Introduction to Stochastic Processes
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Lecture No 48
Properties of states in CTMC

So, last class we started with our CTMC study and we kind of defined the notion that the switch on times, that how much time my CTMC is going to stay in a state or continue to stay in a state before it leaves it. And in the last class we showed that it is exponentially distributed with a parameter Ai. That Ai is state specific. The I, whichever state you started looking at and the time after which it has left that state.

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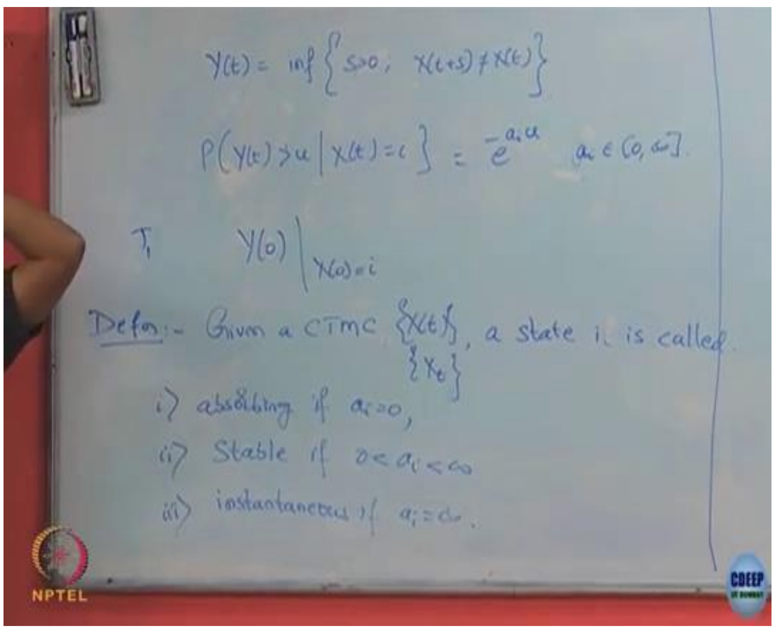

So, basically what we did is, we had this random variable. So, we said that Y(t) is the time, remaining time after which you will leave the current state, okay. And then we have said that suppose if I conditioned this random variable knowing that at time T, I am in that state I. So, here in this definition, yeah what is that? So, here when I define this, this is not state specific. I just looked at, okay. I start looking my CTMC at time T and just see what is the minimum time before it leaves my current state.

And here, suppose if I condition that, okay. When you start looking at your CTMC, you are in a particular state I. And then how much; what is the probability that it is going to take at least give more units of time before you are going to leave. I have shown that this to be e to the power Ai of U for some Ai, which we say can be between 0 to infinity.

Okay, let us say I have this random variable. So, Yt is a random variable here. Suppose I condition this random variable. Like here, I, Xt I is equal to 0. Or let us say more generally, instead of particular T, I am going to look at time is equal 0. So, I am saying that in the beginning at the 0th instance I am in state I, and then I looking at when is the time I am going to leave my state, how long I am going to stay.

So, what is this Y naught is going to give you? What, in this case, what will be the distribution of Y naught? It is basically the first time you are going to leave out of state I. So, you are saying I am going to start from state I. Y 0 is nothing but the minimum time when you are in a state which is different from I. That is basically, in a way it is going to say, when is that your state is changing. In a way this is same as like our random variable T1.

So, you remember how you define T1? T1 is the instances at which the renewals were happening. Here if you are going to interpret, your renewals are when you are going to change your state. Then T1 is the time here that is given by this random variable Y of 0 conditioned that. So, this is, T1 is the basically the time that you are going to leave state I here because I have conditioned on I here. Okay fine.

So, now you see that the amount of time you are going to stay or the rate at which you are going to transit to other states is going to be governed by this parameter Ai here. So, based on that how this value of Ai is, then we can classify the states in my CTMC. So, let us see this, what are possible things we can get.

(())(5:43) the sequence of random variable, sometimes I write X of T like this, sometimes I simply write X subscript by T. Assume that both are indicating the same, okay. And a state I is called. So, I have classified them into two things. So, if it takes the value at the boundaries, either if you take the value 0 or infinity; then we are going to either call them absorbing or instantaneous. Specifically, when it is going to take the value Ai is equals to

0, we are going to call it absorbing. And when Ai is equals to infinity, we are going to call it instantaneous.

So, let us see this. Suppose Ai is equals to 0. What is this telling you Ai is equals to 0? It is basically telling you that, anyway this probability. So, if Ai is equals to 0, whatever U you are going to take, this is going to be 0. Sorry this is going to be 1. So, what; however large U is, however large you are going to take, it is still going to be 1.

That is probability that your Yt is going to be larger than that quantity, that is U continue to stay in that. However, large U, it is going to be 1. Then that makes sense. Like that is why you are going to call it absorbing. That it is basically got stuck in that, whichever you started with. That it is basically never leaving. How far you are going to look, it is there.

And now, suppose you put Ai is equals to infinity now. Now, if we put Ai is equals to infinity, for whatever value of small u this guy is going to be 0. So, however, small value of U you are looking at, that your Markov chain is going to stay in that state is just 0. You put U arbitrarily is equals to 0. Once Ai is infinity, this probability is going to be 0.

That means there is no way like my Markov chain is staying in that, continue to stay in that change. It is just like; as soon as at leaves it is leaving that instantaneously. That is because however, close, U is close to 0, this probability is 0. That is the reason when Ai is equals to infinity; we are going to call it absorbing.

Now, anything in between this; sorry instantaneous. We are going to call it instantaneous. And anything in between this, when Ai is strictly away from infinity or like strictly finite and it is a non-zero quantity, then you are going to call it as stable case, okay.

So, now see like in our DTMC, we had a classification of our states into; there also we had transient or recurrent. In recurrent, further, null recurrent, positive recurrent. There also we have the notion of absorbing state. But here we have another classification of states when it comes to CTMC that is terms of whether; how that parameter Ai associated with state is going to be, whether it is 0, infinity or something in between these two.

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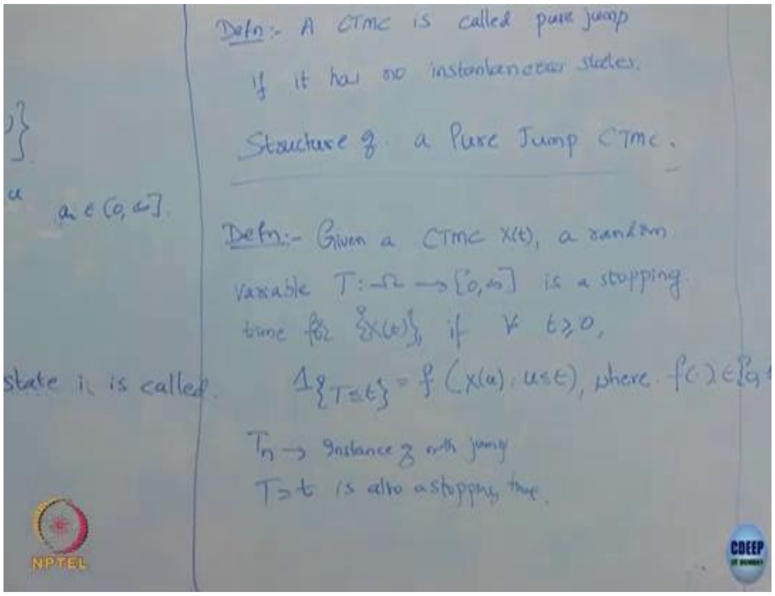

So, we are going to call; so basically now we are going to focus on a CTMC in which the status are likely well-behaved like CTMC is not like which is changing very very instantaneously. That means it is kind of a highly unstable scenario. Every instance your CTMC is taking different states. So, I do not want to have such case. So, henceforth I am restricting to CTMCs in which such states does not arise.

So, it is clear? We are just going to say that my CTMC is well-behaved. It is not going to change its state at a very high rate. It will change its state but at some bounded rate. So, like in this case when it is absorbing, once it hits the state I, it is not at all changing it states. Here, the rate of change is 0 but whereas in this case, it is still changing it states but at some bounded rate. But whereas in this case it is changing its states at a very unbounded rates. If you pick particular states where Ai is equals to infinity. So, we are assuming that my CTMC does not have such states, okay.

Okay, next we are going to focus on; so now we are going to study how these CTMCs which are pure jump structure; they are going to look like. What are the things I need to characterize there? See like in DTMC, we always looked my Markov chains to a time index which was kind of discretized and we know at okay, at some discrete times we were looking it. Here the time itself is a random quantity.

So, suppose in a discrete time Markov chain, you looked at T1, T2; let us say periodic time. Not necessarily periodic but at some discrete times. But now when you going to look at the CTMCs, a event is happening at some times. And these times are random. So, the jump is happening at sometime, next jump is happening another time.

So, if you are going to focus on these events when the state is changing and you are interested something in between. As long as my CTMC continues to stay in a particular state, fine. But when it changes, that is the event of maybe of interest to you and that is happening at random times. So, often, in this case we have to condition on random times.

And naturally we will be asking the same question as we asked in DTMC that is, is my CTMC satisfies a strong Markov property? So, in DTMC we also looked at when we conditioned on random times whether it still satisfies Markov property. If we dissatisfy, then we could call it as a strong Markov property.

So, we want to see if the similar thing holds here. So, for that first we are going to define what is a stopping time for a CTMC. So, this definition is exactly same as that in DTMC. We just want to know that if there exist a function F such that if I want to answer the question that my random variable T is going to be less than or equals to T, I only need to look at my random process up to time T, nothing beyond it. If this is going to happen then we are going to call that time T as my stopping time.

Now, let us say Tn to be instance of nth jump. See, I am using the terminology jump here. That means that at this time my Markov chain is changing its state. It is leaving its current state and going to something else that is what I am calling it as jump. Now, let us say Tn is my instance which my Markov chain is changing its state for the nth time. Is Tn a stopping time?

So, if I need to know whether my nth jump have happened before time T, you just look at your Markov chain till time T and see how many times it have change its state and then you can know this answer. So, you can always have such a function F here which will say yes or no. So, Tn here happens to be a stopping time. And you can also see that if you random time T happens to be just some given T, then also it will be the stopping time, as we did it in the DTMC case.

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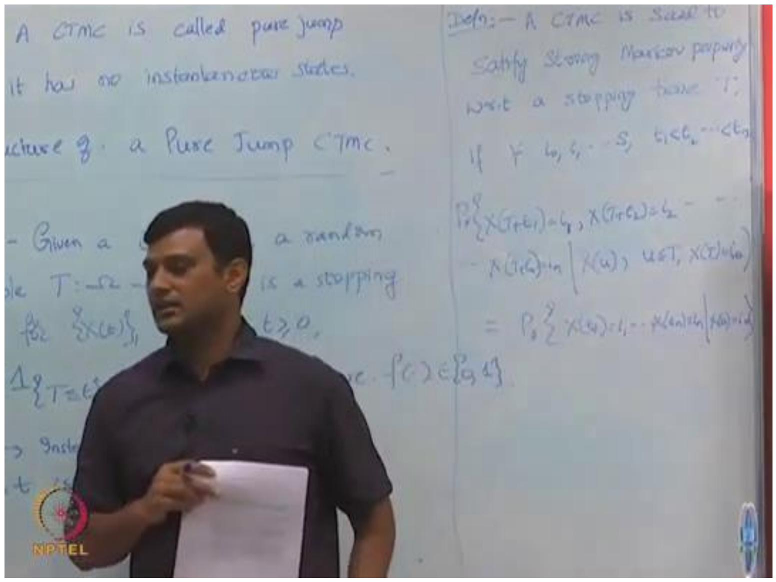

So, okay now with this we are just going to define our strong Markov property like we did in our DTMC which is as follows. Okay, so what we are saying? We are just saying that if T is the stopping time. If you have given your; you have observed all your CTMC till that time T, T is this stopping time. And then you are looking at your Markov chains taking the value I1 after T1 rounds and I2 after further T2 rounds, like this.

This is going to be like as if I start looking my Markov chains from the beginning, that is as if I start my Markov chain from state 0 where it has taken state I0 and subsequently it has taken state I1 at time T1 and In at time T1. It is just like at random time whenever it is happened, I am just pushing it to the origin and then I am pretending as if I am looking my Markov chain from the origin, okay.

So, if you are going to condition on your Markov chain that fine like; as long as that is going to be a stopping time with respect to that Markov chain, it is good like you can pretend as if you are starting from the beginning. Okay fine. So, this is the definition of strong Markov property we are going to assume.

Now, you see that even though my the time is continuous, I am basically looking at discrete events. I am going to; because I have focusing on my pure jump CTMC here, I am going to look at the events where my state is changing. So, these are like discrete events because the things are not happening instantaneously. They are happening after some time. And this time is positive probability 1.

So, you are going to see kind of events that can be possibly indexed with discrete indices. So, for example, first transition, second transition, third transition. They could be happening at different random times. But still these events you are indexing by discrete numbers, like 1, 2, 3 and there are, you can make a countable indices set. So, based on this we can; out of my continuous time Markov chain, I can extract and underline embedded chain, okay. So, let us see what is that.

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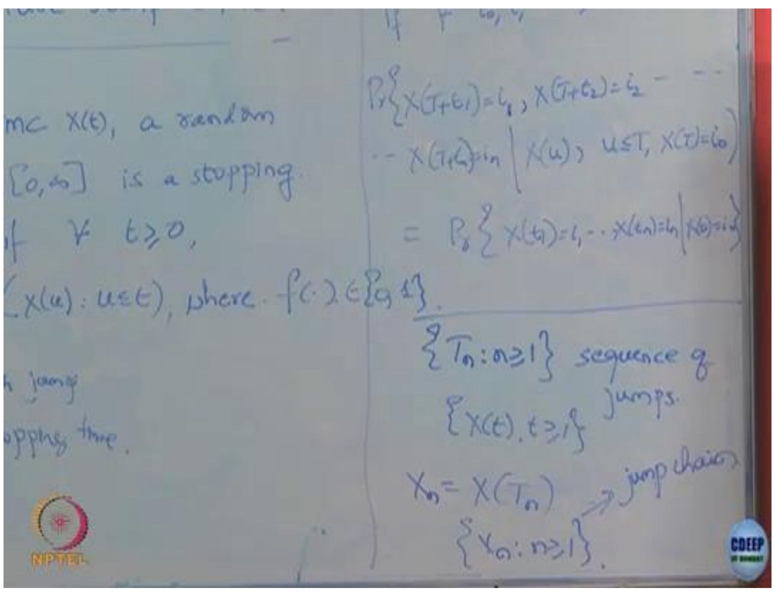

So, I am going to assume; let Tn. This sequence is what, is my nth jump. This is the sequence of jumps. Now, I have my continuous process X(t). Now, this process I am going to sample at this sequence, these random times. So, let us say, now I am going to get; from this I am now going to define Xn to be X of Tn.

And now I am going to look at this sequence Xn and I am going to call this sequence as an; Tn is the? No, X(t) is your process side, continuous time Markov chain. It is just like you set T is equals to Tn. Right, exactly.

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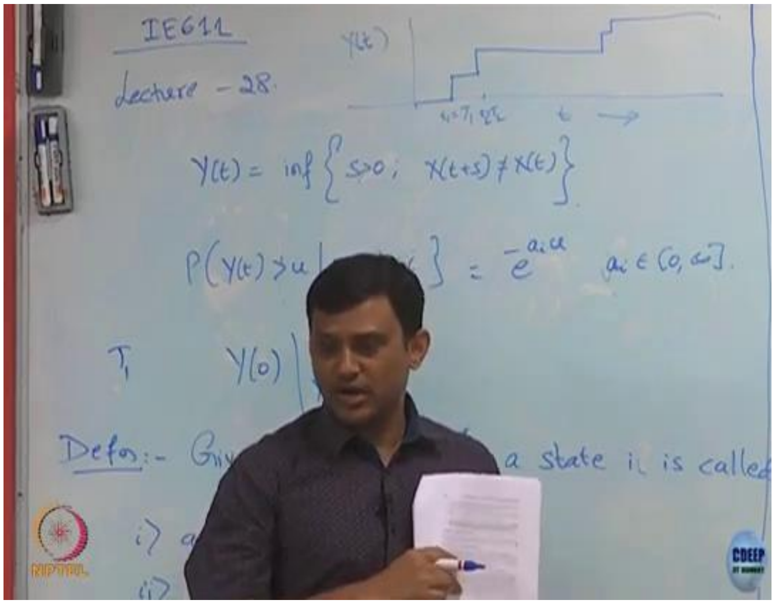

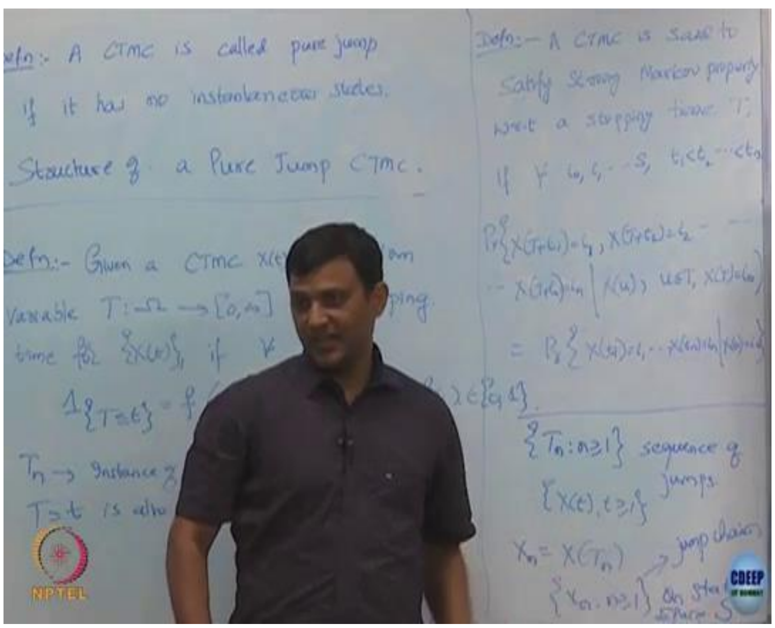

So you are, let us say if you are looking at your CTMC as a function of time. You let us say; initially you started let us say for simplicity from state 0. Let us say you are going to stay here till this point. Let me call that realization T1 which is realization of my random variable T1. And after that you have jumped.

For time being let us assume it is a counting process. So, you have taken a value of 1 here. And after that you will made this sometime here, then go here, maybe take a long time here and then make another jump like this, okay. So, these are all your samples T2 and like this.

Once you tell me these realizations, I am going to sample this process and I am going to derive. So, in the counting process it so happen like every time there is a trigger, we are always counting 1 and it is going 1. But it need not be always the counting process. It could be something else. For example, it is transiting to some arbitrary states. So, in that case it could be something; like these are different different state you are going to transit and that is capture by this jump process, Xn.

So, this is a kind of an embedded discrete time process which is embedded in my continuous time Markov chain. So, this what? This is an embedded process and we are going to call it as a basically a jump chain here. Whether it is a DTMC? We will see that. If we have a, we have a chain now which is discretely indexed. What are its properties and how this is looking at the properties of this jump process? We can say something about the properties of the CTMC is what we are going to study next.

(())(27:39) make an assumption, like here in this particular case. Like whatever. Let us say if it is not right continuous; whatever the value of your Xt at time T1, you are just defining that to be Xn. So, if in this case it is right continuous, you have taken the jump; whatever the new value up to the jump it is happening. Otherwise, would have just taken this value. So, be it, if we want to define it like this. So, whatever that, we are just going to call it an Xn sequence, okay. Okay fine.

So, next we are going to see that; now I have a jump sequence here which is discretely indexed. And let us say this is now defined on the same state space. So, my initially CTMC is defined on state space, S. My jump chain is also defined on the same state space, okay.

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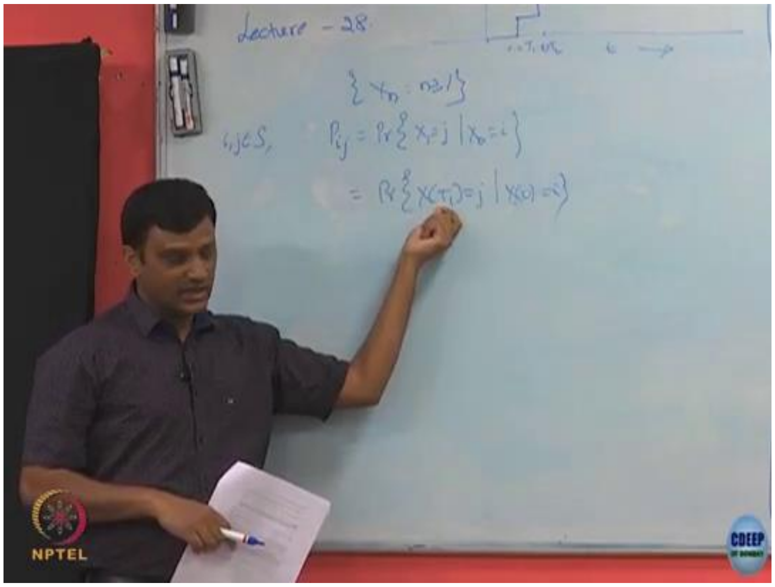

Now, let us say that on this I want to define a transition probability, okay. So, how can I do that possibly? Because let us say, here also like what, I can ask the question that okay initially. So, now I basically have this chain Xn. I can ask what is the probability that this chain takes value J, given that it has started from a particular state, okay that you can always ask this question. Like what is the probability that in one step it takes a (())(29:22)?

So, suppose, let us say on this state I could always ask. Like if I and J belongs to my state space, I want to ask this question that what is this probability. What is probability? This probability could be simply, let us say probability that my X1 equals J given my X naught equals to I, like we did it in a DTMC. Okay so fine.

But I have not started with this. I have started with my CTMC for which I know P of T. what is P of T? Yeah, for every T it is going to give you the matrix right, for your original CTMC. So, how you are going to define it in terms of my initial P of T.

See, like going from state X0 to X1, this is happening in how much of time according to my initial CTMC? T1 rounds. Now, basically what I am asking this question is probability that X of T1 is equals to J, given that X of 0 is I. Now, this one was like embedded process. But now I have written it in terms of my continuous process. Or I can write using my Pt, I could find out what is this probability. So, depending on how your distribution, what is the distribution of this T1 and what is the corresponding values of matrix Pt, we can go on and compute this value. Okay, fine.