition we will just take like a definition and this make sense. Now we have a equivalent notions of xn converging to points here but in the on sets.

So similarly if we have B1, so how you like this to be defined as, what should be the natural?

Students: Null set.

Professor: Why null set? It should be null intersection. So this is like a decreasing sequence whatever the value we have eventually that should be at the intersection of all the points. So this thing here is analogue of xn converging monotonically xn is a monotonical sequence converging to x2 and this is another analogue of x2 is a monotonically decreasing sequence converging to x.

So now let us define, so before I make… so if this function f is continuous at point x2 let us take this x I have a sequence xn which converges to instead of x let me call this y I have a sequence let us call this as some point y, so what does this mean? Limit as xn n tends to infinity equals to y. If F is continuous at y, is this true? If the function f is continuous at y do you expect this to hold this is true irrespective of how this sequence xn is converging to either from the left or right we do not care or it is mix of both left and right.

So any sequence xn that is converge into y should satisfy this property and if this hold then we say that my function f is continuous at y.

So now let us apply similar notion here, so with this we will show that this is let us write it as Lemma. So this is our claim. May be a better way to write is let P be a probability function let Pn be a sequence. So what we are saying is let p be the probability function you are given other Bn sequence and now if this Bn sequence whatever is given is monotonically increasing then this P function will satisfy this equality which says that, you should have going to take this probability of this sequence P B and compute their limit that limit is going to be nothing but probability on the union of the sets.

Is this clear? And similarly if we have if this B sequence is like a decreasing sequence then if you are going to take apply this probability on this sequence and take the limit this is nothing but the limit is nothing but the probability on the intersection of this events. So in a way what we are said here is, so the other way of writing this guy here is.

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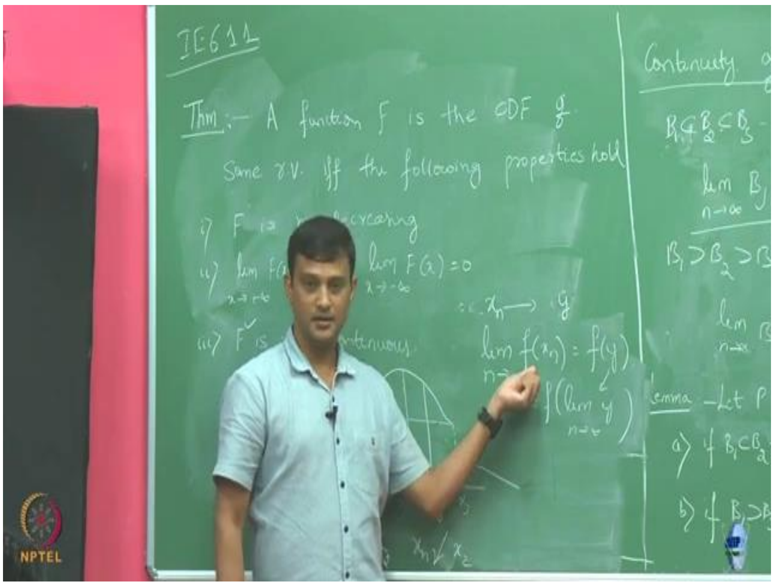

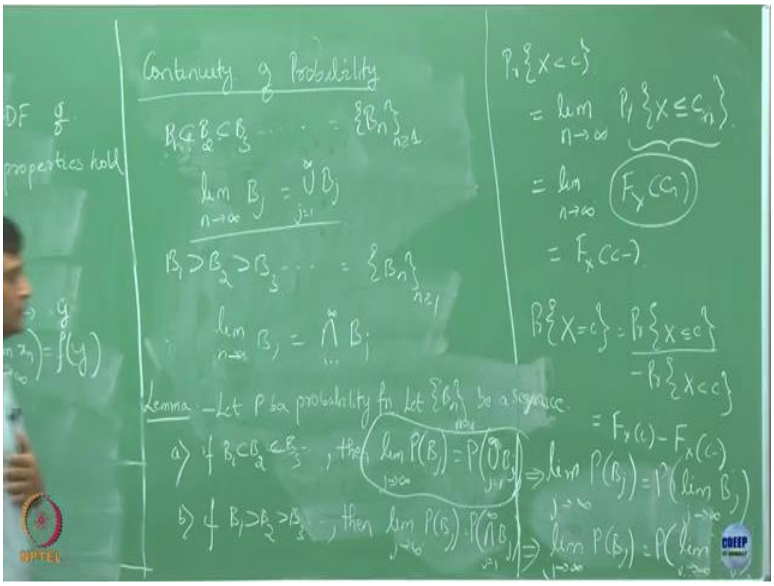

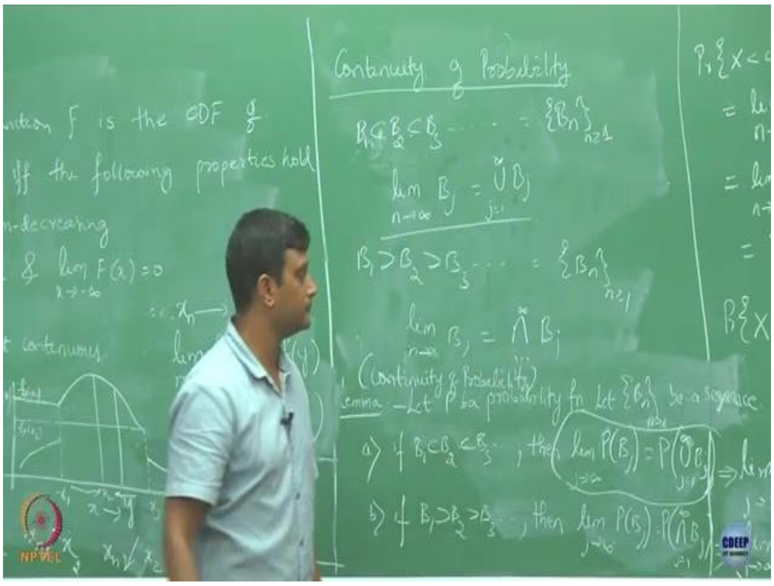

Limit j tends to infinity P of j is equals to limit of j tends to infinity of B j and limit, so what we have basically done is I have just replace this definition this union of j1 to infinity is nothing but what limit as j tend to infinity of Bj. This is our definition now what we are just saying is if you want to take a limit of the sequence of the probabilities this is nothing but take the probability of limit of Bj that means we have basically interchange this probability function and limit.

So now first we took the limit and then we are applying taking a probability taking the limit it is saying that, this is going to be same as first take the limit and then apply the probability on that. So just whenever you have this limit the probability function is such that it allow us to interchange between limit n P it exactly what happen here also, so what was y here, here y was limit was n tends to infinity of xn.

Now so this F of y was nothing but limit of f of xn as n tends to infinity. Or maybe I should have written this one slightly differently way. So if f is a function continuous at y we know that limit as n tends to infinity f of x equals to y but this is nothing what is y, y is nothing but limit is n tends to y so this is the definition of y. So here if f is a continuous function at a point y at that point for any sequence that converges to y, I am able to interchange this limit and f in this fashion.

So this was the definition or this was the implication of my function f being continuous at y and this is what I am also saying similar properties for my P here, p also I am sure we have what I have not assure this but what I have showed the way I have written acclaimed is it shows that I can interchange the limit in this way can I interpret this function P is a continuous function doing the analogy with this guy here. So whenever f was continuous I am able to interchange the limit and the function.

So here I am also doing this function. So because of this, this property is going to be called as continuity of probability. Let us quickly show this, this will, when all of you are with like does this make sense to claim that P is the continuous… probability is a continuous function in some sense because of the analogy we draw from what is the property of a continuous function, fine, so these are finer aspects we need to see which we are going to later. As I said to proof that part. So now let us try to argue, why this should be true?

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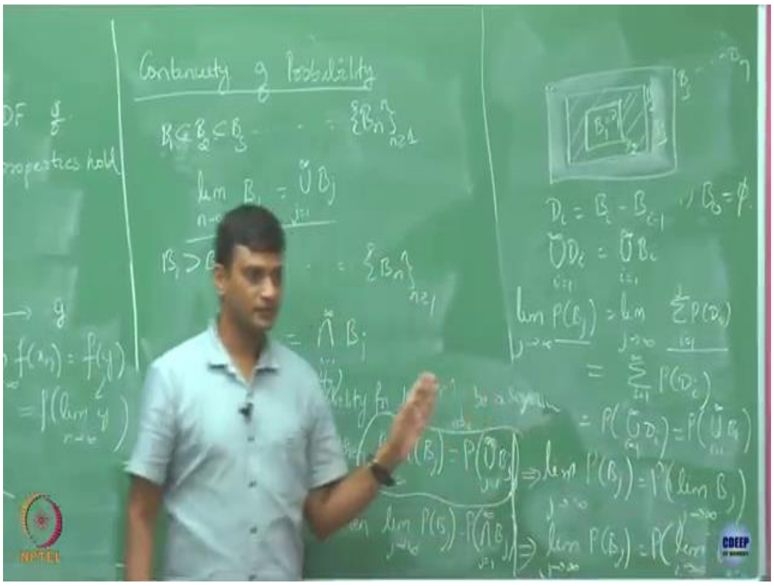

So I have this increasing sets of sequences let me call this as B1 this is B2 and B3 and like that we have. So now I am going to define this to be D1 and the difference here to be D2 whatever that is there between B1 and B2 and similarly what is there between B2 and B3 let me call that as D3. I can keep on doing this for everybody, that means so what I have basically done is I have taken Di to be?

Student: Bi minus (())(19:11)

Professor: Bi, and I want to define it as null set. Is this fine? Can I define my sets like this iteratively . So this is I have another sequence Di and is this true that union of Di is equals to union of Bi?

Student: Yes.

Professor: Yes, no because they are basically capturing the same elements together I mean you just take B1 B2 their union or just take their difference whatever the increment you are going to get from B1 to B2 and B2 to B3 you just take their union we should get the same thing. But what is the difference between these two while the sequence Bi are nested this Di are?

Students: (())(20:23)

Professor: Which will be exclusive because of the way we have defined it. So is that clear I can write this as union of Di same as union of Bi. What I want to show, I want to show this let me take this limit as j tends tau P of Bj I am now going to write it as Pj is equal P of Di (())(21:11) to j is it correct that probability of Pi is equals nothing but the some PDi from I is equal to 1 to j because Bj can be represented as union of Di.

And there it is joined and from the property of Di I know that the finite additivityproperty satisfy right, the probability of union of Di is nothing but summation of probability of Di. So that is what I exactly applied. So this one of the properties or the axiom that we assume P should satisfy. So now by definition this is nothing but I equals to 1 to… so I have just like this, this is a I am just letting G J got infinity I am just letting, just taking this upper limit to be infinity that was the definition of Di . And now I know that these Di are what this joint. Now what property can I exploit here if I want to write it a just one probability here instead of summation of the probabilities, probability of union of?

Student: Di.

Professor: Di, right this just the meaning of some of probabilities of these join sets. Now we already know that this union is nothing but union of Bi because they are covering the same set of element, so this should be equal to than probability that union of Bi i equals to 1 to infinity and this is what we wanted to show here right. So if you want to show this on this set on the decreasing sets what is the changes you can possibly think of how you are going to, so how to now define your Di appropriately.

How we are going to that? Largest one at infinity you do not know, so think about this like how you are going to do so just do it as an exercise, you should be thinking along exactly the same line. So ultimately from this, so all this due to just to convince our self that if there is a probability function I am able to interchange limit and probabilities just like from the whole (())(24:18) just we are going to just take it like that fine whenever I have a probability function I have a sequence of probabilities sets that are monotonically increase I should be able to increase.

I should be interchange probability and the limit in this fashion that is the only thing I wanted to use from this part. Now let us go back here. So do you think this is obvious so let us come back to the properties of CDF this is the cumulative density function right, so accumulation means we should keep on accumulating and what we are accumulating we are accumulating probabilities which are non-zero non negative quantities.

So it should be… this function should be increasing, so how to show this formally. So see this I have said if and only if, you guys understand what I mean by if and only if, so what does that mean.

Student: (())(25:25)

Professor: So what we are saying is suppose if CDF if I am saying CDF then this is CDF for already some random variable right then it should be satisfying all these properties and now if F is some function which satisfy this then there is… it should correspond to CDF of some random variable what is that random variable you do not know right now it may depends on what is the f we are talking about. So in this we are only going to show only if part that is if F is CDF of the some random variable it is going to satisfy this properties, the other direction we will skip.

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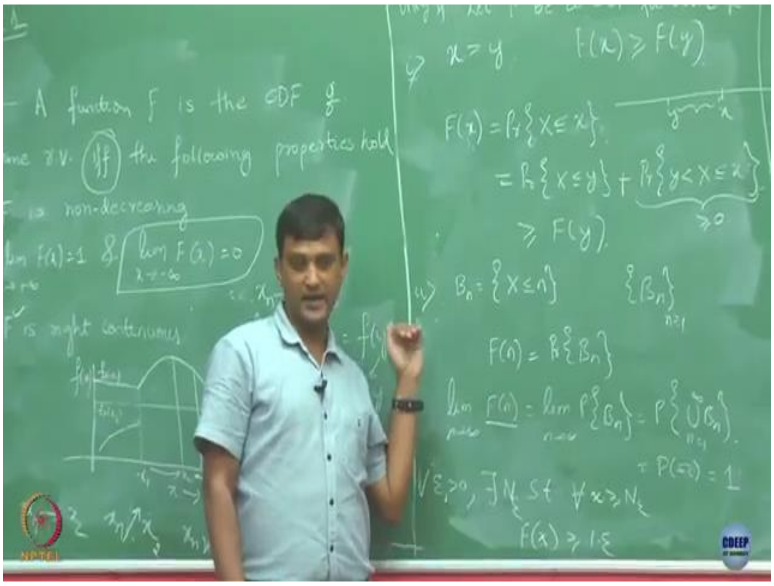

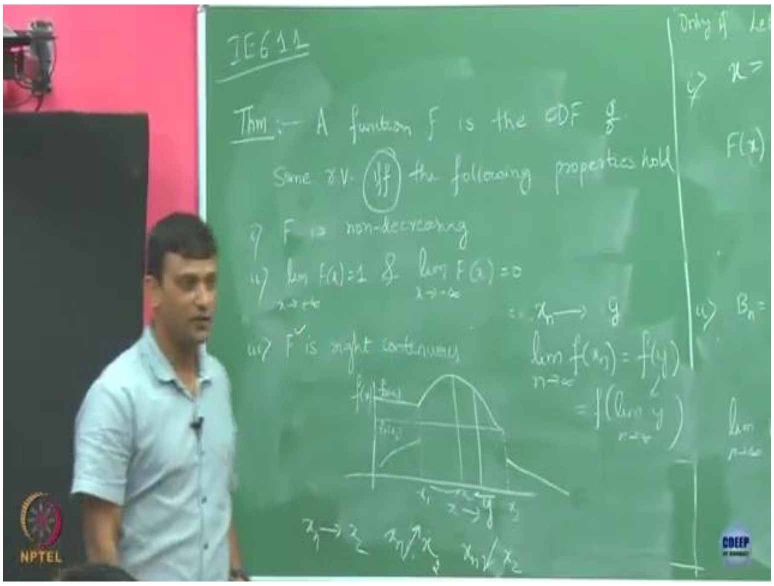

How we are going to show F is not decreasing if you are going to take x greater than y then we need to show that F of x is going to be greater than or equals to F of y this is the meaning of that F is monotonically increasing. So let us take the case only x strictly greater than y because if x and y are same nothing to show right, so let us take x is strictly greater than y and then try to show that f of x is strictly greater than or equal to y how we are going to show this?

So by definition F of x is going to be less than or equal to x and then this is the case if Y is strictly less than 1 it should be the case that x is less than or equals to y plus x is, can I split this probability like this, so what I basically have done is I have taken x is less than or equal to… so we have the way I have done first comes y than comes x. So probability that x takes that value is less than or equals to this is same a probability that x takes the value less than y and probability that detects value in their between x and y.

And now it is obvious that this should be equals to f of y, why is that? So we know that this guy is going to be greater than or equals to 0 and this is nothing but F of y. So now let us try to show the second part. Let us take a sequence Bn x is less than or equal to n. So before I do this when I say only if part we are going to say that let f b a CDF for some x and then I am doing this I am already assuming that f is CDF of some random variable and I am doing this. So let us take for that random variable let us define a sequence like this.

Now if you look at this sequence Bn is it monotonically increasing, yes or no?

Student: Yes.

Professor: So that means that x I am increasing n right so it should include more and more elements from my sample space, so that is why so this is a monotonically increasing function. Now what is F of n? It is nothing but probability of B of n, my definition f of n means probability that x is less than or equal to n that is exactly Bn. Now if I let n got infinity and now at this point I want to exploit the fact that my probability function p satisfies continuity property. So if that is the case how can I write it?

Now what is this union so what is Bn, Bn is excess less than or equals to n and now I am allowing this n to go to infinity so what is this quantity is going to be?

Students: Sample space.

Professor: Sample space right so this is P of omega equals to and I know that is going to be one. And I have done with this part of the proof, yes? From this so does this whatever I have just shown here, does this concludes the proof of this part, yes but I have done it for n integer right but whereas this x can be any real number after that what does not matter.

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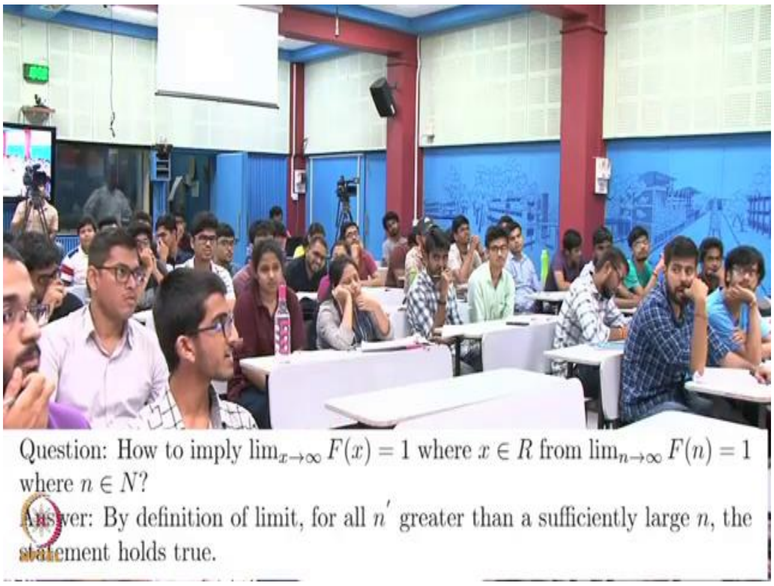

Students: Because there is (())(32:09).

Professor: So what cumulative value, why is like that… so what you are saying fine according to this definition. So F n are now real numbers right so we have this definition according to this what their for all epsilon greater than 0 there there exist some N epsilon such that for all N greater than or equals to N epsilon I have F of n greater than or equals to 1 minus epsilon. This is the definition of the limit.

You take N sufficiently large then the epsilon F of n is going to be less than, now instead of this N now you look for all the points beyond this N, any point beyond this N and my function is such that it is now decreasing right? It is increasing, so all the points above this N should be also be greater than 1 by epsilon, right? So because of that if I am going to took all the point which are greater than N and also if I look at all the points between N, N plus 1 like that you can convince yourself that this is in this true even if I replace N by x any real number which is instead of x we can say that for all x greater than epsilon this is going to happen.

And because of that this is true that I can assume x goes to infinity limit of F of x is equals to 1. So you have to convince yourself that yes I can replace because of this monotonicity property of my function F even though I have shown it only for the integer value I can replace it by a continuous real number and then the very definition of the limit says that this is true. And this is like in a similar fashion you can show this all you need to do is replace n by minus n.

So I wanted x go to plus infinity right? Now I want to go x to minus infinity, so replace Bn by B minus n that means X less than or equals to minus n. So in this case what my sequence will be?

Students: Decreasing.

Professor: It will be decreasing sequence now instead of increasing sequence but even for the decreasing function your probability definition that the limit and probability you can still interchange right? And then you do the same thing but now you are coming from the in the negative direction that is why this should be true. The last property, so what I am saying here is F is right continuous when I say I did not say F is right continued at some point right? I just said F is right continuous. That means I need to show at any point x my function F is right continuous.

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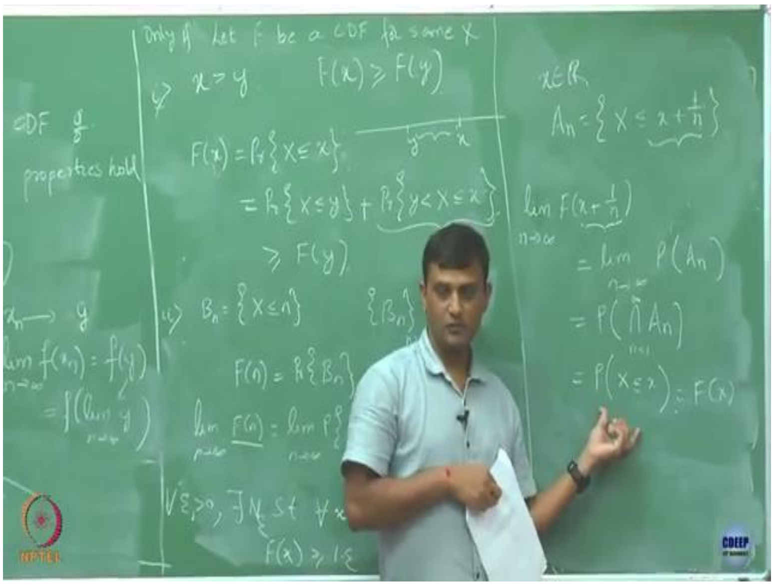

So take an arbitrary point x R and we have to show that for what are the arbitrary point x you are given my function f is right continuous. So there I am going to define a sequence Fn equals to X. So x is the same point whatever you have taken and now for An define a sequence like this now is the sequence the set of An is a set right?

Is this increasing set or decreasing set? It is going to be decreasing right? As you increase and this is going to be shrinking x plus 1 plus n. So now again apply our standard trick of using continuity of probability here. So this is nothing but limit of n tends to infinity of probability of An. I just use that like this quantity here F of x 1 plus 1 by n is nothing but probability of An by our definition An.

And now because this An is monotonically decreasing function and by the continuity property of P how can I write this is nothing but probability of intersection of… right? Now go back to the definition of An, An are looking like this, what is going to the intersection of all this An?

Student: (())(38:18)

Professor: X or 2 X are there.

Student: (())(38:25)

Professor: It is going to be like if I have going to let N go to infinity here the limit this will be simply equals to x right because this is nothing but this is the sequence which is approaching from right to x and now that is why we can say that this is nothing but this is the limit of x less than or equals to this. So now what we have basically done is and this is nothing but F of x by definition.

What I have basically done is I have taken a sequence here which is approaching x from the right, right? So this sequence as n goes from 1, 2, 3, 4 up to infinity this is approaching from right the quantity x here. So I have constructed a right sequence which is approaching x from the right and for that we have shown that this is true.

But to show right continuity what we need to show take it any arbitrary right sequence that is converging from the right and show that this holds. So now by this self my proof that F is right continues is not complete right because I have what I have demonstrated through this is there exist a particular sequence that is converging to x from right and where this holds but what about the arbitrary sequence this is convergence.

Do you think using this argument you can extend the same analysis to any sequence that is converging to x from the right using which property?

Student: Continuity property.

Professor: Continuity I have already used, so do you think we can use the again the monotonicity property to make this argument work for any sequence such at that, this 1 which I have proved it for a particular sequence the same argument can be used to argue that for any right continuous sequence this is true. How is that true? Like this is going to be a same argument like this right here I showed you for integer value.

So here in this proof we showed that we can extent the argument for any x using the argument for integer that holds for integer values. It is also the similar case here now we have a instead of n we have just 1 by n here we have an argument that construct a sequence using this integer valued sequence well not integer value but a sequence, but now you can show that using this you can come up with this argument extends to any sequence that is right converging to x.

So just see that like 1 by n is converging to 0 right at some point it should be falling arbitrary close to 0 if it is a right continuous sequence, so any right continuous sequence also, any right continuous sequence, any sequence that is converging to x from the right. So if that is the case this xn should be also going to be 0 it just like that this 1 by n going to 0. So what are the values between that you can fit as any other sequence you should be falling between one of those 1 by n and 1 by n plus 1 and then you can say that this is true for any arbitrary right sequence that is converging to x from the right.

Fine so I hope that you will convince yourself that this is going to work for any arbitrary sequence that is converging into x from right. So then we are done with all these three points right. So before we leave can you, fine we just define somethings and try to prove their properties right, so do you think where the CDF is going to be useful. Do you think CDF is going to be useful at all or just like fine you just define it and do it whatever you like to do with it.

So the CDF function is already available we already know that probability, so if you have this precomputed CDF function we already know that what is that suppose the way the temperature varies you have a distribution on that and you know already the CDF. So if you want to know what is the probability that my temperature today in Mumbai will be less than let say 30 degrees just look at the CDF and you already have the value.

We just show that like how that CDF what should be the properties of the CDF and this is going to be help lot further as you will see.