Introduction to Stochastic Processes
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Lecture 05
CDF and its properties

So in the last class we ended up by defining what is a random variable. So we said that the random variable is a function that gives real number to my sample space and such that it is f measurable or it is measurable on my event space. So today we will just define a notion of what we call CDF Cumulative Distribution Function and study properties of cumulative distribution functions.

And So to prove this notion some properties of cumulative distribution function, we need to have some understanding of what is, what is continuity of probability is. So we will take a slide detour and study what is continuity of probability then we come back and complete the properties of cumulative distribution functions.

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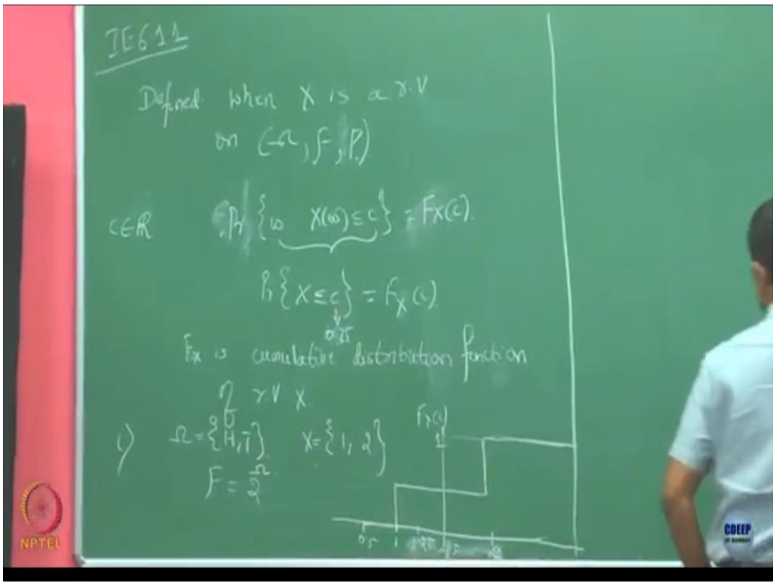

So last time we defined and we say that and now suppose I give a C real number. Then we said that this quantity that X of omega less than this belongs to f. This is what we said as definition of f measurability. Now suppose, I want to know I know this quantity here. This is for a given C whatever this, I know this quantity belongs to my event space.

Already that is if X is a random variable this already belongs to event space and I can if I further know my probability function, I can ask what is this probability. So this is now I have defining for any given C what is the probability. So let us call this f of x of C. So X is my random number and for any C you give me I am going I can define this.

And now this F of X, this function or I am going to write this quantity here. Henceforth simply as probability that X less than or equal to C. So when I write this it is what this means is set of all w such that X of omega is less than or equal to C. This is the shorthand notation I am going to use for this.

Student: (())(03:41)

Professor: So this is the definition. I am further going to call this, this probability as F of X C. So X is a random variable given to you and now on this random variable for any C coming from r I can define a quantity like this. That is what I am going to call it as F of X C and this quantity we are going to call it as cumulative distribution function.

So notice that I am defining this cumulative distribution function on this x which is a random variable. So that is why this function is well defined here fine. So I have now defined something called cumulative distribution function. Let us see how does this look for some of the random variables we know.

So let us say so, we know that let say let us an take an example of a simple coin toss problem. The outcomes are heads and tail but I am going to now on this I am going to define a random variable X which is going to take value 1 and 2. So 1 corresponds to head and 2 corresponds to tail. I can define a random variable like this.

So because random variable is just a map which gives real numbers to your sample points. Now, let say on this I want to and assume my sigma F is just the power set of my omega. So to understand what I this notation 2 to the power omega that means just power set. All possible subsets of my omega.

Now this X is F measurable if I because this is, we have discussed last time that if your F happens to be power set then any function any random X we are going to define on that any random variable with that sigma algebra is going to be satisfying the properties of a random variable definition, function.

Now, so I have an X here. Now let us try to understand how this function F of X looks like. So, how do you expect it? So let us say this is my x-axes. This is my C here and this is my. So if I take any value, so less that is less than 1. So now let us say I am going to take a value of C. So to plot this F of X function you need to find this quantity for all possible values of c. So now let us take a case where my C is less than 1. So if my C is less than 1 here, what is the probability that X is going to be less than or equals let us say C strictly less than 1. Let us say for time being let say C equals to 0.5.

So what is the probability that X is going to be less than or equals to point 5 in this example that is going to be 0. So till what point this is going to be 0? Let us say for time being this is going to be 0 and let say my coin is fair. Both heads and tails are equally possible. Then let us say now I take this C to be 1. What is the value of F of X of 1 is going to be?

Student: (())(08:44) 1 by 2.

Professor: 1 by 2, why is that? Because if I asking X is less than equals to 1 what I am asking is basically that what is the probability that head occurs in this case and that I know is going to be half. Now let us take another point here till 0.75.

So if now I said the C to the 0.75 what is this value is going to be. It is going to remain same. Till what point it is going to remain same?

Student: 2.

Professor: They just want to write it as 1.25. I want to write it as 1.5 and I want to write it as 2. So now this is going to be remaining flat till what point?

Student: (())(09:57).

Professor: At to be what happens? This is going be, and what happens after that for any value greater than 2, this is going to remain 1. So this is the simple case that, that is going to depict this function F of X for this simple example of coin toss.

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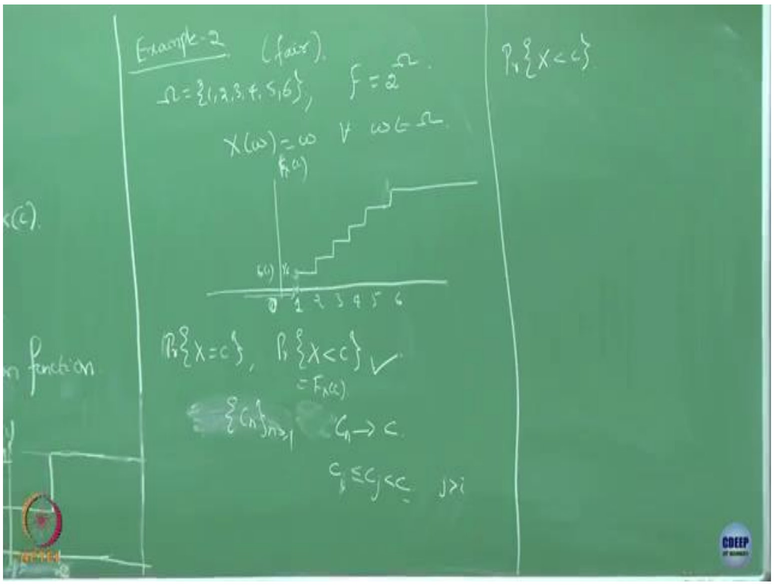

So now let us look at another example, which we should now be able to quickly plot. So now let us say my second example is toss of a coin sorry, toss of the dice. So in that case, my omega is going to be 1, 2, 3, 4, 5, 6 and let us take my F to be 2 omega and let us take my X of omega to be omega for all.. So now how does mu so let us start from 0. So, now how does it look like?

So let us say my coin is my dice is fair. So what is going to happen before 0? So this is my F of X. So it is clear that till 1 this is going to be 0. What happens at 1?

Student: (())(11:43).

Professor: So now it is going to be 1 by 6. Then it is going to remain like this. Then it is going to be like this and after that it is going to move flat like this. So, fine what this is giving you? This by the definition of my cumulative distribution function it is giving you probability that my random variable takes value X less than equals to C for all possible values of C.

Now does this say anything about then what about probability of X equals to C and what is what about probability that X strictly less than C. What we have defined is probability X less than or equals to C. Then why not define it why not define something like a another cumulative distribution function which is defined like this. So I am asking basically question.

Why is that F of X of C, you have defined probability that X is less than or equal to C, why not it define it like probability that X is less than C or let us define F of X of C, simply probability that x equals to C and this is a definition that I had introduced. I have called something as CDF that is probability that X is less than or equals to c. I could have as well defined like this and call this cumulative distribution function. Why not?

Fine so, it so happens that even want these things, everything here it can be just represented in terms of this and we could have defined in different ways, but we have to choose 1. Let us choose this, this looks more appropriate because we are talking about CDF and in this case we want to include everything.

But using this we can even represent what are these quantities. So now how to represent these quantities now in terms of my CDF. So what is this saying? Let us focus on this. Probability my X takes value everything till C but not including C.

So if you want to so, suppose here, here what I have done. In this example, this quantity here at 1. This is what F of X equals to. This is like 1 by 6 but suppose if I want to redefine my function F instead of X less than equals to C I want to if I redefine it F of X, F of X C to be simply probability that X is strictly less than c. Then where it would have jumped here it would have jumped exactly at 1 or before or where it would have jumped.

Whatever it is like. So, how this function would have, let us only take this. So, if I am going to define this to be F of C. What would have happened at C equals to 1 here. At 1 also it would have remained 0.

Student: (())(15:53).

Professor: What would have happened that exactly 1? So then what is the difference between this and this? What is the difference between this function and if I just include less than or equals to.

Student: (())(16:11).

Professor: In which Case?

Student: (())(16:17) If X is less than equal to.

Professor: Yes, In this case.

Student: In this case it is 0.

Professor: So X, let us say X equals to 1, X is strictly less than 1. That means 1 is not included. What is that probability?

Student: 0.

Professor: It is still going to be 0. At still at 1 point it is going to be 0 here and maybe just soon after that it is going to jump. So let us try to understand this, how to represent this quantity in terms of this, Suppose, let us take a sequence. C 1, C 2, this is a sequence, such that let better at let say C n is a sequence such that C n converges to some point C and but it is so, let us take us a case that everywhere C j and I take this sequence such that C n is converging to C and it is monotone.

So let us say here j is greater than i. If you take a index j which is larger than I, C j is going to be larger than C i but it is still less than C. Both of them. So what basically I am doing is, I am taking a sequence here. Let us take this point. I am taking a sequence. Suppose if C is equals to 1, I am taking a sequence here that the limit converges to 1 but none of this points all these points here, they are strictly less than 0. That because there everybody is on the left side of 1. So it is all going to be less than 1.

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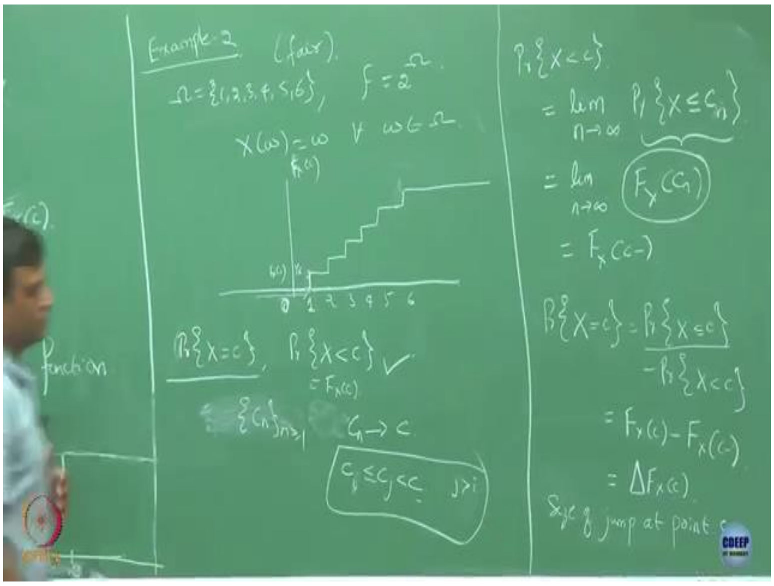

Now in this case, how can I represent this quantity probability that X is less than C. Is this true that this is equals to limit as n tends to infinity. Probability that X is less than or equals to is this correct. Why is this I have bought in inequality less than so this is, sorry, I mean, I am just using this index here.

Everywhere what I am doing is basically I am looking at a sequence which is converging to c. I am taking C n here where every C n is less than C. Strictly less than C because of this, I have this limiting condition. So we will just take it. We will come back to this definition again where I said we have to make a detour. We will make a detour in a bit in a moment from here.

So, we have this and I know that this quantity here is exactly F of X, C of n. This is by definition. Now if you have this, now how can I can represent in terms of F of X C n or like a if I want to get the probability that F of probability that X is strictly less than C, I can express it in terms of this, but now what is and this must be true for any sequence C n which satisfies this property.

So, what it is what we are basically saying that if you want to compute this probability that X is less than C, take any monotonically increasing sequence that converges to C but every point that is it is monotonically increasing from the left side and that is converging to that point C. Then I can write this probability as the limiting sequence of this function F of C is this clear.

Like how I can, if I am going to define my function F of X to include this inequality here and then the probability where I want X to be strictly less than C, I will get it through this limiting case. And now I am going to define this case here as F of X to C minus and now I am just defining this. This entire limit as this. What is this is like if you take any sequence that is from the left approaching C whatever that limit you are going to get let us call it F of X C minus.

What it is saying basically saying that the value of F of X just before C. That is what C minus. Now with this definition can I express P of X of C in terms of F of X C and F of X C minus? How is that?

Student: (())(22:49).

Professor: Now let us say I want to compute this probability. This probability I can always represent as probability that X n minus minus probability of X is less than C. It is just by definition, probability that X is equals to C that means probably less than or equals to C minus probability X is strictly less than C. Now what is by our definition?

This quantity is F of X C minus F of X C minus. So if you want to find what is the value of, if you want to find the probability that X takes exactly value of C, you can express that in terms of your F function by computing F of X at C minus F of X. Just before C which is defined in this fashion.

And so, what is basically this is saying is the value of the function at C value of the interpretation of this quantity here is the value of the function just before C and because of this how can you interpret this quantity. This quantity is like a jump that is happening at the point of C. So we are going to denote it as F of X C.

So, this is basically so, in all these cases in this case what is the jump at C equals to 1.

Student: 1 by 6.

Professor: 1 by 6 and jump at 2.

Student: 1 by 6.

Professor: 1 by 6, 1 by 6 in all this case and in this case it is just like half here. So we will see that like the way we can interpret this as the mass added by the realization C to your cumulative distribution function. So the mass added by the value C equals to 1 is exactly 1 by 6 and similarly the mass added here by the point C equals to 4 is again 1 by 6 and that we are accumulating for all the points and that is why we are getting cumulative distribution function.

So as I said we have chosen to define cumulative distribution in this function, and we could represent the other quantities, which is strict inequality and exact inequality in terms of the same function F. Now we want to study what are the properties. Suppose if you have defined CDF like this, in general what properties it has?

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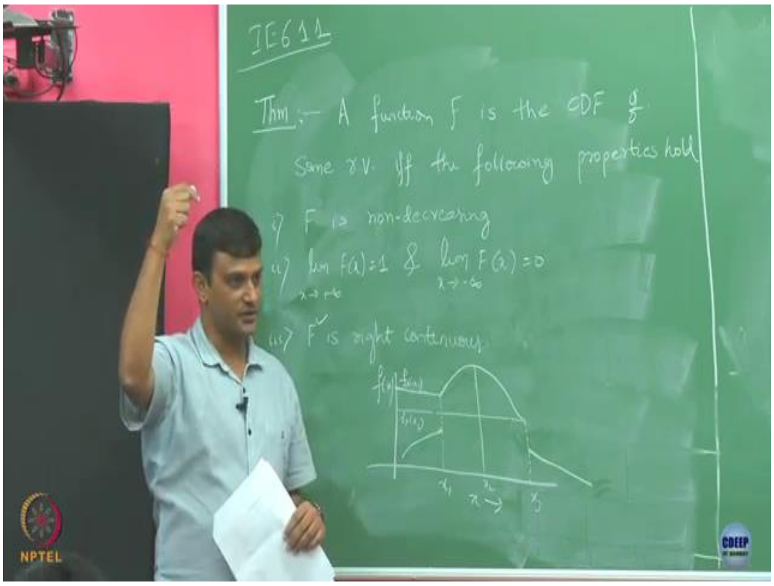

So this is like a, let us call it as a, so we will before we prove this how many of you know what is continuous function? All of you know. How many of you know what is a right continuous function? Those who do not know, please raise your hand.

Student: Right continuous.

Professor: Right continuous function. So, what is the relation between, those who know, what is the relation between a continuous function, a right continuation function and a left continuation function?

Student: (())(28:38).

Professor: What positively?

Student: (())(28:42).

Professor: It is true.. Just one of you give me what is the meaning of right continuous. Suppose let us say I have a function and I have it like this, let say I have a function which is look like this. Is this function continuous at this point x 2?

Student: Yes.

Professor: Is this line continuous at point x 1?

Student: No.

Professor: Is it not continuous?

Student: no.

Professor: But it is continuous, if I going to come from this side? So the value at x 1 is exactly this quantity. Here what kind of continuity is there? It is right continuity and what about this?

Student: (())(30:42).

Professor: I have put it till this point. It is taking this value.

Student: (())(30:53).

Professor: So our claim is the CDF function happens to be a right continuous at all points. See this function here, whatever I have drawn, this line is continuous in in this interval, everywhere it is continuous. Only what are the two points of possible discontinuity is here x 1 and x 2 and x 1 and x 2, they are kind of partially continuous. Like x 1 it is right continuous, at other point it is left continuous. What we are going to say is and when we are going to say function is continuous?

Student: (())(31:44).

Professor: Yeah at what point.

Student: (())(31:49)

Professor: It should be like we are going to say it is function is continuous if it is continuous at all the points. By that definition this function is not continuous at because at two points it is not continuous. Now what we are going to say that if we have a CDF, it is going to be a right continuous at all the points. We already saw this when we draw this CDF for the dice. We have the jumps which were always right continuous. So now let us try to make this at least the other points more formal.