Introduction to Stochastic Processes
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Lecture 41
Example of a Queue

So in the last class, we talked about how to we basically discuss the conditions that are transient and the recurrent states need to satisfy. So, basically we said that how to classify irreducible discrete time Markov chain into positive recurrent basically said how to classify it as recurrent and transient, right.

So, we basically said that, if we have a Q matrix, which is radio, which is basically derived from transition probability matrix by eliminating one particular state and then we looked at the equation of the form y equals to q y and what would we say, if the solution of this y equals to qy such that if y is 0, then what would we say? So, if the solution then we said it is going to be recurrent, right.

So, and also we had said that the y equals to q i equation will have only two, will be solution will be such that, the y will be either all 0 vector or it will be such that supremum over y i will be equals to 1. So, these two solutions are possible when I look at y equals to q i, but among the two solutions, if I get y equals to 0 as the solution, then I said we said that my reducibility MC is recurrent.

Otherwise it is transient and when I know it recurrent, I know further how to classify it as positive recurrent or non-recurrent right. How I do that? I look for a solution of a form Pi equals to Pi p, if I can find a solution which is a probability vector and it has all strict positive values in it, then I know it is going to be a positive recurrent otherwise it is it has to be non-recurrent. So, today we are going to apply this result on a simple Queue model.

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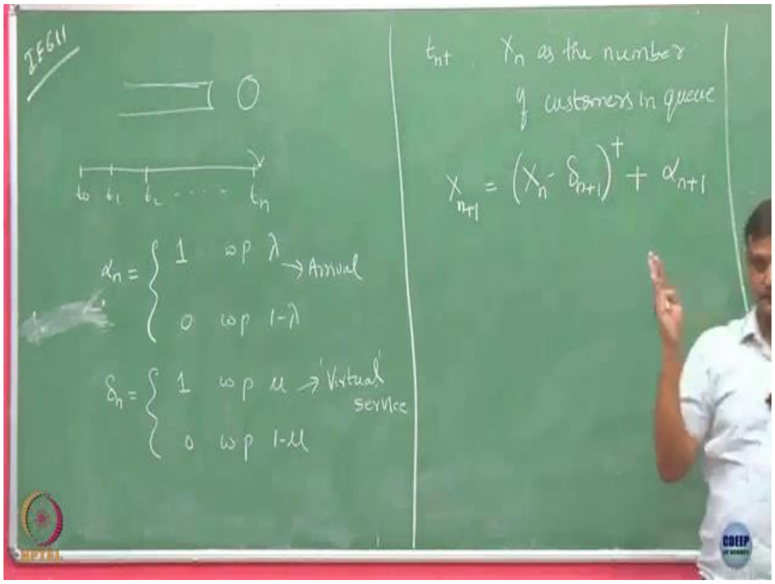

So, let us consider a queue and let say I have it so the, let say I have this periodic times here t0, t1, t2 like that and tn. So in each of this time slot, let us assume that there is arrives a one customer, the certain probability and also, when a customer arrives, if there are already multiple guys in this queue, he is just going to join the queue.

And when there are multiple people in the queue, the guy who is in the front who is going to get served, and when we will assume that in each slot, one guy is going to get served and going to leave the queue. So let us say that in each round n one guy arrives with probability lambda and there is no guy with probability 1 minus lambda. Okay?

And similarly, I am going to say that dn equals to 1 with probability Mu and 0 probability 1. So I am saying that, so if you are in round tn one going to arrive in this round is going to be probabilistic. One guy arrives with probability one, one guy arrives with probability lambda otherwise, no guy arrives. Similarly, let us say there are already some guys in round del n one guy will be getting served in the server, he will complete his service and leave in the nth round with probability Mu, okay.

Naturally if there is no guys in the queue then the server is not going to serve anybody right. So, we are going to assume there is a guy departing the queue only when somebody is being served okay or when the queue is going to be non-empty. So, in a way this is going to capture our arrival. So, this lambda is going to capture our arrival rate. So, this lambda n basically defines our arrival process in a way and then this Mu I will capture other virtual service.

So it is basically a service but I am adding the word virtual because if there is nobody in the queue, I am going to assume there is no need to serve anybody. Only when there is somebody then I need to serve, right. So I am just to occur for this facts, I am just going to call it as virtual service rate as Mu. This is basically our arrivals, right? In round n, this alpha n characterizes whether there is going to be arrival or not. They are going to say this is the arrival in random variable.

So arrivals are random and how that is defined? It is define like this, it is going to arrive with probability lambda or otherwise no. So the sequence of alpha n basically defines your arrival process. Similarly, alpha n defines your sequence of virtual service process, right, so if you are going to take one slot let us say between t1 here we are going to say that whatever that happens the arrival here, and departure in this we are going to take it as this.

This is going to be characterized like this. Okay, so we will try to make it more precise when we are going to write the state diagrams here, okay. At any time slot tn we are going to denote xn as number of customers in queue, okay. So suppose let us say you are interested in this slot tn, at the beginning of tn, I am going to make it actually tn plus here. What does this mean?

Just at the start of tn how many customers are there in the system. We are going to denote it as xn here, okay. Okay, now let us come back to this. So one guy can arrive in this with probability alpha n, next guy can arrive in this with probability alpha n like this. So if and this is going to be with the same probability, right this lambda are constant, the probability that a person arrives in a one slot is going to be fixed, that is going to happen with probability lambda.

Now, this is a Bernoulli random variable, right. In each round somebody arrive, and the entire process is like a Bernoulli process. Now if I am interested in arrival between two customers, let us take any instance when a customer arrive and the next instance or the number of slots after which the next customer arrives, how that is going to be distributed?

Exponentially, it is going to be what? Suppose let us say at some point one guy has come in the next slot, one more guy can come with probability lambda or he may not come with 1 minus lambda. After that if he does not come he may come in the slot fallen that with probability lambda. If he does not come, he may come in the two slots after that, right.

So if I am going to say that, if I want, let us say, if I am going to denote capital T as a random variable that denotes arrival time between two customers, how the T will be distributed as, it is going to be geometry distributed, right, with what probability, with what parameter? It is going to be still lambda. And what about if I am going to say let us say now let us say T is the time slot between departure of two customers from Matthew, how that T is going to be distributed as?

Again geometric with what parameter, it is going to be distributed with Mu, right. Okay, let us say at the beginning of just at the start of tn plus, this is denotes how many customers are there in my queue and then how can I write this process now. I can write this process as xn plus. So one thing is definitely depends on how many people are there in the previous slot, right?

And in this if the next slot if a guy is departing, then I am going to remove one from this, but I am going to add a plus here plus alpha and plus one. So if suppose let us say at any point, my xn is 0, that means there are no waiting, there are no customers in the queue, then even though I have written delta as the probability that somebody leaves with probability Mu.

Then actually there is nobody leaving there right. Even if it is one leaving somebody then 0 minus 1 is minus 1 I will just by plus will make it 0. So it will basically take the positive and it will any negative value, it will truncate it to 0 okay. So, only when this xn is going to be get at least one then somebody can depart and then this can become 0 right.

So, that is the meaning of when I said virtual because I can only serve somebody when there is somebody in the, right, delta n plus 1 is 0 by our interpretation, but the way we have written it, so, when I wrote this delta and process I did not care about whether there were some people in the queue or not, I just thought about like when somebody going from this is going to happen this.

But somebody there is there somebody really to account for that I have to add this part, only when this where it can be a departure that is defined by this process only when xn is at least one okay.

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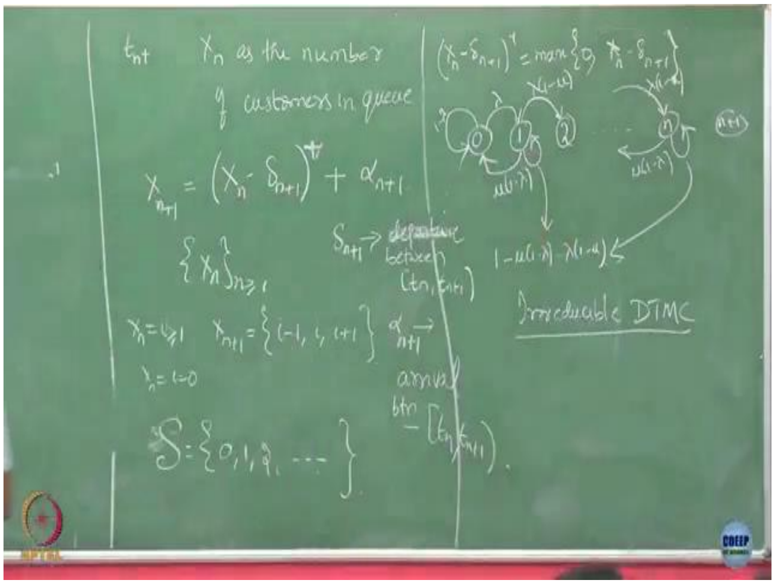

So the way to think about this is more formally xn okay, if this quantity happens to be negative then I will just take it as a 0 okay. Okay fine, so what is this is saying in the n plus 1th round whatever, if whatever I have in the xn^{th} round, if somebody is going to leave, then I will be reduced.

And if something is arriving, then I am going to get increased. So, this is what captures the dynamics of this number of customers in the queue. Is this fine, now the question is if I had a sequence of random variables like this which are defined in this fashion, that is the sequence forms a dtmc? So, if I tell you what is already xn you know what is the value possible values xn plus 1 is taking right suppose, let us say xn at any point n is i.

What are the possible value of xn plus 1?So it can be, go one below, let us assume this i is strictly at least 1 there is some one guy at least in the queue, then it is going to be i minus 1. And can you is it possible for you to calculate what is the probability that it will go to i minus 1? And or i, or i plus 1 right, it can be computed based on the values of this lambda and Mu.

And when this i equals to 0 here what is the possible values of xn plus 1. It can be either 0 or 1, right. So it cannot go negative here, it cannot be minus 1 that is the reason we have put plus 1 here to account for that case. Okay, so okay, let us take when I said, xn plus 1, right, this is at the beginning of this instance and now how I am counting.

Whatever xn plus 1 whatever the value I had here and this delta n plus 1 this is the arrival that has happened in the n plus 1 th round right that can happen anywhere here in the n plus 1th round and what about this delta n plus 1, where this departure, I am actually counting, this is now our convention basically I can just take it this departure all the departure happened just before p n plus 1

Or this we have to this we can take over as our convention. So, this alpha n plus 1 can happen anywhere here but I can take this departure that has happened just before I started my this slot here okay, Okay, then I do not worry about what is the departure happening in this slot. I will just take all the departure that has happened before just beginning of my T n plus 1 slot here.

Okay, now let us try to understand how does the states of the Markov chain look like. So what are the possible statesof this Markov chain?. So suppose let us say at this point I have xn. Now, when I go to this point, I can take into account all the departures and arrivals that has happened in the entire interval, right. So, and that I want to get affected when I am going to count it at xn plus 1.

So if I want to get counted, okay, the way we have defined delta n plus 1, this is just all the departures that have happened just before this, their departure we have taken into account in x n plus 1 but what about, okay, let us see. If I do that, then only it gets captured when I measured at the just the beginning of the Tn plus 1, right okay, so let us write this down.

So the Tn plus 1 is arrival between and I am going to write it as alpha n plus 1. So this is a departure, right and also this is going to be arrival. Again, this is going to be between Tn and I have excluded Tn plus 1 slot in that okay, so now x my state space S is going to be 0, 1, 2 all the way up to this. Now what is going to be the transition probability matrix of this Markov chain, how it is going to look like?

So instaeadof transition probability matrix let us try to draw transition diagram. So my states are 0, 1, 2 lets say n and n plus 1 and then it continues. From stage 0, what all the possible states I can jump to? I can go here, and I can remain here. What are these probabilities? So when I am 0 I can remain at 0, if no arrival happens, right? What is probability of no arrival? 1 minus lambda and this is going to be lambda here.

And now when I am at state 1, what are the possible transitions, what is probability to go from 1 to 2? So I should be the case that a new arrival happens and nobody departures right. And then what is the probability that I go from 1 to 0, there is a departure and no arrival. And is it possible I can stay in the state 1 itself and what is that probability?

So similarly, you can feel all this. Okay, so just to complete, what is this going to be. So let us take it, what are the first, so from n minus 1 I can come to n. What is this probability is going to be? This going to be lambda into 1 minus Mu and what is this going to be, Mu into 1 minus lambda and the self-loop probability is going to be again, going to be this.

So, we have this description we have, you know, it is a Markov chain. It has this transition diagram. Now we are going to see whether it is irreducible and in that case, what type of class it is, this is an irreducible Markov chain? So can you reach any state from any state? Okay, irreducible Markov. Now, what about, can you say anything about whether it is a positive recurrent, null recurrent or transient now.

So, if you want to claim it is positive recurrent, how we are going to do that? So one possibility is if your initial interest is just want to see whether it is a transient or recurrent, one thing you can do is you can pick up a state i and try to see if Pi i equals to 1 or strictly less than 1, is that Pi of i are easy to compute in this case.

Okay, let us instead of taking that down going I mean of course, we can compute a phi of i always, but with lot of taking into account all the possible combinations of how I can reach, I again after a certain number of rounds, but instead of that, let us try to use our results that we have stated and proved, right.

So, if you want to argue that a irreducible DTMC is going to be positive recurrent, what is you need to verify?

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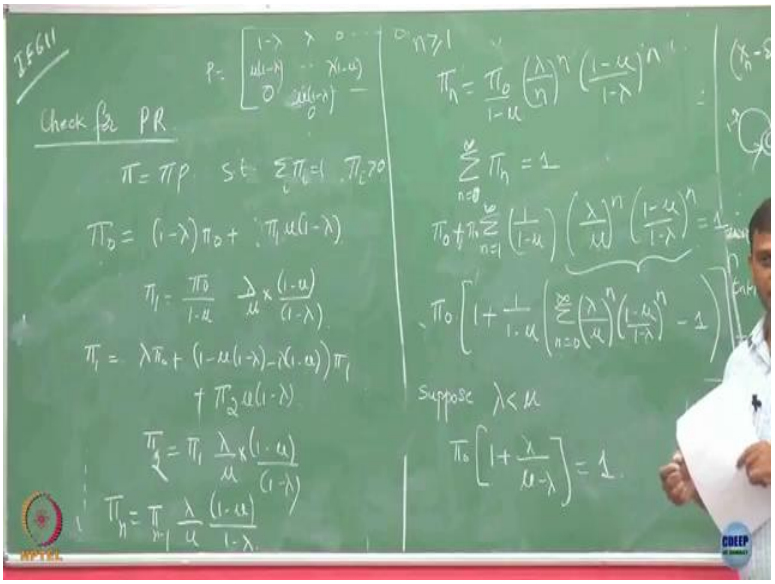

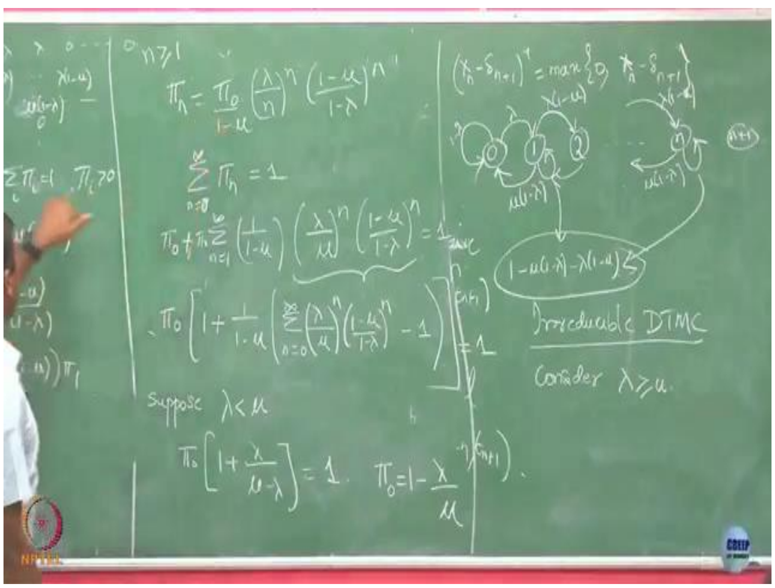

So, you want to check whether, so, check for positive recurrence, okay you want to check whether there exist Pi equals to Pi p, such that summation of Pi equals to 1 and Pi i equals to 0 okay, let us verify whether this I can find such a Pi.

For this transition probability matrix P, the P is given by this transition diagram here okay So this P is given by the transition diagram. Now let us try to see whether I can compute this Pi here, and then check whether it satisfies this property. So this Pi equals to Pi P means we have many equations here, right let us try to write each of those equations. So let us write first in this what is Pi 0 is going to be.

So what is the first equation is going to be, so if you look into this what will the first row consist of? The first row will be 1 minus lambda, and lambda right. So, what is then you expect this to be, into lambda. Now what is your P is going to look like let us write it for something. So 0 to 0 is 1 minus lambda, lambda and all 0s, right. And then what will be second rows going to look like?

Then okay, so this complicated thing right, let us call me I am just writing that complicated things by dash, dash and what is the next thing, lambda into Mu. So what will be the first equation when I am going to write so, Pi 0 is going to be my Pi vector multiplied by this column vector right. So, what is that in that case it is going to be Pi 0 into this and Pi 0 into this quantity, right?

So I have this relation which I will going to write it and after simplification, you can see that if I do like this I get it as Pi 0 into 1 minus Mu into lambda by Mu into 1 minus Mu divided by 1 minus lambda just verify, I have just whatever that equation there I have simplified it.

Okay now let us write the relation for Pi 1. So, what is the Pi 1 relation is going to look like? The Pi 1 relation is now going to multiply the second column here right. So, that is going to be at least lambda into Pi 0 plus that whole thing 1 minus Mu into 1 minus lambda, minus lambda into 1 minus Mu to Pi 1 and what is this term here can anybody say, is this Mu into 1 minus lambda?

So, what is this I want to go from second state to first state right, second state to first state this is going to be lambda 1 minus Mu. Sorry, I want to go this one quantity right. Mu into 1 minus and I know after this it is all going to be 0. So this is going to be Pi 2 into Mu into 1 minus lambda Okay now, I want to plug in whatever the quantity I have for Pi 1 here and write the expression for Pi 2.

Okay, I can do that right like you can again verify that after simplifying I am going to get it as Pi 2 as Pi 1 into lambda by Mu into minus Mu 1 minus lambda here and as you continue to do this we will see that Pi n can be written as Pi n minus 1 times lambda minus Mu and 1 minus Mu 1 minus lambda okay. I have just simplified that you can verify all these things.

Okay so now good I have this then I will do a recursion on Pi n minus 1 like that I can do like this and then actually, after doing this what I can get everything in terms of Pi 0 into lambda by n to the power n and 1 minus Mu 1 minus lambda to the power, is fine?

So, no actually this is going to be n minus 1 here and after this is n but what I will exactly get is, I will have 1 minus Mu here because of this guy 1 minus Mu here, okay. So, if you just do this recursion, you have this Pi knot which is in this format okay Pi n for all n i have written in terms of Pi 0.

So, I will get this equation for all n greater than or equals to 1 right. Now, I will look for this I also need to see whether so to find this. Here mu Pi knot is my free variable right. Once I know my Pi knot, I will get all the Pi ns for n greater than or equals 1, right. So to get this Pi knot what I can do, I know the summation Pi of n is going to be 0, right.

So this is starting from 0, right so now I have Pi 0 plus all these quantities. So n equals to 1 to infinity. I am going to pull this Pi knot out. This is going to be 1 minus new times lambda n to the power n into 1 minus Mu, 1 minus lambda to the power n. I have just simplify the things. and this saw one, this is equal to 1 okay.

Now let us, if you simplify this quantities. What you find is Pi knot plus 1 upon 1 minus Mu so if you, this whole quantity if you are going to. So let us focus on this whole quantity now, what I will do is I have pulled out this 1 minus Mu outside, and what I will do is I will start from n equals to 0 to infinity and lambda minus Mu and 1 minus Mu 1 minus lambda n, and then I am going to subtract 1 from this.

So why I did, so here notice that n was running from n 1 to infinity, right. Now I have letting n go from 0 to infinity for n equals to 0I have got one so to eradicate that subtracted minus 1 from here Okay Is this clear. How did I do this manipulation just like manipulation you can work out yourself.

Okay now let us see, I want this to be equals to 1then I can find a solution, when I can find when the solution here makes sense for Pi knot okay let us see suppose now right now to I have taken lambda Mu to be something, some probabilities right which are between 0, 1 Now, let us assume that lambda is strictly less than Mu okay, so if lambda strictly less than Mu, lambda by Mu is going to be less than 1 right.

And also 1 minus Mu 1 minus lambda is also going to be less than 1. So this product is going to be less than 1 okay. So I have now I have a geometric series where the ratio is less than 1 right, strictly less than 1 so this guy converge, right. So, for this case, I can write this after simplification as.

So, this simplifies a slot and just turns out to be 1 upon Mu minus lambda equals to 1 because I started from n equals to 0 right, so, I have this if you just going to simplify this, what you are going to get, you get Pi knot equals to this lambda. So, you are going to get Mu divided by Mu minus lambda. So, what you are going to get is 1 minus lambda by Mu.

So suppose if that is the case lambda is less than Mu you will end up with a solution Pi knot which is equal to this. Now, we have a complete solution under this assumption, Pi knot lambda by Mu is going to be less than 1 so, this is a Pi knot is a quantity which is less than 1 and also positive. And now if you go back and plug this Pi knot here I will have all the Pi ns which has to be positive.

And the I ensure that those Pi ns add up to 1, right. So when this lambda is less than Mu, I have been able to find a relation Pi equals to Pi p, which satisfies this condition. So, now can I shown that one lambda is going to be less than Mu my irreducible DTMC is a positive recurrent right that is what my theorem said.

Now, let us take the case. When lambda is going to be greater than or equals to Mu. When lambda is going to be greater than or equals to Mu what happens to this quantity. So lambda by Mu is going to be greater than 1, this ratio is also going to be greater than 1. So this whole quantity diverges. So because of this, the only possible solution for Pi knot is going to be 0 and if you plug in zero, all my Pi ns are going to be 0.

So, I will not end up with a solution here. So, the solution in that case happens to be Pi equals to Pi P happens to be 0 in this case, right when I have lambda, so somebody is asking right when is it possible, the only possible solution to Pi equals to Pi P can be 0. So, here you have a situation okay when lambda happens to be greater than Mu, I will end up with this.

So, in that case, it is not going to be positive recurrent, right. So, so now recall that the our results says that it is positive recurrent if and only if this condition happens. In the case where lambda is greater than or equals to Mu this condition does not happen. So, it cannot be positive recurrent. So, maybe when lambda is going to be greater than or equals to Mu either it has to be a transient or null-recurrent right okay, now how to verify that it is transient or null-recurrent.