Introduction to Stochastic Processes,
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Lecture 45
Application to DTMCS

Okay so, today let us windup our discussion on renewal theory whatever we started in the last class.

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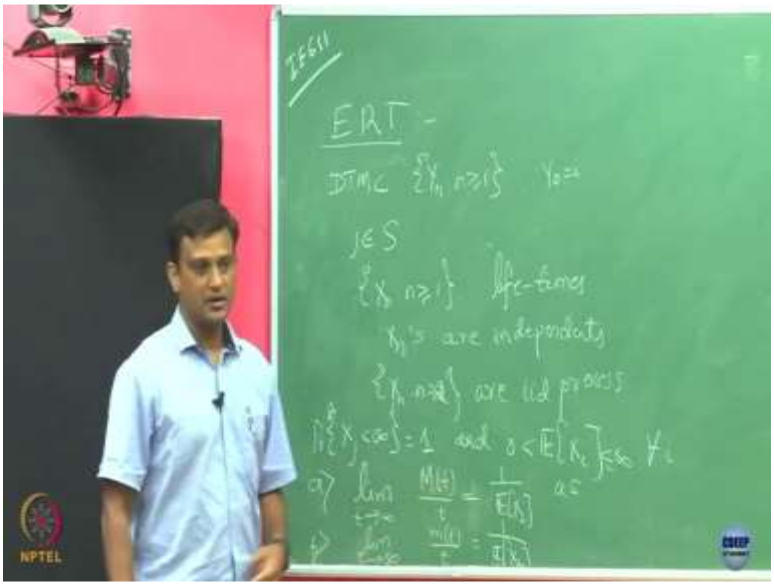

So, the last class we talked about this Elementary Reward Theorem, what we call as the ERT what did this theorem say, if you have a process, so okay, just as a recap what we did, we started with a DTMC and we you said okay let us say started an initial state i and I am interested in some particular state J where S is my state space and I am interested in my DTMC going back to this particular state again and again.

And then we have this sequence of random numbers which we called as lifetimes right and we said that this lifetime are such that these are all going to be independent. So, Xns are independent and then we said that if you look at Xns for n greater than or equals to 2, that IID, is an IID process.

Now if you consider this case then our ERT theorem, elementary renewal theorem said that this is under the hypothesis that if probability that Xj is equals to is 1 and so what did we say, we said that limit as t tends to infinity M of t by t is equals to what would you say? This limit goes to?

Student: 1 by n.

Professor: 1 by expectation of 2 almost surely and then we also said that t is also. We have basically said this, these results said assist the average number renewal in the interval in any interval that converges to this rate which is given by 1 upon expectation of X2, right we discussed this last time if X2 is going to be the expected value of X2 is going to be large that means I am not coming back to returning to my state in small number of terms may be I am going to take large number rounds to comeback.

In that case, I do not expect this quantity to be large right if this guy is going to be large this guy is going to be small, this is because this is the average number of renewals in the interval 0t, okay fine. How is this theorem we said we will not going to a proof of this but let us take it we understand intuitively as this should be correct. Now what is the use of this theorem?

Okay so, if you recall I had stated in earlier result which we will use heavily in the study of DTMC and that result was about when is a state a j when if I know that a state j is recurrent when is a transient sorry when it is a positive recurrent and when it is null recurrent. So, we has stated one theorem related to this, right. So what is that theorem?

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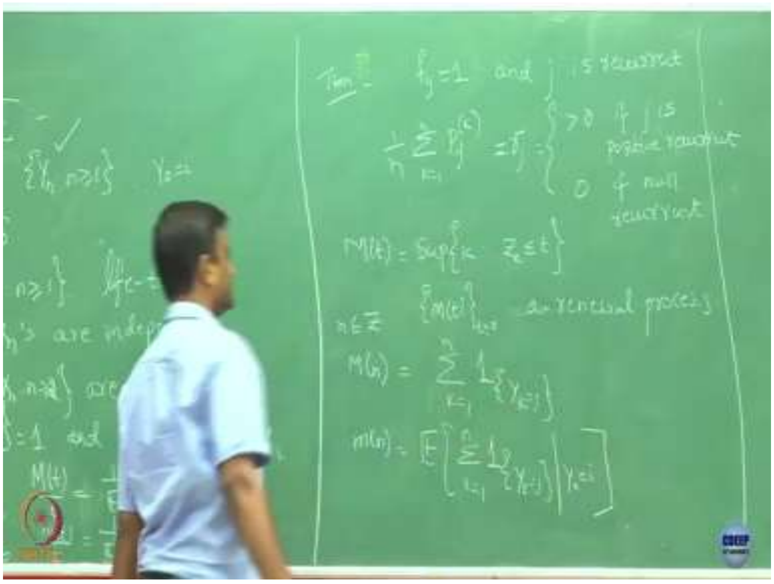

So, in a DTMC suppose let say Fi j equals to 1 and j is the current then we said something about this limit right, what was this? If j is recurrent we said this quantity we called it something like gamma j and we said that this guy is greater than 0 if, and we said that this is going to be 0 if null recurrent.

Actually this theorem is a direct consequence of this elementary renewal theorem. Okay let us see why this is true. So, remember this theorem where all we use, we use this theorem in the proof of in variant distribution, right. So, when we made a claim that for an irreducible DTMC it is going to be positive recurrent if and only if Pi equals to Pi P. In the proof of that theorem we used it and we have also used it in many-many theorems.

Now why this is true? Okay so to understand this now let us comeback to our setup here again. One thing we have defined is M of g is equals to what we have defined it to be M of k such that Z of k less than or equal to t. And we call this M of t as what? What? We call it actually a renewal process and this is defined for any t. So we have called it as a renewal process.

And this is for defined for every t, you give me arbitrary time t than I am going to tell you how many renewals have happened in the interval 0 t. So, what was, what is that M of t is telling, basically M of t is telling how may renewals have happened in this interval 0 to t.

Now, yes this t here is continues time but so if you to skill I may also be interested in knowing what happens at some M of n way, n is my some integer valued, I can ask this question right. If this is going to be defined for every possible t that is real I could as well ask for a particular integer value. So, then what is M of n? So that is in that case I can just write it as indicator that Y k is equal to j starting K is equal to 1 to n, is this right?

So, basically the number of times I have state, hit state j till time n right, till the nth instance, okay. Now let us assume that for time being all this, this yn are such that the n, they are of unit. So, when I say n equals to 1, n equals to 2 like this these are like separated by 1 unit. So, 1 second, 2 second like the separation between any two integer corresponds to 1 unit of time okay. Now, so for this also I can define I am assuming that I am starting from some initial state y equals to I.

So, I can denote the expected value of this to be k equals to n integer K j given. So, M n is nothing but expectation of this M of n earlier we have defined this small m of t which was the expectation of this m of t but now I am looking this function only at the integer value. So it is fine I can define it in similar fashion. Now let us look at the average of this quantity.

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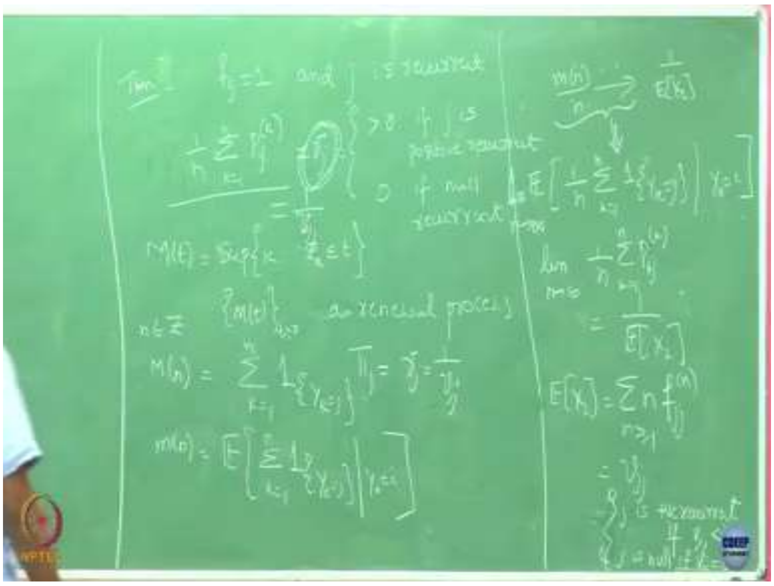

M of n by n as n goes to infinity. So, as n goes to infinity what you expect m of n by n go to is it going to be the same limit as this? Or what I am doing is instead of looking at all possible values I am only looking at t taking integer values. So, if I am going to look at this m of t by t instead of all possible t I only look at the integer values is this limit is going to be different from this limit or it is going to be different limit.

It is going to be the same limit right. So, it has to be expectation of X2. Okay now let us understand what is this quantity? Now let us focus on this, if I have mn by n this is going to be what? So, this quantity here is expectation of 1 by n summation k equals to 1 to n integer Y k is equals to j given y knot equals to i.

I am just writing the expression for this, now what I want I am interested in knowing the limit of this, so I am just writing this quantity in this. Now limit as n tends to infinity is expectation suppose if I interchange this expectation on limit what is this quantity is going to be when I take this expectation inside then expectation of indicator becomes what? Probability of given y knot equals to I right.

So, is that same as saying probability of i j super script k, right? Then you see that you will end up with something very similar to what we have here. Okay before that fine if I do that I already see I am going something what I want but how can I interchange this when can I interchange expectation and limit here, can I do that here?

So, we have been asking this question at many times right, we were are all many times facing with this problem of interchanging limit and expectation, can I do it here? And when can I do it? We know when we can do it we have seen couple of cases where we can do it, but the question is, is this corresponds to one of those cases?

So, which of this results like we know dominance converges theorem, monotone converges theorem, bounded converges theorem which one we can apply here, bounded converges theorem, so if apply I should be able to expect interchanges expectation and limit. So, because of that I could as well write it as things, right. So, actually I really, I really do not need to worry about that because, the expectation is already inside.

So, why I need to worry about interchanging them? So, I think you guys did not notice this, so I could directly say anywhere this is a finite summation right, I really do not need to worry about interchanging that, so I can already always plug in this and what I get is basically 1 by n P i j of k, k equals to 1 to n. why this is i because on condition that it is going to start from y knot equals to i. Now we know that this limit is equal to 1 upon expectation of x2.

So, we now are saying that this gamma j whatever we have we had earlier is nothing but that was 1 upon expectation of X2 okay. So, now let us look at what is this expectation of X2. What is X2? X2 is the number of time it took to visit my state j for the second timer ight, what will be its expected value? So okay, what is this value of X2 can be, what values it takes, X 2 takes what values? Integer value, right.

And what is the probability it will take some value n, 1 by n, why? So, what is, what is the meaning that X2 is equals to n that means my Markov chain visited return to j again after exactly n rounds, not before that right. We have use that rotation for that, what was that? Fjjn, right, we are talking about starting from state j going back to that. And that one so because, so that, this guy is going to be Fjj of n, is this correct?

The expected value of x2 and this also we have denoted as what Nu jj. And what would we say when this Nu jj is going to be finite we state this state j to be, are you sure? When v j is finite this now we say that j is positive recurrent if vjj is less than infinity and we said that j is null if vjj is infinity. Now let us compare this, this quantity here what we actually showing is this gamma j here is nothing but 1 upon Nu jj right.

So is that correct this gamma j is nothing but 1 upon expectation of x2 that we have shown it to be one upon Nu jj. Now when this state j is positive recurrent I know that this Nu jj is finite. So, this quantity has to be positive, when Nu jj is infinity that is when j is null recurrent then Nu jj infinity then this quantity has to be 0 right. So, basically what this result we use earlier is nothing but a simple consequence of our elementary renewable theorem okay.

Now, let us further go back one step. So, what we have basically saying is the gamma j we had define is nothing but Nu j if you now recall the theorem we showed for positive recurrent, we said that my irreducible DTMC is positive recurrent if and only if Pi equals to Pi P, and how did we show that such a Pi exists? We showed that, that Pi was nothing but this gamma j when we actually show we needed to show a Pi exist such that Pi equals to Pi P.

And that Pi was we have exactly took that Pi to be this gamma j. So, on your Pi j is nothing but the reciprocal of 1 by the it is just like 1 upon Nu jj right. So, now how to interpret this, this expectation is what this expectation is about the number of rounds to return to state j, right this is the expectation on that. Now I am going to return to that state often that means this expectation is going to be small right.

Then what is this quantity is going to be, pi j, corresponding j? Large that means I am going to see that more and more right. So, that probability that my Markov chain is visiting that state j is going to happen with high probability, the probability of that is going to be high that is because the number of times to visit that state again is going to be small. So, I will be frequently visit coming back to that state quickly because of that I am going to see that state again and again many times.

So, the probability that, so also recall that this Pi j is the stationery probability that my Markov chain will be in state j, right. So, because of that I will if this guy is going to be too small I am going to see my Markov chain in that state j for a large fraction of time, that is the probability of me seeing my Markov chain in state j is going to be high okay.

Now, so you should be able to bit more comfortable in applying this results like if you want to interpret, how frequently I am going to visit a particular stage and how that is going to be related to its number of mean visit to that time.

You should be able to connect all this results and derive any properties, any relevant properties about the Markov chain, okay fine. Now let us come back to our renewal process, the way we started defining our renewals process is by taking a underline discrete time Markov chain, right we said that let us yn be a discrete time Markov chain starting in some state i and now I am interested in visit to particular state j. And for that j I will construct the renewal process okay.

So, then we did like that the number of visits to number of slots it took to return to the same state j that was all integer valued right. But it is not necessary that we have to build a renewable process like this, renewable process is about anything where you are interested in something happening again and again right. For example in the battery case we said, okay when the battery life ends, battery life may end at any time it need not be integer valued something.

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So, one could be taking a renewable process as simply something where you have the lifetimes are such that they have identically distributed and all I am saying is identically distributed I am not saying Xn has to be integer valued they could be possibly continuous valued okay such that are at iid. Or more generally I can just take anything so instead of making this special case I will just say hence forth for simplicity I can take a sequence of are iid including the first one.

Okay so, when I talk about my DTMC case I said, I am interested in returning to state j particular j given that I started in a particular state i, I could as well say why start with i, I would start with j and then look at returning to that state. In that case I do not need to make a separate distinguish between the first cycle and the subsequent cycle right, I can they are all returning to the same state starting from the same state.

So, in that case I could just say that these lifetimes are all iid. So, I will not make a special distinguish, distinction about the first cycle all cycles I am going to treat the same including the first one and in that case I will simply write Xn are iid. Now actually we have already looked at renewal process where this X1s were exponentially distributed, what was that process we call, what was that called, Poisson, right.

We have already looked into a Poisson process where inter arrival what we call basically inter count times are all exponentially distributed and if this, if this Xns are exponentially distributed with parameter lambda then actually we had said that this is nothing but a Poisson process with rate lambda, right.

So, this is nothing but, so in this process what we are saying, so how did we describe our Poisson process basically we said that Poisson process is counting process which is basically counting something and the time between two counts is exponentially distributed with parameter lambda right.

So there count can happen at any time, it is not has to happen at a some discreet times slots. So, in that way Poisson process is actually continuous time process. So, for example one case we say that okay when you guys enter into the class, one guy came that is the first count happen, after sometime second guy came that is the second count happened and after some time third guy came like that.

So, here you guys are not coming at some discrete times right, you are coming at some sometime. So, Poisson process is about continues time process and now but also it is basically counting something that I discrete and inter arrival times they are all exponentially distributed and this exponential distribution is again continues random variables. Now the question is how to define, okay so now let us we should be so this is basically Poisson process is also a kind of a renewal process in that sense right.

It is basically looking at the counts as something happening again and again and it is just keeping tracks of how many counts has that has happened so far.

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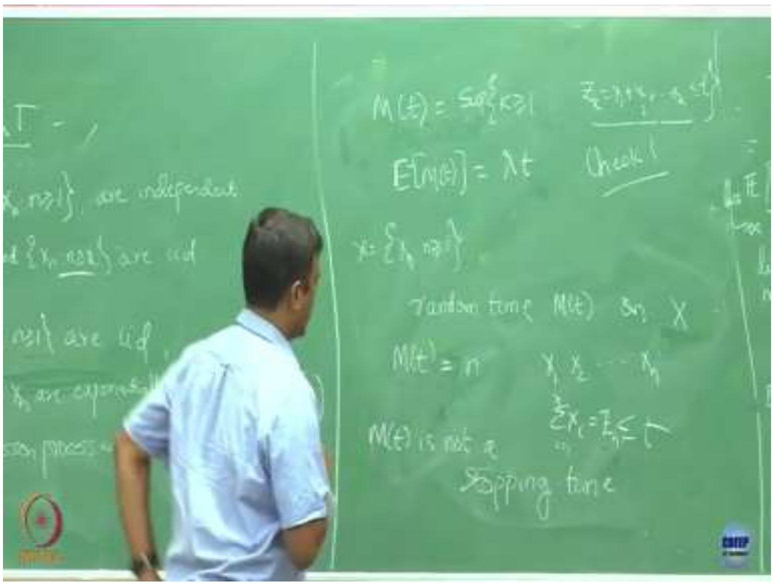

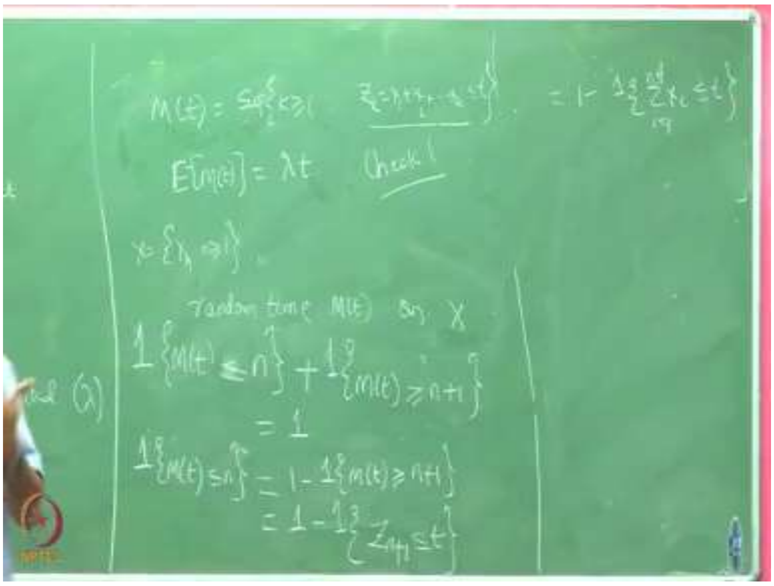

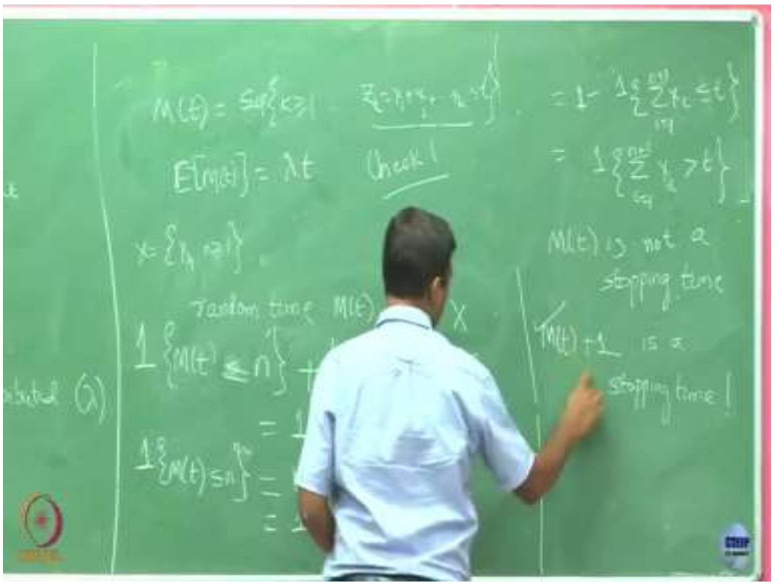

Now let us define M of t for this, that is number of the counts that has happened in the interval 0 to t. So, this is as usual so, no this is a general thing this is for any renewal process so I can also define the same thing for my Poisson process right, and then it is Zk here which is nothing but x1 plus x2 all the way up to Xk being less than or equals to t. I can define this right, for Poisson process also.

So, Poisson process is characterize in terms of this inter arrival times and that is going to define my count instances that is Zks. So, once I know that I already know I can define my renewal process like this. Now what is this basically M of t? M of t is the number of arrivals of the interval 0t right. So, for a Poisson process number of arrivals in the interval 0t what was that, Poisson right, and what is the mean of that 1 by lambda t, lambda t right?

So, what you expect this to be? Lambda t, right. So this is by correcting so if I have defined a, and this is a general process I defined I can defined this anything but if you are going to do this on a Poisson process you are going to get this, now and this is by the argument saying that this is just by interpretation M of t is nothing but the number arrivals or counts in the interval 0 t and I know that is Poisson distributed with rate lambda t and that has to have a bit.

But you can also using this notion that Zk is a sum of this you should be able to again derive the same thing, okay you have to check that. Another thing for a Poisson process let us say these are my lifetimes right, with respect to this process, life time process I can define a stopping time that stopping time is M of t, like sorry I am going to not stopping time I am going to refine a random time M of t on x.

So, M of t is what? Basically M of t is always integer valued, right even though it is define for every t but M of t is giving you integer valued numbers and this is all itself says it is a random quantity right. So, now I am saying let us take this sequence and define a random time on that. Okay now I want to ask the question, is M of t is a stopping time on this sequence? Is this random time is a stopping time on the sequence Xis or Xns.

So, suppose if I know that, okay basically what I need know, if I want to answer the question is M of t is less than or equal to n is it enough to know x1, x2 xn is this true. So, suppose let us say you been given x1, x2 all the way up to xn. So, if you know x1, x2 all the way up to xn and you have been also given t, okay if M of t is less than or equals to n what I know this sum has to be less than or equals to t right.

But does it also say you that the n, n plus 1th lifetime has not happened within this first, within this t, yeah but all we know is x1 plus x2 is going to be less than or equal to t. So, what is this because so this is nothing but if I sum all of them xi is going to be, is going to be what is this this is going to be n, right? This I know as zn and I know that z n has to be less than if M of t is less than or equals to n I know Zn has to be less than or equals to t that we know beside n plus 1.

But you do not know anything about you have been not told at x n plus 1 as this x1 plus 1 if you add to this that will go beyond t or it is still going to remain within t. May be or may not be right, so because of this without knowing xn plus 1 I cannot say that Mt is going to be less than or equal to n is this clear? That is what we want to answer whether M t going to be less than or equal to n the question is can we answer this based on this information.

Student: (())(36:40)

Professor: oh sorry, I want to ask this question, right, whether M of t is equals to n can answer this question completely. So, it depends completely unless I know that the x n plus 1 that is going to happen has to happen after time t that the sum if I add xn plus x1 plus to this sum that is going to be larger than t I cannot say this right. So, if the only, if happens that guy after adding that if that guy is exceeds t then I know that till n things have been incorporate n plus 1 is going to stay outside okay.

So, because of this we cannot really say that this is going to be a M of a is going to be a stopping time with respect to M with my sequence Xn okay. Stopping time, okay just a minute I am think we can also argue that, so let us try to make this more formal right. Okay so you are right so to make to apply for the stopping time we have to just not worry about equality we have to answer this question.

So, let us try to answer this. What we can do is, is this correct? So, I am saying that at any time t M of t has to either less than or equals to n or it has to be greater than or equals to n plus 1. This, one of this must be true and this should be equal to 1. Now to I am going to write this bit n plus 1 and what is this going to be, 1 minus M of t is going to be greater than or equal to n plus1 right.

1 minus of this, what is this is going to be? So, this is going to be basically we have already said that if M of t has to be greater than or equals to n plus 1 in terms of Z n plus 1 what we know? We know that this guy has to be z n plus 1 has to be less than or equals to t we have argued that these two are the same events right. So, this both must be equals okay.

Okay now let us see this Z n plus 1 is nothing but summation of first n events right, this is correct, I have just apply the definition of let us say n plus 1 here. Now if I want to look at this 1 minus of this, this is basically 1 minus of this means the compliment of this event right what is the compliment of this event, the compliment of this event summation of I equals to 1 to n plus 1 xi is greater than or equal to t.

Is this correct now, the question that one the question that is M of t is, is less than or equal to n has balled time to asking the question whether the sum of first n plus values has exceeded t, right? You see that so what we want to know that n arrivals has happened in the within the interval 0 t to ensure that I will need to be confident that the nth plus 1 has not completely occurred within the interval 0 t.

It must have ended after time t, so that is what exactly it is capturing, this is going to be the same as asking the question whether the first n plus 1 Xis their sum is going to exceed t and so answering this question is equivalent to this and to answer this question how many Xis I need to know, n plus 1 right?. So, just knowing n random variables is not enough for me here, so you can also just like that think it till now that we have written this.

If you want to know that with in my interval 0 to t less than or equals to n might have happened and no more that much be the case that n complete renewals might have, must have been completed in that and n plus 1 must not have been completed within the interval 0 to t had n plus 1 had also been completed in the interval 0 to t then M t would have a taken the value n plus 1 right.

It would have been greater than n. So, if M of t has to less than or equal to n it must be the case that n plus 1 renewal might have completed in the after time t. So that is exactly it capturing. So, it is because of that M t is a not a stopping time. Now so, you somehow you feel by this information that if I want to know that n arrivals or renewals has happened in the duration 0 to t I need to know not n variables but actually n plus 1 renewal life right or n plus 1 Xis

So, it so turns out that M of t is not a stopping time but if you just look at M of t plus 1 is a stopping time. Why is that? This is also actually obvious, so if I want to check whether M of t plus 1 is a stopping time replace M of t by M of t plus 1 right. That is basically asking that whether M of t is upper bonded by n minus 1. So in that case I have to replace n by n minus 1. So, in this case just summation happens to be i equal to 1 to only n.

So, in that case it only depends on the n random variables but again this is argument but convince yourself that, if I want to check whether M t plus 1 is stopping time that is the case.