Introduction to Stochastic Processes
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Lecture 39
Properties of Invariant Probability Vector

So, in the last class, we looked at a nice property of irreducible Markov chain that is positive recurrent. We said that it is going to be positive recurrent if and only if there exist a probability function Pi such that it is the solution of the equation Pi equals to Pi P. So, in that in the last class we just showed existence of such a Pi and we argued that that is going to unique, but one of you pointed, who is that? You right, so about the uniqueness. So, he has a concern that whatever the method we started with we had some way of construction of bns.

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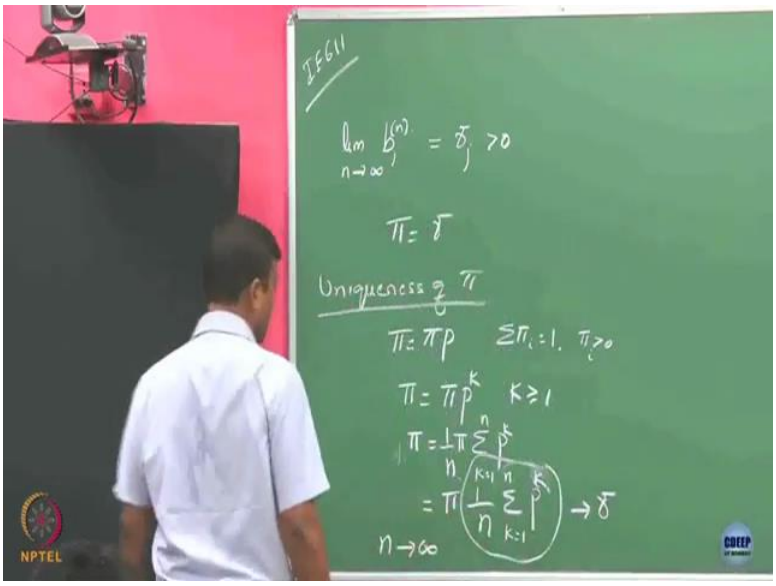

So, we had this bns and this was a vector and we showed that this as n goes to infinity, this was converging to some gamma. So, if I am going to take some (())(1:17) we said it is a this was positive. But this is we know that this is unique, but you are starting with some method where you are going to start with some particular sequence and saying that that is going to converts some later.

So this whatever and we somehow translated this we take Pi equals to gamma. But after normalizing this gamma by dividing it by summation of gamma S. So, by this method, I mean, what are the gamma we got, yes this is an unique solution for the method we started with. But if you have started with another method, maybe you would have constructed another sequence and looked into its limit, maybe what is the guarantee that that would have also given you the same gamma? For uniqueness, I want that.

Irrespective of what method you are going to construct, when you look at the limit, that limit should be the same that is what I wanted to show. So, in a way the last construct, we said for that method the gamma you obtained is unique. But we are going to say that that was a kind of hint for this, now we are going to say that that is going to be the unique, whatever method you are going to use.

So, now we are going to argue that if there any other Pi exist. Let us say which also which satisfies the relation Pi equals to Pi P, that must be necessarily the case that that Pi is exactly equals to this gamma. So, let us try to argue that. So, in the last class it was clear that this Pi is a positive probability vector, we have because all this gamma is are positive.

Now, how to show uniqueness? Now, let us say that Pi is any probability vector with strictly positive elements in this and then let us says that is a solution of Pi equals to Pi P, some Pi, which is going to satisfy this. Now, we are going to argue that this Pi is nothing but this gamma, this Pi must be necessarily this gamma.

Now, how we are going to show that? Here this Pi is such that summation of Pi i equals to 1 and Pi i is equals to 0. So, let us assume that this Pi is such that is a probability vector strict positive elements in it and it satisfies this relation. If it satisfies this relation, I already know that by recreating this recursion, I should be able to get this for all greater than K equals to greater than or equal to 1.

Now, what I do is I will add this for K times. So, this is true for any K equals K. Now, I will take this for K equals to 1 to up to n, and then add all of them. So, if I add all of them on the right-hand side, I am going to get n Pi and on the right-hand side I am going to get, and let us divide them by n.

Now, this guy on the right-hand side, if I just slightly reorganize this, I know already how to handle this. We have already dealt with how to handle this, we have already said that if this is going to be. So, if only if we are going to look into this quantity, this quantity is going to be finite if my state is transient and in that case if I am going to divide it by n any way it will go to 0, we have shown that this quantity in the limit as n goes to infinity already goes to 0, if my state is?

Student: (())(5:57)

Professor: At? I am now looking at 1 by n and null recurrent. But if it is not either of these two then it goes to some constant which we said as gamma j. So, now, just do this we will let, n goes to infinity, so the left hand side is anyway constant or just changing it the right hand side as n changes and where this limit is going? We have already shown that this is gamma.

So, then this Pi is actually equals to gamma only and that gamma is what? That gamma is a specific vector, which is the limit of my sequences. Now, what we have just argued is? This Pi, whatever this Pi that satisfies this relation, that Pi is equals to this gamma.

Student: (())(6:54)

Professor: Right, there is a Pi multiplication here, but when you look at component wise that what we showed. So, now this is in a vector format, you take a specific component j in this and now look at for that. When you look into that this quantity is going to be turned out to be a simple gamma j and now this is a probability vector. So, it will just add to one and you will just get the constant gamma j for that and now just look at the vector, just the same argument we did last time, except that I have just writing compactly it for the vector here.

So, now, we have done the, what we started proving was only the only if part. We are trying to we have tried to show that the necessary condition, that if my Markov chain is irreducible and positivity recurrent then the Pi that exist here that is there exist a Pi which is a solution of this and which is also unique.

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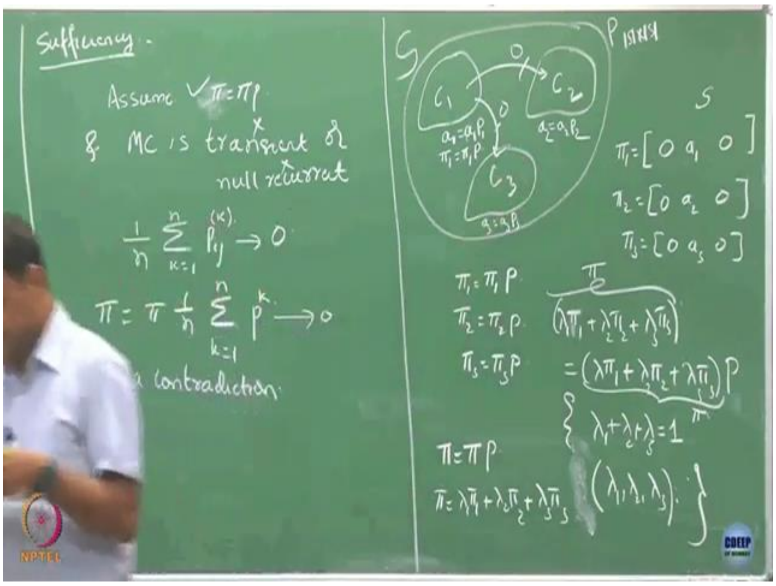

Now we want to say that suppose let us say we want to show the if part, that is a sufficiency. What we want to show now? If indeed such a relation holds Pi equals to Pi P for some positive probability vector, then my irreducible Markov chain is?

Student: Positive.

Professor: Positive recurrent, how we are going to show that? So, we are again going to show this by contradiction. Suppose, assume that you have irreducible Markov chain and let us say it is, it should be of one type, either it should be transient positive recurrent or null recurrent. Suppose, let us assume it this transient or null recurrent.

So, I am assuming, so, assume Pi equals Pi equals to Pi P and then Markov chain is transient or null recurrent, then for this we know that this quantity here goes to 0, as n goes to 0 and again going back to this previous step here what I have, that is Pi equals to Pi 1 by n and K is equals to 1 to n, P to the power K and if I apply my limit as n goes to infinity, what does this say?

This says this guy goes to 0, but what is this? We have assumed this Pi vector to be? Positive, strictly positive. So, this is right away they are going to give us a contradiction. So, it must be the lost case if this relation holds, then none of this should be possible, a contradiction.

Student: P is going to 0 and n is going to?

Professor: Infinity. So, this relation should be true for every n. So, this is this relation whatever we wrote this should be true for any K. So, I can just add all of them and get it for any n and then I let n go to infinity. So, now our part is complete. So, if you handled with a irreducible Markov chain, then how you are going to ensure that that is going to be positive recurrent?

What you first do is? Take this transition probability matrix, check whether there it has a solution Pi equals to Pi P, if it has Pi that solution and all the components in that are going to be strictly positive, then you conclude that this is a positive recurrent matrix. Now, what we did is? We took a irreducible class, and we just focused on that, what if my Markov chain has many communicating classes, which are all closed? For example, how does this result extend?

So, let us say, I have, so, this is my stage space s, I have one communicating class 1 on communicating class 2 and 1. So, let us say my Markov chain reduced into these 3 communicating classes and let us say, each one of them is a closed communicating class. Now then, what would we say? if that is the case, just take focus on this and on the states here, then you can think of your Markov chain on this state space is irreducible, because your entire now state space is one communicating class, and then we did the study this.

But suppose, your Markov chain had multiple things like this. So, how we are going to extend results like this? Now, is this the case that if I look at my solution on the entire space for Pi equals to Pi P, will that Pi is going to be unique? Let us do.

Student: (())(13:35)

Professor: It could be unique for c 1, c 2 and c 3, that we have already shown. Let us say, let me take a 1 here, which is the solution of this to this. So, be watchful here when I write P 1 here, this is the transition probability matrix reduced to the space in this class. So, for this entire thing there is one transition probability matrix which is of size, mod S into mod S. But now, this P 1 is that portion of this matrix which corresponds to this state space that is again a transition probability matrix, because you have just restricted yourself to some particular rows here.

And similarly, P 2 equals to. Now, all of this even a 1, a 2, a 3, are according to us unique, because we have just focused on that particular communicating class. Now, what can we say? So, now extend this to this kind is equals to Pi 1 equals to Pi 1 or entire P. Now, what is this Pi 1? Pi 1, is same as this a 1. But on the places where the states are not included in this state, I am just going to append 0s there getting a sense of what I mean by this?

So, Pi 1 is still on the entire s states, but it is it is going to be some a 1 here and 0s other where. So, that means this this some portion it is a 1 that is corresponding to the states in this class and I am just appending 0s in other places, similarly Pi 2, I mean, the position where this a 1 and a 2, lies need not be at the same this is just at for my representation.

So, if you are going to extend like this, you can still check that, Pi 1 here, is going to be a solution of this equation still, this is going to be the case. Because transition from this class to this class or this class to this class, those probabilities are going to be 0 or this or not there. So, because of that even if you extend it like this, this P will be such that for the portions corresponding to 0s here, the corresponding elements in P is also going to be 0. Because the cross probabilities are going to be 0, just checked that this is in this going to be true for all of them.

So, let us say, this is I have all this are solutions according to our. Now, all of you know what I mean what is a convex combination of two vectors? So, let us take this is a probability vector, this a probability vector, this is a probability vector, let us take their convex combination, will it be a probability vector?

Student: (())(17:51)

Professor: This is going to be a probability vector. So, now if I take lambda 1 Pi 1, lambda 2 Pi 2 plus lambda 3 Pi 3, this is going to be, where this lambda 1 plus lambda 2 plus lambda 3 is 1. I take lambda 1, lambda 2, lambda 3, which they are positive and they sum up to 1 and they have taken their convex combination like this.

So, let us call this another Pi, so, this is the same Pi here, what I have in the in term Markov chain? I have now based on this Pi 1, Pi 2, Pi 3 by taking their convex combination I have obtained a another probability vector which satisfies this equation Pi equals to Pi P. So, this Pi is now, I have such a Pi, where Pi is lambda 1 Pi 1 plus lambda 2 Pi 2 plus lambda 3 Pi 3. But now, is this Pi unique here that satisfies this relation Pi equals to Pi P?

Student: No.

Professor: What?

Student: Lambda 1, lambda 2, lambada 3, (())(19:18)

Professor: Exactly, so this lambda 3 this convex combination, if you change this, so what are this? They are this lambda 1, lambda 2 lambda 3, you take any such things such that, you take any lambda 1, lambda 2, lambda 3, such that it satisfies this, then you will get another Pi here different, different Pi which satisfies this. So, on the whole thing this solution Pi equals to Pi P need not be unique, there could be many Pis that could solve this and specifically this Pis can be obtained as a convex combination of the Pis we obtained on each of these communicating classes.

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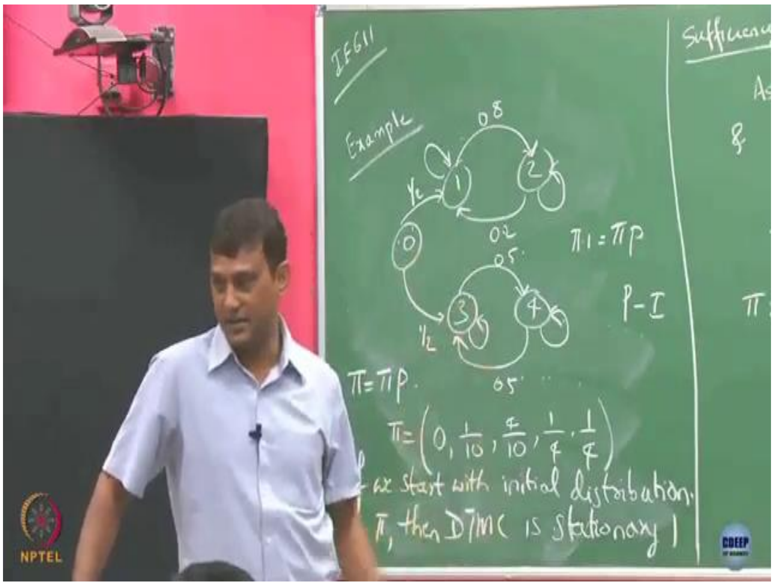

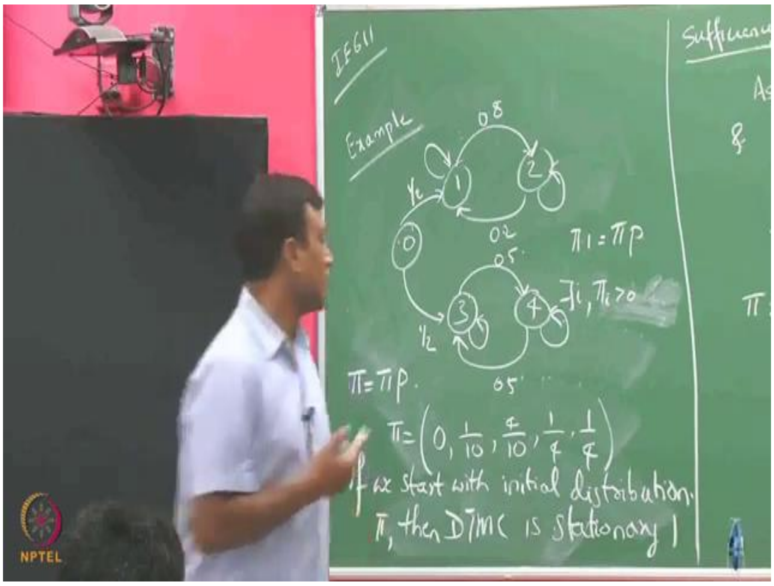

Then, so let us look at another example, an example so for. So, let us I have some 5 states like this, starting from 0, 1, 2, 3 and 4. So, let us draw this transitions. So, let us say I have a Markov chain like this, so think of this initial state as you start throwing some coin here, you toss a fair coin, when the probability half you enter state 1, the probability half you have enter state 3.

So, when you enter state to 1 after that you either go to remain in state 1 or go to state 2, you go to state 2 with probability half, 0.8 and similarly, when you are in state 2, you go to state 1 with 0.2 or remain there itself. Now, this five state Markov chain, in how many classes I can divide this into and what are those classes?

Student: (())(22:20)

Professor: So, will 1 and 0 communicate?

Student: No.

Professor: No. So, 0 and 1 cannot be in the same group, in the same class and similarly, 0 and 3 can be in the same class?

Student: No.

Professor: No, they cannot be. But, can 0, 1 and 2 can be in the same class?

Student: Yes.

Professor: And the 2 and 3 and 4 can be in the same class?

Student: (())(22:38)

Professor: So, then what are the possible classes?

Student: Three classes.

Professor: Three classes. To a class 1, class 2, and this has to be in the separate one. Now, what about class 1? So, let us call this class 1 and let us call this class 2 here.

Student: (())(22:55) 1 and 2 are close.

Professor: 1 and 2 are close. Now, so we can have such a transition like and based on that you can see what are the different classes here? Now, suppose that you want to solve for this equation, it is still not one irreducible class. But suppose let us say you solve this and obtain some solution. For this case it looks like the solution is going to be 0, 1 by 10 and 4 by 10, 1 by 4 and 1 by 4, just check this, I am just dumping this on the board.

If you are going to start your Markov chain with this distribution as your initial distribution, you right now, this I have just told you this is a transition, I have not told you anything about the initial distribution. Suppose, you solve this and you start this is your initial distribution, we already said one property, what was that property?

Student: (())(24:20)

Professor: So, x 1, x 2, x 3, at every point it is going to remain in the same thing. Now, then you can also then in a way that can also implies that, if we start with initial distribution Pi, then my DTMC is stationary. So, we have already discussed what is a stationarity of a process. So, if the Markov chain is such that, if you are going to start with your initial distribution simply to be this invariant distribution, then it is going to be stationary.

Student: (())(25:21) is positive?

Professor: So, I am this is not a irreducible Markov chain here. So that was the case when it is irreducible. So, now I am not saying that this is going to be reducible, this is an arbitrary Markov chain with different, different, different possible set of classes. Now, you just still take this as the solution of this, provided this Pi solution happens to be a probability vector, you take it and make it as your initial distribution and then run it, then your Markov chain is going to be stationary.

So, we have only ensured that my Pi is going to be unique when I have a irreducible positive recurrent one. But here it is neither positive, sorry it is irreducible, I have not yet verified whether this states are recurrent or what. So, I have just said take a solution like this. So, in that case I am not guaranteed to have any unique distribution.

So, that is the thing, like if there are multiple solutions, this satisfies this, and if you are going to start with those different, different possibilities, then the your stationarity, your probability that your Markov chain is going to take a particular state in a particular time that is going to be different.

It depends on what is the initial distribution you have started with and that you can have multiple possibilities in that case, in case if you have many possible Pis in that case. Let us say let us try to see a case where such a thing will not happen, when 1 is a possibility. Can you think of any case where this Pi equals to Pi p cannot be a solution?

Student: (())(27:28)

Professor: So, at least there will be always a Pi equals to Pi P solution will be there, because why? We know that we already argued that Pi is a stochastic matrix. So, Pi has a eigenvalue of 1 and this Pi is what we are basically saying? That this Pi is nothing but the eigenvector corresponding to that 1, we will always have, but the question is, will this add up to 1? If whether it is going to be going to be, if not if it is not going to add up to 1, is there a way you can make it and add up to 1, how?

Student: Normalize.

Professor: Normalize this, then you will have always one such Pi which is going to satisfy this relation, then you can start with that. I can think of a case when it is possible, I have 1 as a eigenvalue, if that eigenvalue will be such that 0 is the only possible eigenvector. Yes?

Student: (())(28:31)

Professor: Yeah, so, we have to think about an example for that, then in general I do not know like under what condition it holds, we have to basically construct an example. So, in a way what this says? If suppose, Pi is equals to 0 is the vector.

Student: (())28:56)

Professor: No, if rank is not full it can have multiple solution, the question is, can 0 be a solution?

Student: (())(29:02)

Professor: No, we are not saying p equals to 1, so this is Pi equals to Pi P. So, let us write it as Pi into 1 into Pi P, this is my, the value. So, suppose this is all 0, this is anyway it will be satisfied.

Student: (())(29:20)

Professor: That is not an issue here.

Student: (())(29:22)

Professor: So, think about this, is there any transition probability matrix where my Pi equals to 0 is the only solution? For so, carve with a proper, so, even forget transition probability matrix, just construct a matrix with this stochastic, we will have eigenvalue 1. Is there a matrix whose with eigenvalue 1 will can have only 0 as the eigenvector? If that is the case? Yes then we will have.

We have an eigenvector when we say by default, let us say it makes sense only when the component at least some components are going to be positive. If that is not the case, then this is going to be this relation is going to be satisfied for any constant for any eigenvalue thus.

Student: (())(30:14)

Professor: What all you are asking now is? Suppose, if I have a such a relation, is it always the case that Pi i is equals to 0 for some i, for some i? As long as one component is going to be positive that is fine. I will have I will come up with a vector which will not have all 0. So, what is that matrix? Or is it that the case that whatever P you are going to start with you will end up with this?

Just think about this, I think we should be able to argue that whatever P you are going to take I will end up with the Pi in which at least one of the component is nonzero. So, because of that, I can always discard the all 0 solutions. So, it looks like Pi here is an eigenvector, but we have to just make sure that our definition of eigenvector is consistent in the sense that, that exclude the case that all the components being 0, we just need to ensure that, just talk about this. So, now what we have? We have just dealt with the case when I have a reducible class, how to say whether it is going to be positive recurrent?

So, now if it is not positive recurrent there are, that means if it is not possible recurrent, that means I will not be able to find a solution Pi equals to Pi P where Pi is going to be my positive is a probability vector with positive component and by the way, notice that we have although also argued that, if one of the component is positive in my vector Pi, it must be the case that all the components are also positive. It is not that only int his solution only one component is positive and other components are going to be 0.

So, we showed it, when we discussed the properties of this statement, when we said that if one component is going to be strictly, it means that all other components are also going to be going to be strictly positive. How to ensure the now, what are the other property? Do have any other properties to say that if this is not going to be, I will not end up with such a Pi, which is a solution of Pi equals to Pi P and we will have all strictly positive element.

I now, end up with some Pi to Pi p solutions here with some of the components to be 0. But that does not say that my irreducible my DTMC is a positive recurrent class. For my DTMC to be positive recurrent, I want that the solution of this Pi equals to Pi P will be such that all the components in that are going to be positive.

So, I first when I have such a big matrix if I want to see that first thing I will do is okay is this irreducible communicating class, if that, try to find Pi equals to Pi P solution, see, if all its components are going to be strictly positive, then you are done, you know it is a positive recurrent.

If you happen to find this Pi equals to Pi P solutions with some of the component is 0, then you know this is not a positive recurrent, it has to be either transient or null recurrent. How to verify this? So, next we are going to look for a condition when we can say that this is going to be transient?