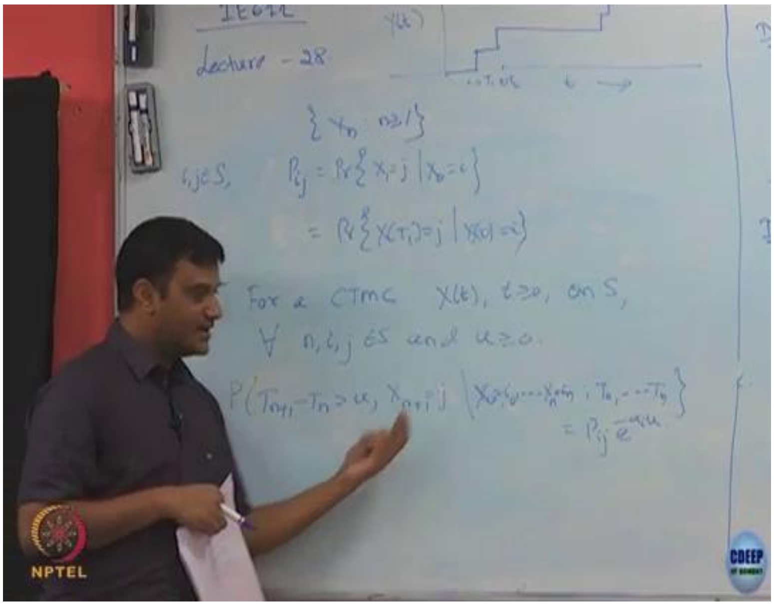
Introduction to Stochastic Processes
Professor Manjesh Hanawal
Industrial Engineering & Operations Research
Indian Institute Technology, Bombay
Lecture No 49
Embedded Markov Chains

(Refer Slide Time 00:17)



Now, we are going to state a relation, which is kind of going to combine, what is the next going state I am going to take, given that I have already spent this much of time, I have already spent in my given state. So, this is already what is this state, you are just going to state j whenever, the jump has happened.

When you started from X naught from state i but, you may also ask the questions, like on this state at I have already been there, I have already spent this much of time and after this, what is the probability that I go to state j let us formalize that. I am going to ask this question now. So, what we are asking is here.

Suppose, you have been given that, the jumps are happened at time T0, T1 all the way up to Nth jump and you have been told, what are the states that has been taken when this jumps has happened that is X0 corresponds to the state at time T0 and Xn correspond to the state at time Tn of your continuous time Markov chain. You understand the meaning of conditioning like this?

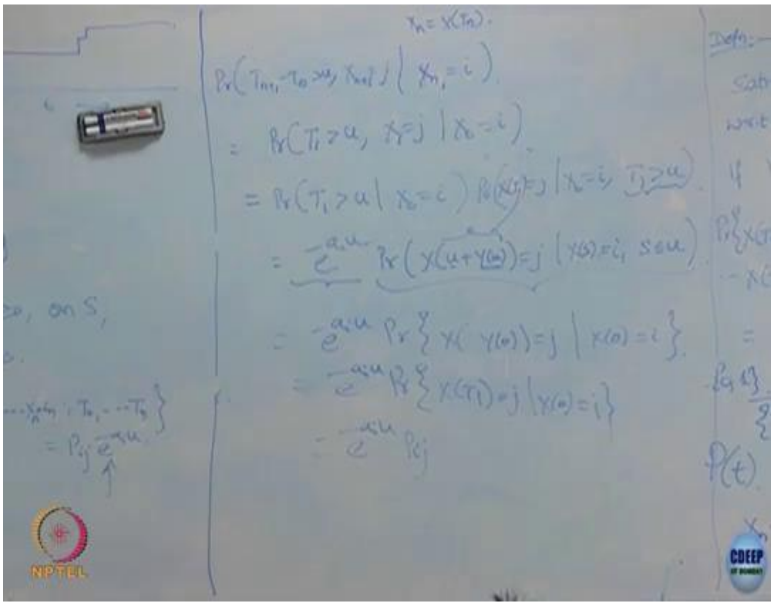
So, basically saying that, these are the instances at which my jump is happening and these are the corresponding states that has been taken when the jumps have happened and now, we are saying, fine it looks like, so far n jumps are happened and after that you are asking the question, before the next jump happens, that is Tn plus 1 it is going to take at least u units of time and after that the jump I am going to make is to a state j.

And now, what it is saying that this probability is nothing but, the probability that you are going to stay in that state this is a to the power aiu, that is what we have shown and then from that state the probability of going to Pij.

So, in a way what you are saying is, see, when I define Pij, I define it for one full life cycle, like T1 before this is for all T1. But, here you are further asking this question that, before you starting from this state i when n jump happened, you are going to state j after spending at least this much of time.

It is saying that, then if you want that extra conditioning, that not only I want to go from i to j but, I also want to ensure that before going to state j, I am going to spend at least u units of time then multiply by the probability and in a way it is saying that this is a design that these two are independent because it is just multiplying both of them. So, let us the quickly write down a couple of steps in this proof.

(Refer Slide Time 05:08)



Probability that Tn plus 1 minus Tn being larger than U Xn plus 1 equals to j is given. So, first thing it is a CTMC continuous time Markov chain. So, I already know that it is going to have a satisfy Markov property. So, once I know the nth jump is going to happen at this time

I do not need to know any of the things which is happened before, to answer, question about my feature.

So, this is about the future after nth event has happened right I am going to this is about that. So, to answer this question, I do not need to know what has happened before the nth jump. So, I am just going to condition it on this event.

Let Tn some number which is given to me that is a random quantity. So, basically, this is exactly, this is nothing but, Xn our rotation Xn, we have defined Xn to be X of Tn. Now, I am going to use my storm Markov property let us say and then, just now, let us say I am going to use my time homogeneity property if I can use it, along with my storm Markov property I can shift, whatever happened at Tnth instance as if it has happened at origin and I can start looking from that point onwards. So, Tn I can assume it to happen at the 0th time in that case. I could write it as now, T1 greater than u given, X1 equals to j given, X0 equals to i.

So, now everything this Markov chain, I will just shift it origin and now from the origin I am asking this. So, I am basically saying at the beginning I am in state i now, I want to T1 I should, before I jump to next state I should have spend at least time u and my next stage should be j.

This one I am going to write a probability T1 greater than u given X naught equals to i probability what. So, I have just applied my chain rule here, and I could write like this and what is this quantity here? We are saying that you are state i in the beginning and you are going to spent at least u times on it, before you are going to make a jump. What is this probability? e to the power aiu that is the definition we have. So, good, we already have this stuff.

So, now let us try to deal with this term. So, instead of X1 I am going to write it as X of T1 here, X1 is nothing but X of T1 and then X naught equals to i T1 greater than or equals to u and then what I will now do is, I know that at least my T1 is going to be at least u that has been already told. So, I am going to replace this T1 here, probability that X of u plus some y of u.

And, now I have been also when I have been told that T1 is going to be greater than or equals to u that means I have basically no my chain all the way up to u. So, is this correct?

So, what I used, I have been told that my T1 is greater than or equals to some u I know that my Markov chain, sorry, my CTMC is continue to stay in that state i at least till u and this T1 I have split it into two parts, because T1 I know is at least going to be greater than u yes that much of u and after that this is my residual time, it is going to take at least larger than this before, you are going to change your state. So, I can write my T1 as these two component I can do that because T1 is already been told me to be larger than this u.

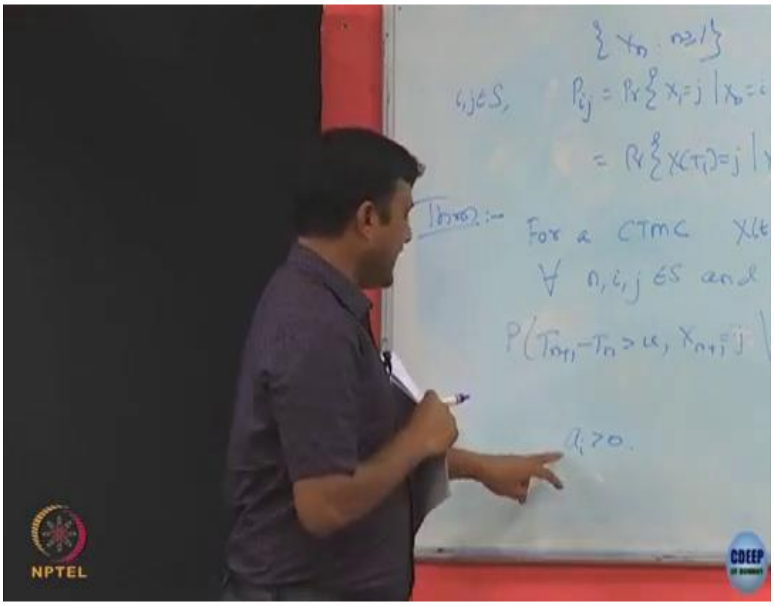
So, now what we will do is, this has been given to me like this in this what has been observed now, I am going to. So, anyway that, this part I have dealt with I have to deal with this probability here, this probability we will assume that by assuming the homogeneity property, expert in the homogeneity property that if I shift everything the whole process by u amount.

So, u is a fixed quantity for me I am going to shift all this process by u amount the probability should not change. So, e to the power aiu and this one is like X of. So, this is I will set u to be 0 then this is like u of 0 taking value j and now, this is like X of 0 is equals to u what I have done is basically I have shifted my entire process to the origin, that is why I am going to get my y of 0 that.

But, I mean this is all this manipulation but, now what you will see that once what is y of 0 condition that you already know that X of 0 equals to I what is the, how much time this is going to be? It is going to be T1 like, if you are going to be how much more time you are going to need before you leave it minimum time. So, this is again X of T1 and this is j, this is X of 0 and according to our definition this is nothing but Pij.

So, what we have basically done here is to decouple these two process that amount of time the probability that I am going to stay in this process at least for u amount and after that if I am going to look at probability of jumping to another state that is going to be still governed by my Pij process. We can make some more observations here. So, now let us say that. So, this is what we have shown.

(Refer Slide Time 13:26)



So, now let us see that, what is that it is going to happen when ai is greater than 0. What is ai? ai is basically giving the rate at which I mean the amount of time that ai governs before you are going to leave state i. Suppose, let say ai is going to be greater than 0 that means probability that you are going to stay at least for some amount in the state i is going to be positive, if ai is going to be positive that like, it is not like it is, it is not at least instantaneous, instantaneous is the case when ai is equals to is equal to infinity let us see that this is so ai greater than 0 allows me the possibility of taking ai to be infinity also. So, whatever.

Suppose, ai equals to 0 what is this probability is going to be, suppose let say I have started from stay time and now, I am looking at going to state j after the first jump. So, if my ai is going to be positive it must be the case that I would be jumping to some j other than i because if my j is by definition my X1 is the state. So, here maybe the importance of right and left and continuity comes, suppose if X1 when time T1 has already happened I am already assigned the state which is the new state that has been taken.

So, by the time T1 by definition I have move to a next state. So, if this quantity is j if my j is equal to i, what is this probability is going to be? It is going to be 0. Because, by T1 by definition I am saying that I have left that state and gone to other state. So, only if j is not equal to i this guy is positive if again j is equals to i like, then this is a 0 quantity.

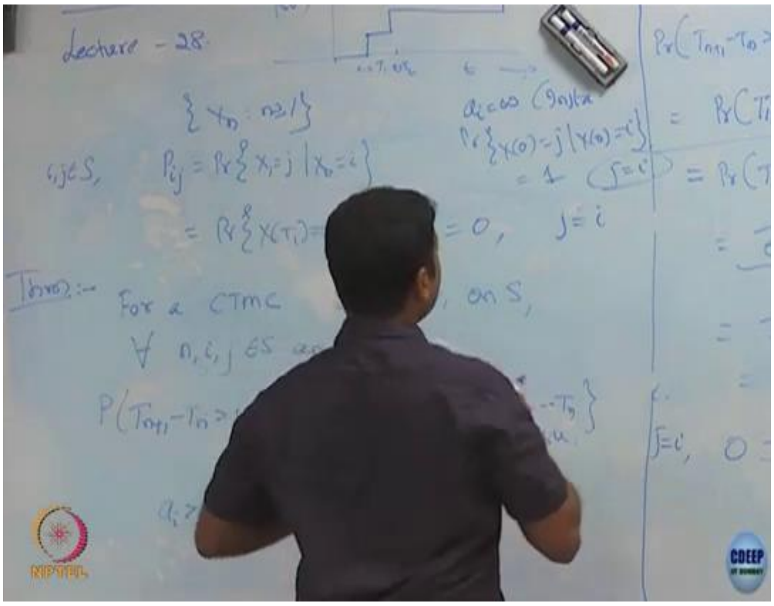
So, if for j equal to i let say this guy is quantity is 0, and if ai is greater than 0. At least, what is this quantity is going to be some unless let us say this is also not equals to infinity. So, ai is some, let say finite quantity then this guy is not a 0 quantity, then what is the only possibility it must be the case that Pii is equal to 0. So, what is coming from the definition of pij? Why is that?

So, what did we say, T1 is something, jump has happened. So, if I am looking at a some state, which is other than i that is not going to happen, that is by definition, so fine so we do not need to so, I mean the same arguments we are applying here. So, we are going to say that this is going to 0 if j is equals to i and this can be some nonzero quantity only if J is not equal to i.

So, let us understand this, suppose now, so by this definitions it looks like this should hold for all values of ai whether the state i is absorbing or instantaneous. So, I really need to worry about whether, this ai greater than or equals to 0.

So, suppose my state is i is instantaneous if my state is instantaneous, what is T1 going to be for that? T1 is going to be 0, like it is by definition of yT, it is going to be 0. So, in that case T1 is going to be 0 here. So, I am just looking at, why is that done X of 0 state i? Then, it is going to be 0. I would ideally like that instantaneous like.

(Refer Slide Time 18:42)

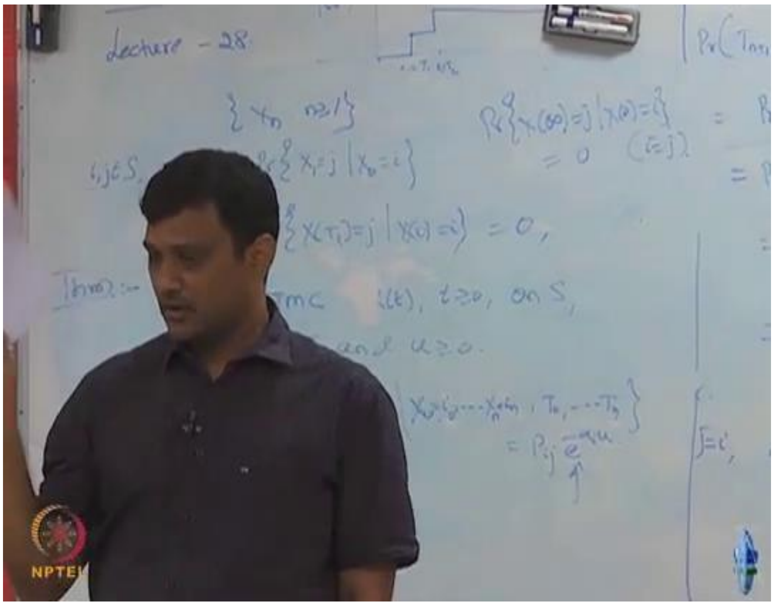


Suppose, if my ai is equals to infinity that is my instantaneous case. So, in that case like this is like a probability of T1 is like almost surely going to be 0 in that case, because T1 in that case like i would be in this case and what this quantity I would like to be?

I would like it to be 1 actually if j is equal to i. So, we are start instantaneously i. Oh no, in that case it is going to be 0 only because, u are like living that state very fast like even T1 is kind of 0, it is like instantaneous you have quickly leaving that. So, if j is equal to i that is going to be 0 again so, that is fine.

Now, let us consider a case, where my state is absorbing. When is my state is absorbing, I said ai equals to 0. So, when my state is absorbing. What is my T1 is going to be, my T1 is going to be infinity like. So, in that case when I will looking at?

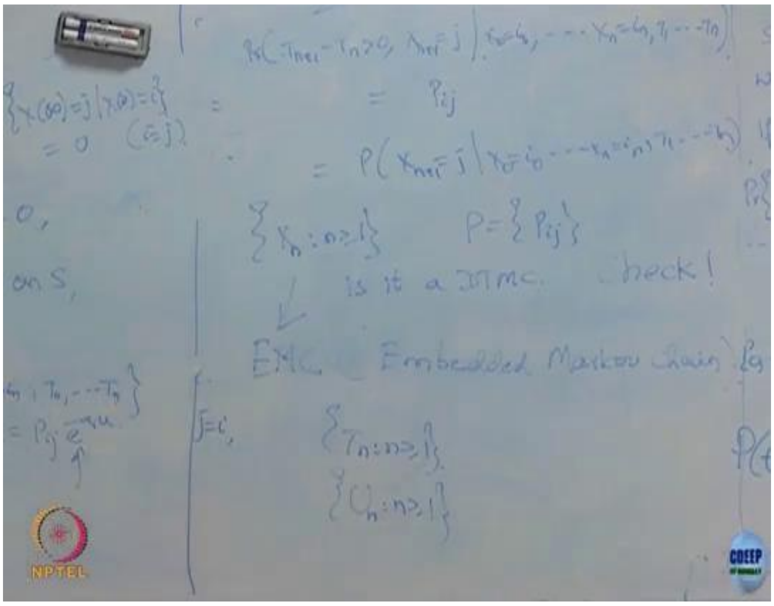
(Refer Slide Time 20:02)



So, I will be looking at basically push to i and here, what we are saying is, is this still consistent here, and I have i equals to j, that is fine, like even when I am very after large amount of time, I am going to look for a different state but, here that different test is happening only at infinity time, so fine, this is also fine.

We do not need to really worry about whether ai is strictly positive or whatever. So, it appears like as long as any ai I am going to take this Pij happens to be 0, if j equals to i. So, let us see that what the assumptions or whatever the arguments, we have are going to be consistent here. Now, in this definition whatever I have here.

(Refer Slide Time 21:07)



If, I am going to just said u equal to 0, we already just saw that probability that my Tn plus 1 minus Tn greater than 0 then X1 plus 1 is equal to 0, all the way up to X0 X1 to Xn what are these values i0 give, T1 all the way up to Tn, because u equals to 0 this is simply going to be now Pij by our definition, because e to the power aiu that has (())(21:56) to 1 and we are going to get this and now suppose, I am in the case of pure jump process, because of the pure jump process I am not leaving my state instantaneously.

So, there should be gap between my nth and n plus 1 process jumps, because it is a pure jump process. So, I mean this is fine. So, this probability is nothing but, simply then probability that Xn plus 1 is equal to j X naught equals to i0 and all the way equals to i n plus T1 Tn.

And, this is a way like same as what we have got here, it is just like after shifting the Tn process to the origin. So, here I am just say Xn plus 1th jump is going to take state i given that all the way up to this, this is my Pij, which is exactly nothing but, this. So, now with this definition of Pij embedded process, which we called as jump process with this, let me call this as a metric transition probability matrix is it a DTMC.

So, I have for my continuous time Markov chain, I have derived and embedded chain, which has this transition probability Matrix Pij, which are defined like this with this Pij’s. We are on in the setting let us assume time homogeneity and all, is my Markov chain is my chain, embedded chain or my jump chain is a DTMC. So, you need to check this.

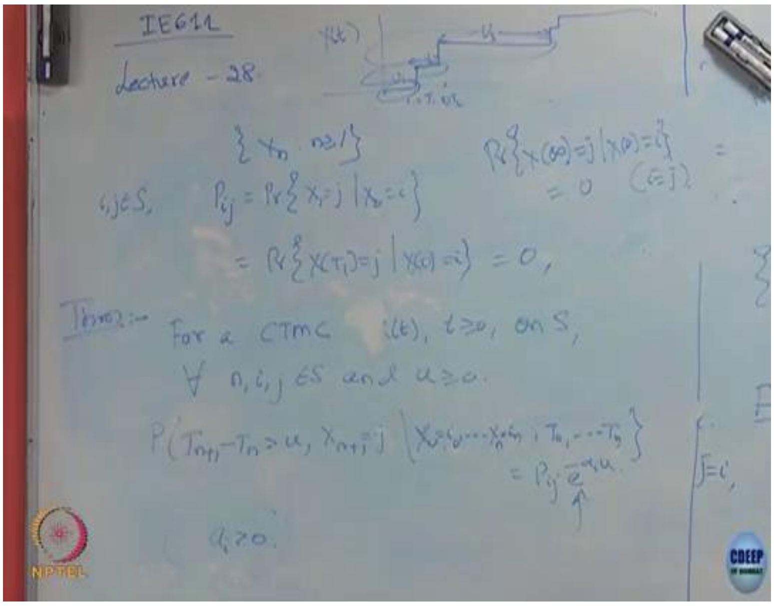
Indeed it is true like, you can show that, that if I have an underlying, if I focus on this underlying jump process then whatever the jump chains I have is going to be a Markov chain with respect to this transition probability Matrix P.

And hence, fourth, we are going to simply we call, going to call it as a EMC that is Embedded Markov Chain. So, we are looking even though looking at, looked at a continuous process Markov chain but, what we actually did is we extracted a discrete version of this continuous time Markov chain, which is called as Embedded Markov Chain and what we are going to focus on here is kind of renewals here, what are the renewals here?

I am this particular state, I am going to take another state, before that I am going to spend some amount of time in this state and after that I am again going to jump another state after spending certain amount of time. So, between these two jumps you can think that has a cycle and jumps you can think it as renewals. So, basically what is this? This underlying embedded chain is a, if you are going to look at this sequence, these are what?

These are the jump instances that from this can drive your life cycles like, lifetimes, like the way we did earlier. Now, what are the distributions of this life cycles, is there, did this life cycles have any distributions? So, what would we say? You have my.

(Refer Slide Time 26:49)



Let us state this I have already my CTMC, my event happened at these points, I looked at this time. So, this is some, this defined my T1, this entire thing define my T2 and all the way this is defined up to my T3, but I could just focus on this u1 this interval, u2 this interval and u3 this intervals and each one of them I could think of a cycle. Now, this interval, what did govern it, what govern the length of this interval?

Suppose, let say at this point I am at state i, what did govern the length of this u3? That is u3 but, what did govern it? Did it have any underlying distribution, that govern? What is the length of this u3 is? We just said it already. What did that govern it? So, what is this? Basically this is a sojourn times, the way we defined it u3 is the amount before, you are going to make a change in your state. So, u3 is the sojourn time and we have just said that, that has a exponential distribution, but with what parameter?

Is it a parameter ai which is state dependent that state i dependent. So, here what we have is? We have basically renewals, but, the length of that renewal cycle depends on from which state it started with. So, when I defined the actual renewal process. How did we define?

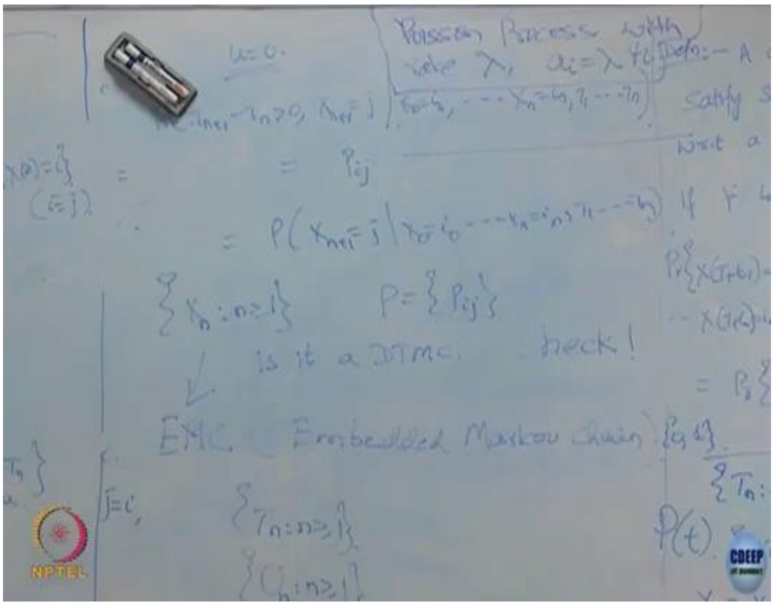
Let us consider a sequence of ui’s, u1 all mutually independent and then we said u2, u3 are all also identically distributed but here if you are going to look at u1, u2, u3 this sequence are they mutual independent? They are, like once you know you are in this state Markov Chain previous to that does not matter and I am going to stay the next state, I am going to focus on that that is independent.

So, they are mutually independent. Are their identically distributed, are they? They are not, because, every cycles the nth of the distribution is going to depend on from which state they are going to start. So, I have renew a process here, like but, unlike in the our process way all my life cycles were have has identical distribution, except maybe the possible first cycle subsequently everybody has same renewal it is not here, it depends on which state you are going to start it. Now, let us focus on Poisson process.

So, we know that Poisson process is a CTMC we showed in the first class itself of this lecture about CTMC that, if you are going to take a Poisson process it satisfies Markov property. Now, what is the state space of your Poisson process? It is first going to be natural numbers we are just counting, when the first one happened this is a counting process. So, it has to be all natural numbers.

Now, if I say I have a Poisson process with parameter lambda, what are my life cycles distributions are exponential with what parameter? Lambda. So, what is ai’s in my Poisson process? It is going to be Lambda and does it depend on which state you are in? It is not like.

(Refer Slide Time 31:34)

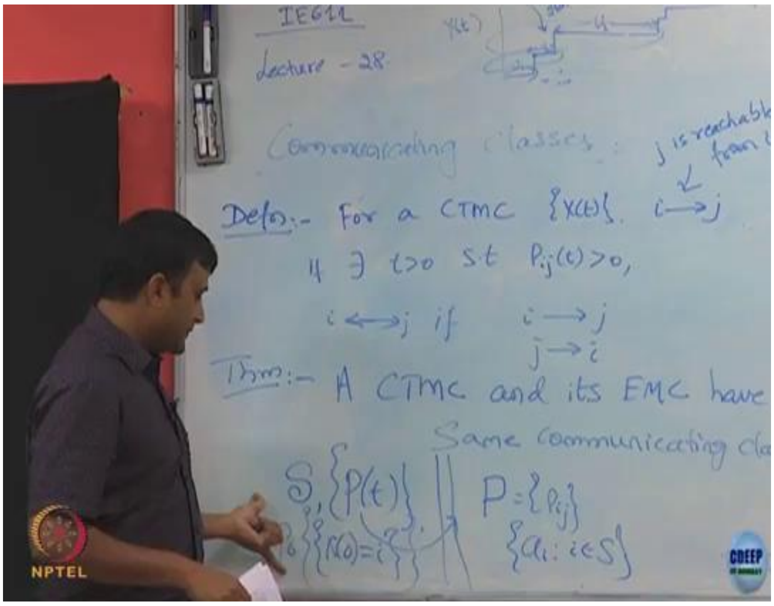


So, you can just verify that, if you have a Poisson process with rate lambda that means you have basically ai is equals to lambda for all i. So, we have pressure, Poisson process is a pressure CTMC in which all my ai’s are that common parameter lambda but, if you have this different parameters ai then, it is a more general CTMC.

So, Poisson process is a special case of my CTMC in which all my ai’s are the same value and depending on my sequence of ai’s maybe I can get a different, different continuous time Markov Chains. So, when the theory of this CTMC it is going to, we are going to develop in a way very parallel to what we did before DTMC.

So, now to understand all the properties of CTMC, now we are going to only look at the properties of my DTMC because, I know DTMC well, we already studied all its properties. So, now let us see whether, there is a notion of transience recurrence in my CTMC also and how to define them.

(Refer Slide Time 33:18)



So, is there a notion of Communicating Classes in my CTMC? So, do you think they should be analogous question of Communicating Classes for the CTMC? So, let us see. So, what is this notion i goes to j, there we will have the same (())(34:33) as in DTMC i arrow j means i is reachable from, sorry, j is reachable from i.

So, we are going to say j is reachable from i, if there exists some t positive such that Pij of t is going to be positive and similarly, we are going to show that we are going to say that i and j are reachable from each other if i is equals to j and j is equal to i.

So, This is exactly what we had in our DTMC, but we are just now looking at earlier we wanted some positive n such that Pij of n is positive but, now we want Pij of t to be positive. So, like in DTMC, we can also argue that, this equivalence this relation is an equivalence class and it partitions all my states my CTMC.

Now, here is the theorem, how to and we can define my communicative classes in terms of this equivalence classes set of all states which have satisfies this equivalence relation. Now, it so happens that the communicating class of my CTMC should be same as the communicating class in the underlying Embedded Markov chain.

So, a CTMC and it is Embedded Markov Chain have the same communicating class. So, we know that, both my CTMC and its underlying Embedded Markov Chains they have a same state space whatever the communicating class my Embedded Markov Chain has that is also going to be the communicating class for my CTMC.

So, if I know my DTMC well the underlying Embedded Markov Chain I already know about my CTMC. So, I am not going to prove, this is just a brief proof in the book just look into that, it is just like a under a simple intuition that in your DTMC, if you have to reach from state i to j there should be some same finite amount of time in some finite amount of time we should be do with some positive probability.

That means if there is an, that is like that many finite number of intervals should happen but, that finite number of intervals you can translate in a jump process to some finite time in your continuous CTMC and you can come up with finite time under which you can go from state i to j. So, that is made bit more formal in the proof just look into that.

So, now we know that my CTMC has an Embedded Markov Chains and it has association transition probabilities Pij's which I can derive, now to define my CTMC, what all the parameters I need? So, what are properties, what you feel so far, are the characteristics of my CTMC’s I have this P of T.

So, my CTMC has this transition probability sorry, my state space it has associated transition probability matrix for P of t, and let us also say I have my initial distribution. Now, this P of t from this I could derive my P which is my Pij in addition to this work or any other characteristics of your CTMC that is associated with each of your states?

ai’s, like there is another parameter ai so depending on your ai’s your CTMC could be different and it is not like, this ai’s are independent from this Pt’s, they will be these Pt’s are going to influence how this is ai’s are going to look but, ai’s are one of your important characteristics you just saw that for a Poisson process ai is simply lambda, if ai’s are different that is going to give a different CTMC.

Now, in general and we also give a probability in which ai’s and Pij’s governed my transition from a given state to another state after spending certain amount of time in that state. So, in a way Pij’s and ai’s kind of summarize the information I have in my Pt and using this possibly I can describe my CTMC.

So, now this is the question we want to make more precise give me CTMC, P of t and my state space and in my initial distributions I have, I can find out this. So, my CTMC defines these characteristics. Now, the other question is it possible that if you just give me this matrix P and this ai’s can I have a CTMC that will define my CTMC, you understand the question?

So, this is, let us think of this, these are two set of parameters we initially say that, let me for concreteness also, all this distribution this is my initial distributions are known, from this we could derive these parameters. So, these parameters we said that is these Pt's and this X naught initial distribution they completely define my finite dimensional distributions. Once, I have understanding of my final dimensional distribution I know how my CTMC behave and these are like extracted features of this.

Now, the question is this a complete characterization of my CTMC is it that if I just give you this parameter, you can uniquely recover a CTMC with these parameters? In that case all I need to do is instead of giving this Pt which is a, which I have to specify for every t I will just specify these few parameters a transition probability matrix and this ai parameters then is it sufficient to completely characterize my CTMC.

I just want to say this in two minutes so, that I finish what I planned today. I am just going to state it so that we can start with a fresh topic from next class.

(Refer Slide Time 43:04)

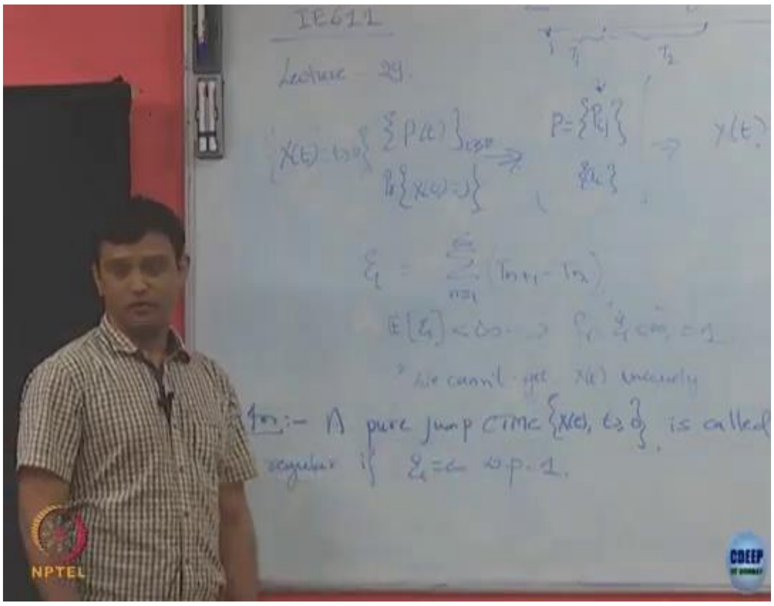
So, it so happens that suppose, you have this quantity. So, what is this? This is the sum of your life time intervals. So, this is the like a if you take n equals to 0 this will give the length of my first interval and then n equals to this will give me second interval, let us say this is going to be Psi, if somehow, the expected value of this spans my entire line then from this parameters I could recover my CTMC complete characterization. So, in a way let, what are these?

These are lifelines, this can be characterized in terms of my ai’s and Pij’s, because I do not know, what state I am going to start with if I know which state going to start with I am going to jump and how much time I am going to spend on those states I could compute this T n and Tn plus 1 using these parameters. So, this is going to be define these distributions.

And, if it so, happens that, the expected value of this Psi happens to be infinity. Then P that is my transition probability matrix a we just going to call this as a enough to reconstruct our Xt. So, I will just leave it to read you, it is a pretty straightforward why, why if this happens you can able to reconstruct your CTMC using these parameters. Otherwise, why you cannot?

If, it so happens that, if my Epsilon Psi is finite then we will not be able to reconstruct in a unique way, your CTMC using these parameters along but, if you so, happens that, this quantity happens to the infinity, then these parameters are enough to reconstruct your CTMC in a unique fashion.

(Refer Slide Time 46:37)



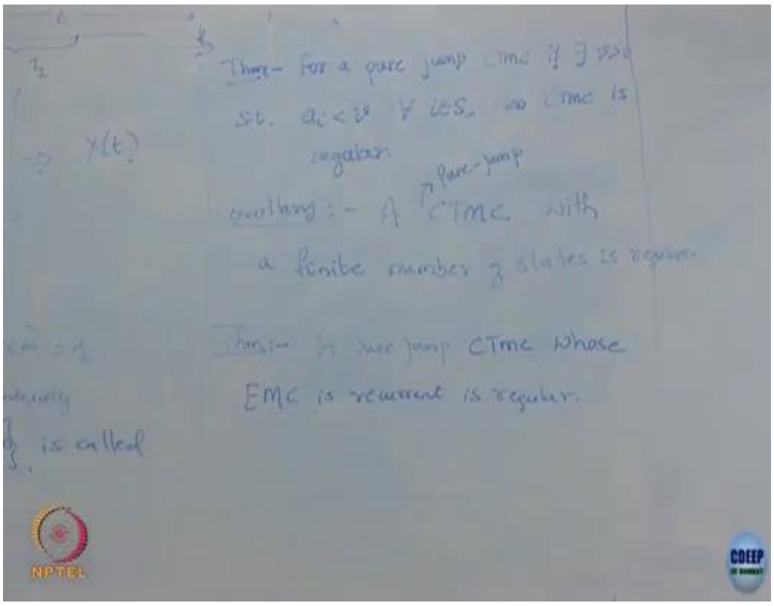
But, now we are going to say, the pure jump process is called regular, if Psi equals to infinity with probability 1 now, suppose Psi is not equal to infinity, is not finite here that means Psi is extends entire real line, then that means this difference the lifetimes, they has spent the entire real line. So, then looking at this process I could define a process of every possible t if it has spanned the entire real line, had it been finite I could do the process only for finite t but not entire thing.

But, if it happens to the infinity the probability 1 I could be probability 1 I could define it for all the point Xt that is what, this condition is requiring it should be the case that if this Psi equals to infinity with probability 1, then we are going to call it as a regular 1 and we will be just focusing on that. So, let us take an example. So, if I have a Poisson process is it a regular process? Let us say poison with rate lambda, Poison with rate lambda, what?

So, just take the expectation of that, this is expectation of this difference let us say I could interchange this infinite summation and expectation after some argument then expect the difference of this is nothing but, expected value of an exponential random variable with rate lambda. So, we are just adding lambda infinitely many time, that means this quantity is already in.

So, if I have a Poisson process with lambda strictly positive that is already a regular process. Now, what else could be a regular process? It so, happens that any CTMC, which takes values in a finite state space is going to be a pure jump process, which is going to take value in a finite state space is always going to be a regular process. Why is that? So, now.

(Refer Slide Time 49:07)



So, I am going to just write it, before I say that so, this is a statement for a pure jump CTMC, if there exist some new positive. Such that, this quantity is ai is going to be new for all i then CTMC is regular (())(50:02) on this like, we are just saying that if all the ai’s so, remember this quantity is ai’s which governed your sojourn times, they can be any quantities between 0 to infinity based on that we have said they are either absorbing stable or instantaneous.

But, suppose if this ai quantities happens to be, uniformly upper bounded then it must be the case that your process is a going to be always regular process, this holds true, this needs a couple of line up of proof but, let us skip this. So, let us understand that as long as my ai’s are all bounded it is going to be a regular process.

For example, in the Poisson press, we are going to say Poisson process with rate lambda, you know that ai’s equal to what for the Poisson process, for the Poisson process ai equal to lambda for all i. So, lambda is an uniform upper bound in that case for your Poisson process so, this condition already holds and it is we already argue that Poisson process is a regular process.

Now, as a corollary to this so, like most of the things I am going to now discuss in the CTMC we have, we are just going to state the results and with respect to Poisson process we will understand the result but, even though they holds in a more, not necessarily just for Poisson process like more general Continuous Time Markov Chain.

So, as a corollary to this if, a CTMC I am going to say always we are assume that pure jump CTMC with finite number of states is regular, is this obvious this corollary now? So, I am saying that my CTMC takes only finite number of states and it is a pure jump CTMC. So, ai is not allowed to take infinity it can be either 0 or some positive but, bounded value.

So, then in that case I can always come up with a uniform bound on finitely many ai’s and so, in that case I will have always. Such a nu, I will find it and that is why I can appeal to this theorem and I can claim that my CTMC is regular, another result is. So, you need to like go through quickly the proofs of this given a book like, they are not lengthy they are just like a couple of lines. But, the point is you need to know this properties.

So, that you can, you are free to use this properties (())(53:30) in whichever problem you like but, you need to tell clearly before you apply in this results you need to make sure that all the hypothesis are satisfied then you can appeal to that. For example, if you want to apply this theorem, you need to first tell me, what is that new uniform bound on all this then you can say fine, then I am going to claim, it is a regular but before that you need to give me the value of nu, it just saying there exists.

But, if you are going to use this value you need to some demonstrate existence for such a new value. I (())(54:11) pure jump CTMC whose, Embedded Markov Chain is recurrent is regular, does this theorem make sense? Intuitively at least, what we are saying? Take your CTMC look at its Embedded Markov chain and if that Embedded Markov Chain happens to be recurrent then your CTMC is also regular, does that make sense?

So, like now, let us say once I said my Embedded Markov Chain is recurrent, that means I am going to revisit my state infinitely often times. So, that is the definition of my recurrence I am going to keep revisiting my states in my Embedded Markov Chain that state again and again.

If, I am going to coming back to that state again and again, that means you are going to first leave it in your continuous Markov Chain and then coming back to that state and this is happening repeatedly, that means in some way, it must be the case that this process is like spanning the entire real line you come back to that j at some point and then again leave it and go back whenever, you go, you live it you are supposed to come back to that state again, because, it is a recurrent state.

So, this is happening and because of that the lifetimes in that states, they are going to add so many times to this and it is like pure jump process. So, you are going to see that the expected value of Psi is going to be finite this is just like broad level idea and you can make this bit more formal.

So, in either of this case, either you can show that, there is a uniform bound on ai's or you are state space is finite or you can show that my Embedded Markov Chain is recurrent then it is automatic that your process is going to be regular. In that case all you need to focus on is, all you need is this information about all to generate your process Xt, which is uniquely represents the process with the underlying parameters, fine.