Introduction to Stochastic Processes
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Lecture 43
Introduction to Renewal Theory

So, in the last 2 lectures what we have, we are going to cover, we can called a Renewal Theory. So, this renewal theory as we will see that is nothing but as much slight more generalization of what the things we have already studied okay. But it kind of captures the essence of, many applications be seen practice and also justifies, how we can model them using some of the tools like Markov chains and also what renewal theory we are going to study well.

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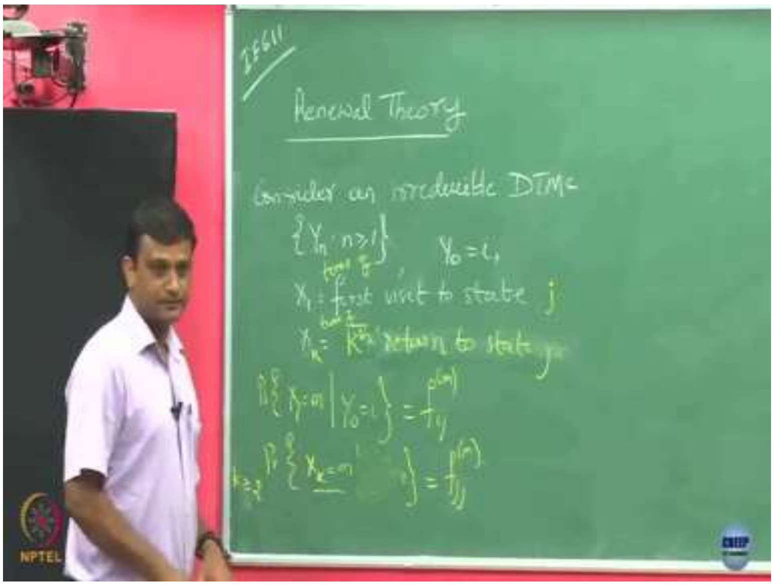

Okay so, in a way we have already touched upon renewal theory when we try to study Poisson processes. You remember, when we talked about Poisson process we have a counting function, which kind of try to count when the things happened. And one can usually think of account happening as basically something happening again.

For example, let us say, you are building a system or you are interested in a particular component of the system, let us say specifically, you are interested in the battery of some system. When you fully charge it and put it in the machine it is going to operate for some time and after that, maybe it get discharge and some red light in that blinks and then you know that this guy has gone below.

Now, it is not in good shape either you need to recharge it completely or you need to replace it. You do it then it is going to work for some time and again maybe the red light blinks and now it is time for it to replace. Now, you may be be interested in knowing, how long this guy is going to operate in the long term.

So, what will be the average lifetime of this battery? That is how such questions you will be interested and here renewing is like when you have to renew the battery or when you have to replace the battery. So, such questions can arise in many-many cases right, not just like you are replacing is a component.

It could be even thought of like, when some new event happened, you can think of something has happened. When again the next event happened, you can think that as the event has renewed. And then see based on this you can try to analyse, what is the rate of these things renewing, what is the inter arrival time between the renewals and all these things. So, let us try to make the things formal now.

Let us say, I have a DTMC, Y n. Now based on this, I am going to define let us say if you are going to start from a particular state y equals to i, you are going to start it from a particular state i. Now, I can define x 1 to be first visit state i and I can also like this can also look like the Kth visit to state i.

So, let us say some i, first visit instead of that let us say let us take some other state. I start from i and now I will be interesting when is the first time I go to state j. And then K, Kth return to state j, after that when I come to state j, after that I will be looking at from j again then I go back to j for the second time and then once I hit j again when I will go back to j for the third time liked it.

So, it is X K here denotes the Kth return to state j. okay now, what we know, probability that X 1 equals to m given that y knot equals to i. If you have denoted f i j to superscript m right, starting from state i, when I am going to hit j for the first time. So, this is the time of, see what I am writing right now, X 1 is now the time taken to visit to a state j for the first time.

Okay so, when you so starting in state i, you have taken let us say I am asking what is the probability that you take when I write X 1 equals to m right that means, you have taken m rounds to hit state j for the first time, that is the meaning of time. So, this should be, I should be adding time. So, X 1, X K here are denoting the time when you start from state i, how much time you are going to take to hit state j for the first time that is X 1.

And once you hit state j, when you are going to again hit state j next time that is X 2 like that, so the times are denoted. So, is this now clear that this term is exactly equals to this. And after this, if I am going to say, let us take any K equals to m, given y knot equals to, what is this going to be? So, when I say let us say when I said y knot equals to I right, so I have started with something.

When I say the Kth starting, so I am going to hit to hit state j for Kth time that is the number of slots I took to return to state j. It is exactly return to j for the Kth time in m number of rounds. What is this can be, or like here I really do not need to condition upon this right because when I say X m equals to m, I am asking for returning to state j for the Kth time in m rounds.

That means before that although already hit state j, and from state that j and here let us certain this K is greater than 2. For time being just take K equals to 2. I am asking X 2 equals to m. So, it is then going to be f j, j to the power m superscript m, right. Because you already know it has hit state j and from there you want to go to the next stage j again in m rounds. And now is it does it depend on what K is or this is going to be a same for all K.

It is going to remain the same once you hit state j, I mean, fine everything in the past, I can forget. And from there I want to see in how many rounds I again go to state j in m number of rounds and this is going to be that problem.

Student: (())(8:35).

Professor: So, because we have already saying, I am going to hit take m rounds to hit state j for the Kth round. Suppose K is going to be greater than A, we are already saying Kth, you are asking for the Kth return. That means it should have previously returned. Otherwise K if being greater than or equals to 2 does not make sense. So, that is why, we have written these 2 cases separately here.

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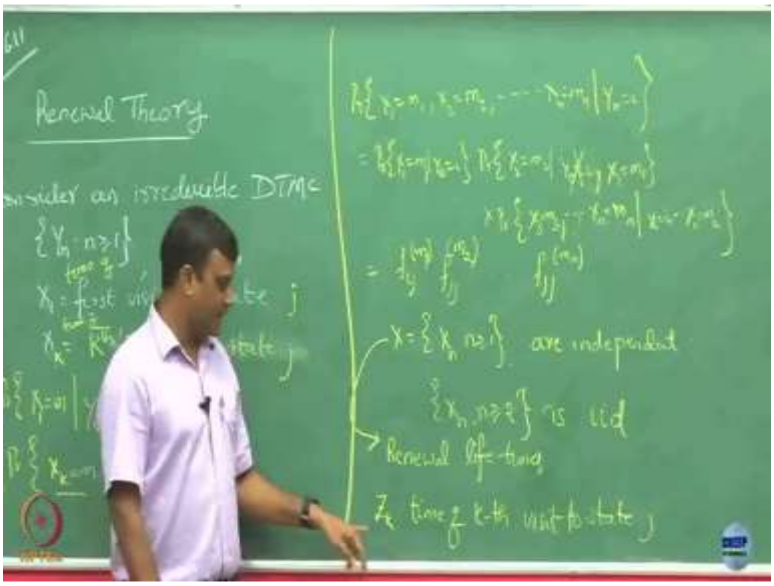

Okay now, suppose let us consider this case, X 1 equals to let us say m 1, X 2 equals to m 2 and all the way up to, let us say, X n equals to m n given y knot equals to i, what I am asking? I am asking starting from state i in the first in the initial round, what is a probability that you take m 1 rounds to hit state j for the first time.

After doing that, you take another m 2 rounds to again return to state j and then again, like this, and in the you are going to take m n rounds to hit state j for the nth time. So, I can always write this as, you can expand this and what and you can keep on doing this right. So, when I, so this is y knot equals to i comma. So, when I say y knot equals to i and X 1 equals to m 1, what does this mean?

I started with state i in the 0^{th} round, then I hit state j for the first time in m 1 rounds and then I am asking from there what is the probability that you are going to hit again state j in m 2 rounds. So, then this is does not matter here right because this already telling that you have hit state j again. So, can now let us try to write this. What does this quantity is going to be?

We have already written this, and what is this? M, it is simply going to be m 2 and here it is m 1. Because we are asking you took exactly m 2 number of rounds to hit state j again. Given that you have already returned to state j. Now like this, you can keep on doing this iteration and you will end of it, you going to do this. So, now if you look into this, basically this joint distribution has split as this probability, this joint probability has split into this individual.

So, each time is like here a probability right as we have already expressed here. So, now if I am going to and you see that this joint distribution has split into the individual probabilities, what does this mean? So this means X 1, X 2, X ns are all independent right. And further, if you forget the first one look at the others. They have the same as f j term. What does that mean? They are not only split, the probability not only split, but each of the term looks identical. That means the distribution of each of this term is same right. They are all f jj. So, that means definitely the sequence if I am going to look at this process, what is this process?

This is a process of inter visit times sorry, what is this? So, this is the duration of the returning, duration of the returning to the same states right. These are all independent. However if you look instead of X n starting from n greater than or equals to 2, if you look at n greater than or equals to 2 or this process, what is this process is?

It is not only independent now, if you look at only the sequence after n greater than or equals to 2, they are also further identical. So, they are IID. This process is IID and this set of sequences are independent. What you are going to call this X hence forth as lifetimes. Okay we call this as a lifetime process, or we are going to call it as simply renewal lifetimes. Is this clear, why we are calling this renewal lifetimes?

My process here, in this case, my process and what I am interested in, returning back to state j. And I am looking at when I am returning, I am looking at how much duration I took to return? In that in the parlance of the battery example, we talked about when I replace a battery and I can think of when the battery dies. I am going to replace it.

So, from starting point till the time it dies that is the time duration or that I can think it as a lifetime of that battery right. So, that is why, we are going to call this as renewal lifetimes. So, X 1 is going to tell me what is the duration? What is the lifetime of my battery when I first recharged it, X 2 will denote it, once my battery was dead when I recharge it again, how much time it lasted before it died again. So in that way, these are going to talk about the lifetimes of the renewals.

And now, we are also going to denote Z K here, I am going to define Z K as time of Kth visit to state j. I am hitting state interested in returning to state j again and again. But I may be returning to state j, first time or the second time or the third time right, I am also trying to keep track of how many times I have returned to state j.

So, Z K is telling me exactly at what time I return to state j for the Kth time. Okay for example, in the battery case, Z K can denote when is the when is that my battery died for the Kth time right. So, every time it dies, I recharge it. And when is that it, it dies for Kth time.

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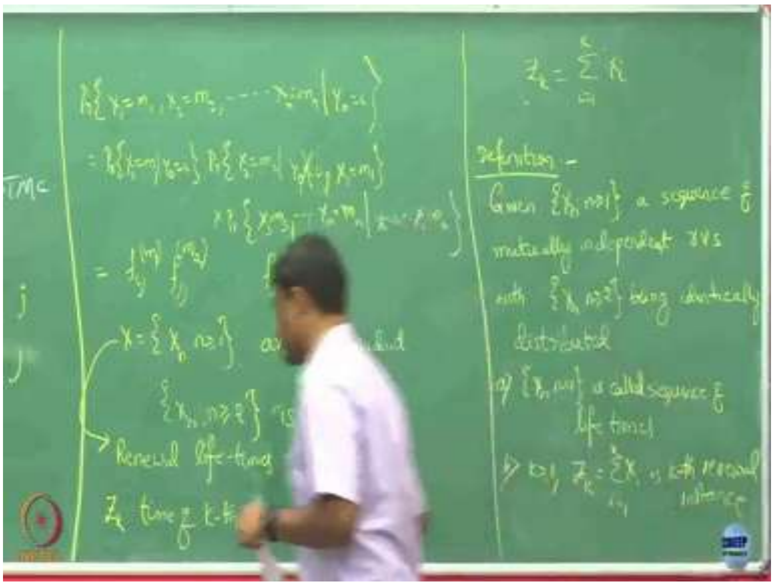

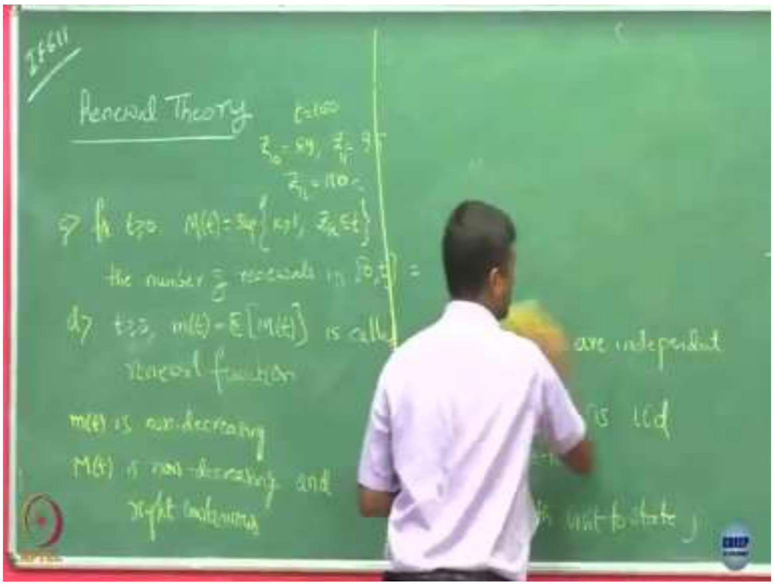

Now, is there a relation between that Z case and X case? Z K equals to sigma X i, i to k, right. So X 1 denoted the time it took for the battery to die then X 2, so basically X 1 told me after how much time I replaced the battery. X 2 told me next time when I replace my battery, so, if you just add all of them, you will get time K when you Z K time Z K when you replace your battery for the Kth time.

Student: (())(17:47)

Professor: Why equal? It is a random process. They are identical but their values, their realizations could be different right. So, here what, this X 1, X 2, X 3 they are all random variables. Yes, it is true that X 1 has the different distribution than X 2, X 3 and others. But you should take a realization maybe X, the sample, maybe the battery. So, when you started for the first time, the battery, let us say lasted for 1 week and died and then you recharge it and then what happened?

So, okay let us say, you have a some battery, a priori you do not know in what state it is? It may be half charge, maybe quarterly, charge three fourth charge whatever it is given to you. And you started using it. So it is going to die at some time. And once it dies, you are going to completely recharge it and again look at when it is going to die again. Okay so, clearly the first part can have a different distribution from the second and the subsequent ones because when you initially started it, the battery was in some arbitrary state.

But when you are going to recharge it, you are always recharging it when the battery completely died. And now, if you even when the battery dies when you recharge it, maybe it last first time for the 3 weeks, when it dies you again to recharge it maybe this time it went for 4 weeks.

And again you recharge it maybe this time it just lasted for 1 week. They are because this is a realization, I am not talking about exact where this, the mean value or something right. This X 1 and X 2 are random variables right. So, they can take different realizations. So, even though X 1, X 2 they have identical distribution. When you take a sample, so let us say X 1 and X 2, they have the same distribution, take one sample from X 1 and take another sample from X 2, do they need to be the same value? That is what I am saying.

Okay now, based on this notations let us define what we mean by a renewal process now (dependent random variables with Xn being identically distributed we are going to just call this, sequence of lifetimes) and we have already defined for k is equal to 1 z k equals to kth renewal instance. (())(22:39). So we have already see, we have already motivated this, that suppose, we have a sequence of random variables.

Let us take them to be independent. But let us take the sequence starting from n greater than equals to be identical. Then we such a process we have when the, such a sequence we are going to call as sequence of lifetimes. And a Z K, defined in this fashion, which is the sum of the first K, X i's, we are going to define it as the renewal instance. So, Z K denotes the Kth renewal instance.

And now you take a t, any real amber t. Now, you define M of t, capital M of t to be supremum over K greater than equals to 1 such that Z K is going to be less than or equals to t. So, let us understand what is this function is saying, when you give at t, it looks at all the instances when the renewal happened within the time t and takes the maximum value of that K.

So, let us say, so it is you see that this guy Z K is a increasing right. Like maybe the first renewal happens at of 1 week. Second renewal happens in the 3rd week. And the fourth renewal happens in the 10th week like that. And let us say you are given some 100th day. So, you are going to take that as 100th day t 100 and look at all the renewals that has happened before that 100 day and see what is the latest renewal in that.

So, if you let us say 100 day you have taken and 100 day corresponds to how many weeks? Let us say whatever, some let us say on the 10th week, a renewal happened and the next renewal happened only in that 20th week. So, then what is this M of t is going to be? It is going to be, so let us say t equals to 100 day, I am counting them here it could be continuous, but let us take t to be 100.

And let us say my Z 10^{th} happened in on 89^{th} day . And the 11^{th} happened on 95^{th} day and let us say and the 12^{th} happened on 110^{th} day. So, in that case what is M of t is going to be? It is going to be 11 right. So, it will not include this because 110 is going to be larger than 100 in this case.

So, basically what this is telling is that a number of renewals that has happened in the interval 0 to t right. And now, this small M of t is basically looking at the expected value of this M of t. So, notice that this M of t here is a random quantity and how to interpret this M of t. So, if you give me a realization of Z of K, then I can define my M of t on that. So, like I have given you one realization right, this realization could change.

And on that realization this M of t is defined and this depending on the realization this value of M of t can also change. So, that is why this is a random quantity. And now I will be taken this quantity as the expectation here and this m of small m of t, I am going to call it as renewal functions.

Okay, now, let us try to understand each of this. Now we have defined so many processes right. Okay anyway, I started with X n based on that I have defined Z K, and based on that are defined M t and further, small m of t. So, how do these functions look like? So, what kind of random variable X n is? X n is discrete time. But what is the value that X ns take? So, all these X ns are defined on a given Y n right.

You started with DTMC Y n, okay so let us say DTMC Y n on space some S, let us say this S is, countable infinity okay. But still, this Markov chain is discrete time. Now what we are doing is, we are fixing a particular state and then looking at the time it takes to visit that time. Okay and again, return to that state j. So, then, in that case, what is the value that X n takes? But this time is in terms of the discrete valued.

So, it is going to, so let us say it is this X n is says, I took 100 slots to visit state j again or I took 20 slots to visit state j again right because this now because this Y n is discrete time, the number of time slots you took to go back to the state again is again in terms of this discrete counts right.

So, because of this, this X ns are again discrete valued random variables. So, like these X ns can take 10, 20, 30, 31, 32, whatever, but not like any value between 31 and 32. They are basically queue in the count of how many slots you took to return to the states again. So, that is why these X ns are discrete valued.

Student: Sir, it cannot give state j between 2 slots.

Professor: No, our Markov chain is changing only at these discrete points right. So, it change in slot. So, my Markov chain in changing, but our count have, we see the Markov chain only at the beginning let us say at the beginning of the slot or like we are looking it into the days, day 1 this is the value, day 2 this is the value and day 3 this is the value and we are looking at how many days it took to go back from this particular state to this state.

So, because of that it is going to be discrete. I know if this X ns are discrete what about Z Ks, they are also going to be discrete. So this is, but that discrete time and discrete valued random variables okay. So, are X n so the X ns are also they are also discrete time and discrete valued random variables.

What about (M k) M t? Is it discrete in time or continuous in time? Z Ks are discrete but now I am passing to this M any real number and then asking in the interval 0 to t, how many renewals happen, right. So, M t is continuous time but it is taking discrete values okay. And anyway M t is simply a real number at time t, because this is expected value. So, for every t it is defined okay.

So what is the meaning of M of t, this is the expected number of renewals that are going to happen in the interval 0 to t okay. So, is M of t is increasing function, if I increase t is the value M of t increases? Right, this is obvious because this capital M of t is itself increasing, if you increase t more renewals will be included in this and M of t is going to take a larger value okay.

So, we just saw that, so M of t is not decreasing and this is because and yeah what about M of t, is not decreasing. And see that I have asked Z of K to be less than or equal to that means I have included t in this not the renewal just happens to K before t. I am asking for the renewal that has happened t time t that includes t okay so, because of this, this function M of t is going to be right continuous okay.

So, you recall this right continuity also happened when we looked at other CDF, right. So, there we looked at probability that X is less than or equal to, so because of that inequality thing that was turning out to right continuous and, so here also, you can see similar things for non-decreasing and right continuous.

Now the question is, I said, a battery being renewed every time right. So, suppose instead of battery, let us think of I am counting some events, so when an event happened and then when the same event happens again, I am going to type the time between these 2. So it may happen that sometimes instead of one event happening, two events happen simultaneously.

So, for example, I can think of something like, okay I am going to treat somebody entering let us say I have a 1 big queue that is being served and the jobs are coming into my queue. So, jobs could be simple customers whom I need to send sell tickets, whatever. So, a customer can enter or it may happen that a customer himself is not entering, he is entering with his family. So, it is not 1 person, maybe a couple is entering.

So there are actually 2 guys entering the system and they are entering simultaneously right, but how I am going to count this? Both are entered simultaneously, I am going to treat them as 2 events actually. I am going to give two events but going to take them that has happened simultaneously. That means suppose 1 guy came and we thought her, her relative also came together. Let us have that guy coming was Kth event and her relative is the K plus 1 event. But the difference between them is 0, okay.

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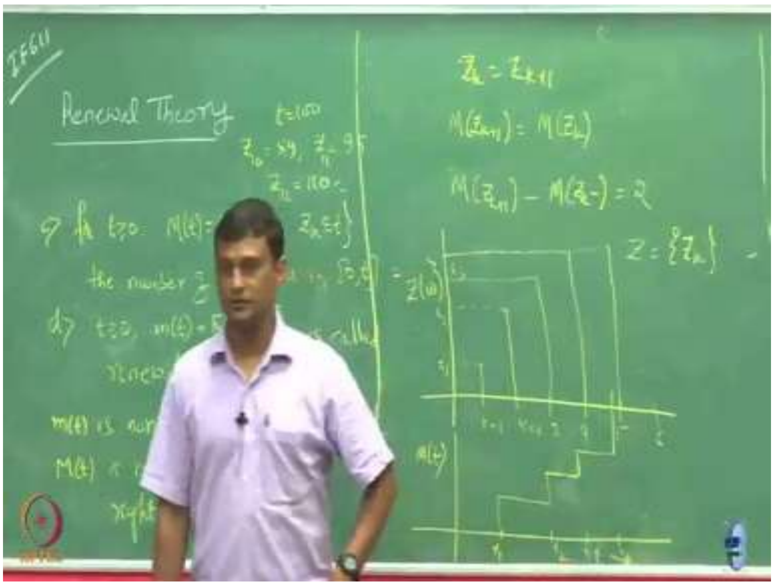

So, because of that, it may happen that, when both of them happen simultaneously Z K is going to be same as Z K plus 1 and also notice that M of Z K plus 1 is equals to M of Z K because both Z K and Z K plus 1, so then I what I mean here? When I wrote M of t right instead of t, now I have put Z K plus 1 here, Z K is also some number right, some basically time okay. Yeah so, this is also going to be some time. And now, I am asking this function, taking the value of this M at that particular time.

Now, that both of them has happened at the same time. Its value will be what? What will going to be the value of M Z K plus 1? So, here replace this t by Z K plus 1, what is going to be this value?

Student: K or K plus 1.

Professor: K or K plus 1, it is going to be K plus 1 right because both Z K and Z K plus 1 will be included in this, because they have the same time. So, because of that these 2 values are going to be the same. But however, if you look at M Z K plus 1 and M Z K just before, when I read Z K minus right, just before the Kth arrival happened, so what is this value you value we expect to be? It is going to be 2, right? Because 2 count has happened.

So, just to understand this, let us realize this. Okay, for counting let us take this Z, my Z is my is Z K process, right.

Student: (())(38:22)

Professor: M of, no, because Z K plus will also get included right because it will also have the same time. So, let u say one let us take one sample. I am going to take for some particular omega. Okay so, this is a discrete time, so let us talk. So, let us say K equals to 1, K equals 2, 3, 4 and 5, 6. So, let us say first arrival happened at sometimes. I will get some value like this.

The second value happens a bit later than this, third value is higher than this but let us say the 4^{th} value happened just before this and then this 5^{th} value happens at a same as this. Let us call it this is exactly equals to X 1. Let me call this, this is some realization, let me just call it t 1, this is t 2 and this is like t 3, this is let say t 4 and this is also t 4, it is also going to remain t 4 only.

Okay now, if you are going to draw its M function, so I have given you one realization of my renewal instances, based on this can we construct how this M of t function look like? So, till t 1 let us take time to be till this point t 1. So, this t 1 is going to be same as this t 1. So, till t 1, what is no Z does have, no renewal has happened. The first renewal is happening only at the t 1. So, before that there is nothing here right because but then in that case it is a supremum of an empty set, because there is nothing here.

Student: Sir, because t 1 only, also, is renewal of t 1 also we can include.

Professor: yeah we will include but just before t 1 okay right, so let us consider the case just before t 1. So, just before t 1, there are no renewals. So, let us define that term to be 0. So, all this function is going to remain 0 till this point and what happens exactly at t 1? It is going to be 1 and now, let us take this value t 2, let us say t 2 is somewhere here.

What is this way, how this function is going to look till the point t 2? It is going to remain horizontal right and what happens at the t 2? It goes to 2 and then what happens at till t 3? Horizontal and again it jumps to 3 and then what happens at t 4? t 4 till this point and how much will be the jump here?

At t 4, how much will be the jump? I t is going to be 2 units, right because 2 things have happened simultaneously. So, that is why, we wrote that K plus 1. So, at this point just before this Z K minus 1 in this case is just before this and it is going to include 2 events, okay.